Theory of the Wing Lattice Composed of Arbitrary Airfoils.

By

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This paper deals with the problem of the two-dimensional flow of perfect fluid through the wing lattice composed of airfoils of an arbitrary section. Three methods being developed, the first one is an extension of Munk's airfoil theory, the second one using Fourier expansion and the third one an exact method of conformal representation. These theories have been developed since 1945 and, in this paper, they are explained briefly and also chronologically.

1. Conformal Representation of the Wing Lattice of Flat Plates.

The wing lattice composed of flat plates in the z-plane with pitch d, chord length c, $d/c=\lambda$ and stagger angle γ is transformed into a unit circle in the ς-plane by the following well-known relation,

$$z=\frac{d}{2\pi}\left\{e^{-i\gamma}\,log\,\frac{1+\kappa\varsigma}{1-\kappa\varsigma}+e^{i\gamma}\,log\,\frac{\varsigma+\kappa}{\varsigma-\kappa}\right\}, \tag{1.1}$$

where κ is a constant determined by λ and γ. Expressing $z=x+iy$ and $\varsigma=e^{i\theta}$, we get

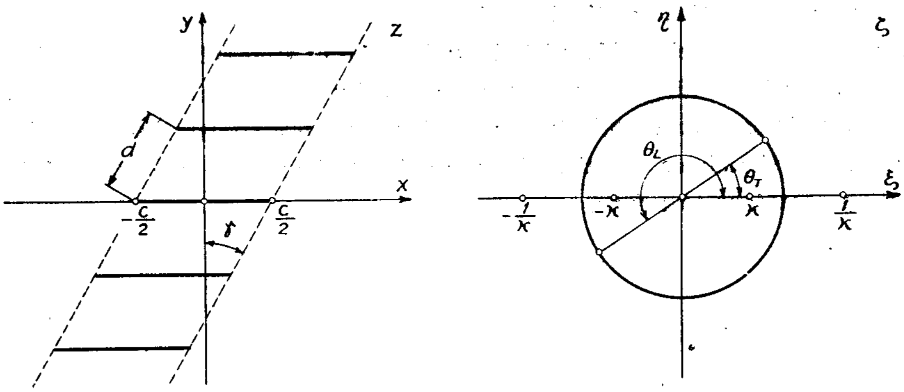

Fig. 1

$$x=\frac{d}{\pi}\left\{\cos\gamma\tanh^{-1}\frac{2\kappa\cos\theta}{1+\kappa^2}+\sin\gamma\tan^{-1}\frac{2\kappa\sin\theta}{1-\kappa^2}\right\}+md\sin\gamma,$$
$$y=md\cos\gamma, \qquad (1.2)$$
$$m=0,\ \pm1,\ \pm2,\ \ldots\ldots$$

The trailing and leading edges correspond to $\varsigma=e^{i\theta_T}$ and $e^{i\theta_L}$ respectively, where

$$\tan\theta_T=\frac{1-\kappa^2}{1+\kappa^2}\tan\gamma,\quad \theta_L=\theta_T+\pi. \qquad (1.3)$$

2.1. The Boundary Condition on the Surface of the Airfoil.

In all the theories explained here, camber and thickness of the airfoil are assumed to be very small.

The boundary condition on the surface of a rigid body is that the normal velocity component of the relative flow must vanish. In the present problem, the uniform flow of mean velocity V passing through the wing lattice has a normal component v_n and a tangential component v_t on the surface of the airfoil as shown in Fig. 2. To satisfy the boundary condition we must cancel ont this normal component by some means. In the first theory, a method distributing sources and sinks of appropriate strength $q(x)$ on the surface of airfoils is applied.

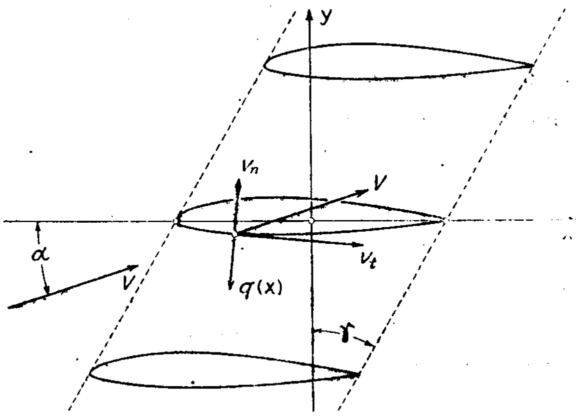

Fig. 2

2.2. Flow due to Source Distribution and Circulation.

The complex velocity potential due to a source of strength Q placed at point $\varsigma=e^{i\theta}$ is

$$W_1=\frac{Q}{\pi}\log(\varsigma-e^{i\theta})-\frac{Q}{2\pi}\log\varsigma,$$

hence the velocity of induced flow u at a point x_0 on the surface of a flat plate due to a soulce of strength Q placed at a point x is given by the following equation assuming x and x_0 correspond to $\varsigma=e^{i\theta}$ and $e^{i\theta_0}$ respectively.

$$u=\left(\frac{dW_1}{d\varsigma}\cdot\frac{d\varsigma}{dz}\right)_{\varsigma=\varsigma_0}$$
$$=-\frac{Q}{4d\,\kappa}\cdot\frac{\kappa^4-2\kappa^2\cos 2\theta_0+1}{\{(1+\kappa^2)\cos\gamma\sin\theta_0-(1-\kappa^2)\sin\gamma\cos\theta_0\}}\cdot$$
$$\frac{\sin(\theta_0-\theta)}{1-\cos(\theta_0-\theta)}\cdot \tag{2.1}$$

If sources of strength $q(x)$ per unit length are distributed on the lower side of each plate and corresponding sink distribution on the upper side, then the velocity of induced flow u_q at the point x_0 is given by eq. (2.1) as follows,

$$u_q(\theta_0)=\frac{\kappa^4-2\kappa^2\cos 2\theta_0+1}{2\pi\{(1+\kappa^2)\cos\gamma\sin\theta_0-(1-\kappa^2)\sin\gamma\cos\theta_0\}}\cdot$$
$$\int_{\theta_T}^{\theta_T+2\pi} q(x)\frac{\{(1+\kappa^2)\cos\gamma\sin\theta-(1-\kappa^2)\sin\gamma\cos\theta\}}{\kappa^4-2\kappa^2\cos 2\theta+1}\cdot$$
$$\frac{\sin(\theta_0-\theta)}{1-\cos(\theta_0-\theta)}\,d\theta\,.$$

In the neighbourhood of the trailing edge i.e. $\theta_0=\theta_T+\varepsilon_0$, where ε_0 is a very small angle, we get

$$u_q(\theta_T+\varepsilon_0)=-\frac{(1-\kappa^4)^2}{2\pi\,\varepsilon_0 K}\int_{\theta_T}^{\theta_T+2\pi} q(x)\frac{1+\cos(\theta_T-\theta)}{K'}\,d\theta\,, \tag{2.2}$$

where $K=\kappa^4+2\kappa^2\cos 2\gamma+1$ and $K'=\kappa^4-2\kappa^2\cos 2\theta+1$.

Next we consider the flow due to the circulation of strength Γ around each airfoil. The complex velocity potential in the ς-plane is

$$W_2=\frac{i\Gamma}{4\pi}\,log\frac{\varsigma^2-\kappa^2}{\varsigma^2-1/\kappa^2}\,, \tag{2.3}$$

and the velocity u_Γ at the point x_0 due to ciculation Γ is as follows,

$$u_\Gamma(\theta_0)=\frac{\Gamma(1-\kappa^4)}{4d\,\kappa\{(1+\kappa^2)\cos\gamma\sin\theta_0-(1-\kappa^2)\sin\gamma\cos\theta_0\}}$$

and in the neighbourhood of the trailing edge

$$u_\Gamma(\theta_T+\varepsilon_0)=\frac{\Gamma(1-\kappa^4)}{4d\,\kappa\varepsilon_0\sqrt{K}}\,. \tag{2.4}$$

2.3. Circulation and Force acting on the Airfoil.

At the trailing edge, both u_q and u_Γ become infinite in the order of $1/\varepsilon_0$ and by the condition of Kutta-Joukowski, we get

$$u_q(\theta_T)+u_\Gamma(\theta_T)=0\,, \tag{2.5}$$

and from this relation Γ is determined.

Let the velocity of mean uniform flow be V, the angle of incidence be α and the ordinate of the airfoil be y, then the strength of source and sink distribution which satisfies the boundary condition on the surface of the airfoil is given by the following relation,

$$q(x)=V\sin\alpha-V\cos\alpha\frac{dy}{dx}\,. \tag{2.6}$$

Hence from eqs. (2.2), (2.4), (2.5) and (2.6) we get

$$\Gamma=\frac{2d\,\kappa(1-\kappa^4)}{\pi\sqrt{K}}\int_{\theta_T}^{\theta_T+2\pi} q(x)\frac{1+\cos(\theta_T-\theta)}{K'}\,d\theta\,. \tag{2.7}$$

The force acting on the airfoil is perpendicular to the direction of V and its magnitude is $\rho V\Gamma$, where ρ is the density of the fluid. Let the force be

$$P=\frac{\rho}{2}V^2\,c\,C_L,$$

then, the lift coefficient C_L is given by the following equation.

$$C_L=\frac{8\lambda\kappa\sin\alpha}{\sqrt{K}}-\frac{4\lambda\kappa\cos\alpha\,(1-\kappa^4)}{\pi\sqrt{K}}\cdot\int_{\theta_T}^{\theta_T+2\pi}\frac{dy}{dx}\cdot\frac{1+\cos(\theta_T-\theta)}{K'}\,d\theta\,. \tag{2.8}$$

2.4. Examples.*

In the case of a flat plate $y=0$ and $\frac{dy}{dx}=0$, hence

$$C_L=\frac{8\lambda\kappa\sin\alpha}{\sqrt{K}}$$

In the case of the parabolic section of camber f

$$y=\frac{4f}{c^2}\left(\frac{c^2}{4}-x^2\right)\text{ and }\frac{dy}{dx}=-\frac{8f}{c^2}x,$$

and after somewhat lengthy calculation, we get the following result.

$$C_L=\frac{8\lambda\kappa\sin\alpha}{\sqrt{K}}+\frac{32\lambda^2 f\cos\alpha}{\pi c}\log\frac{1+\kappa^2}{1-\kappa^2}\,.$$

* The definite integral in eq. (2.8) can be calculated by Simpson's rule and diagrams which facilitate calculation were prepared but they are not shown here.

3.1. Flow satisfying the Boundary Condition on the Surface of the Airfoil.

Now we explain the second theory. We express the contours of the given airfoils as follows,

$$\frac{x}{c}=\frac{\lambda}{\pi}\left\{\cos\gamma\,\tanh^{-1}\frac{2\kappa\cos\theta}{1+\kappa^2}+\sin\gamma\tan^{-1}\frac{2\kappa\sin\theta}{1-\kappa^2}\right\}+m\lambda\sin\gamma,$$

$$\frac{y}{c}=a_0+\sum_{n=1}^{\infty}a_n\cos n\theta'+\sum_{n=1}^{\infty}b_n\sin n\theta'+m\lambda\cos\gamma, \qquad (3.1)$$

where $\theta'=\theta-\theta_T$.

The normal and tangential components v_n and v_t on the surface of the airfoil are given by the following equation,

$$v_n=V\left(\sin\alpha-\cos\alpha\frac{dy}{dx}\right),$$

$$v_t=V\left(\cos\alpha+\sin\alpha\frac{dy}{dx}\right).$$

The corresponding velocities v_r and v_θ on the circle in the ς-plane are

$$v_r=v_n\frac{2d\kappa\sqrt{K}\sin\theta'}{\pi K'} \quad\text{and}\quad v_\theta=-v_t\frac{2d\kappa\sqrt{K}\sin\theta'}{\pi K'},$$

and after some calculation we get

$$v_r=V\sin\alpha\frac{2d\kappa}{\pi}\left\{\sum\kappa^{2n}\cos\gamma\sin(2n+1)\theta-\sum\kappa^{2n}\sin\gamma\cos(2n+1)\theta\right\}$$
$$+V\cos\alpha c\left(\sum nb_n\cos n\theta'-\sum na_n\sin n\theta'\right),$$
$$v_\theta=-V\cos\alpha\frac{2d\kappa\sqrt{K}\sin\theta'}{\pi K'} \qquad (3.2)$$
$$+V\sin\alpha c\left(\sum nb_n\cos n\theta'-\sum na_n\sin n\theta'\right).$$

To satisfy the boundary condition, we add a flow expressed by the following complex velocity potential,

$$W_3=\sum_{n=1}^{\infty}\frac{C_n}{\varsigma^n}+\sum_{n=1}^{\infty}\frac{C'_n}{\varsigma'^n},$$

where $\varsigma'=\varsigma e^{-i\theta_T}$, $C_n=A_n+iB_n$, $C'_n=A'_n+iB'_n$ and A_n, B_n, A'_n and B'_n are constants. Normal and tangential components v_r^* and v_θ^* of this flow are

$$v_r^*=-\sum nA_n\cos n\theta-\sum nB_n\sin n\theta$$
$$-\sum nA'_n\cos n\theta'-\sum nB'_n\sin n\theta',$$
$$v_\theta^*=-\sum nA_n\sin n\theta+\sum nB_n\cos n\theta$$
$$-\sum nA'_n\sin n\theta'+\sum nB'_n\cos n\theta'.$$

If C_n and C'_n are so determined that $v_r+v_r^*=0$, then the boundary condition on the surface of the airfoil is satisfied. Determining in this way the tangential component becomes as follows.

$$\begin{aligned} v_\theta+v_\theta^* = &-V\cos\alpha\,\frac{2d\,\kappa\sqrt{K}\sin\theta'}{\pi K'} \\ &+V\sin\alpha\,\frac{2d\,\kappa\{(1-\kappa^4)\cos\theta'+2\kappa^2\sin 2\gamma\sin\theta'\}}{\pi\sqrt{K}\,K'} \\ &+V\sin\alpha\, c\,(\sum nb_n\cos n\theta'-\sum na_n\sin n\theta') \\ &-V\cos\alpha\, c\,(\sum na_n\cos n\theta'+\sum nb_n\sin n\theta'). \qquad (3.3)\end{aligned}$$

3.2. Circulation and Force.

The velocity $v_{\theta\Gamma}$ in the ζ-plane due to circulation Γ is obtained from eq. (2.3) and

$$v_{\theta\Gamma}=-\frac{\Gamma(1-\kappa^4)}{2\pi K'}. \qquad (3.4)$$

From eqs. (3.3) and (3.4) and by the condition $v_\theta+v_\theta^*+v_{\theta\Gamma}=0$ at $\theta=\theta_T$, we get

$$\Gamma=\frac{4d\,\kappa\,V\sin\alpha}{\sqrt{K}}+\frac{2\pi c\,V(1-\kappa^4)}{K}(\sin\alpha\sum nb_n-\cos\alpha\sum na_n),$$

hence

$$C_L=4\pi\left\{\sin\alpha\left(\frac{2\lambda\kappa}{\pi\sqrt{K}}+\frac{1-\kappa^4}{K}\sum nb_n\right)-\cos\alpha\,\frac{1-\kappa^4}{K}\sum na_n\right\}. \quad (3.5)$$

The velocity w on the surface of the airfoil can be obtained transforming $v_\theta+v_\theta^*+v_{\theta\Gamma}$ into the z-plane and

$$\begin{aligned}\frac{w}{V}=&\cos\alpha\left\{1+\frac{\pi K'}{2\lambda\kappa\sqrt{K}\sin\theta'}(\sum na_n\cos n\theta'+\sum nb_n\sin n\theta')\right. \\ &\left.-\frac{\pi(1-\kappa^4)^2}{2\lambda\kappa\sqrt{K^3}\sin\theta'}\sum na_n\right\} \\ &+\sin\alpha\left\{\frac{1}{K}\left[\frac{(1-\kappa^4)(1-\cos\theta')}{\sin\theta'}-2\kappa^2\sin 2\gamma\right]\right. \\ &-\frac{\pi K'}{2\lambda\kappa\sqrt{K}\sin\theta'}(\sum nb_n\cos n\theta'-\sum na_n\sin n\theta') \\ &\left.+\frac{\pi(1-\kappa^4)^2}{2\lambda\kappa\sqrt{K^3}\sin\theta'}\sum nb_n\right\}. \qquad (3.6)\end{aligned}$$

4, 1. Method of Conformal Representation.**

** An improved method of conformal representation was developed by Assist. Prof. G. Kamimoto, recently, and it will be published in the near future.

In the third theory, we apply the method of conformal representation. Generally, the wing lattice given in the z-plane is transformed into a unit circle in the ς-plane by the following relation,

$$\frac{z}{c}=\frac{\lambda}{2\pi}\left\{e^{-i\gamma}\log\frac{\kappa\varsigma+(1+\varepsilon)}{\kappa\varsigma-(1+\varepsilon)}+e^{i\gamma}\log\frac{\varsigma+\kappa(1+\varepsilon)}{\varsigma-\kappa(1+\varepsilon)}\right\}$$
$$+C_0+\sum_{n=1}^{\infty}\frac{C_n}{\varsigma'^n},\tag{4.1}$$

where $\varsigma'=\varsigma e^{-i\theta_T}$ and ε, C_0 and C_n are small quantities if camber and thickness of the given airfoil are small.

We take x-axis passing through the leading and trailing edges of one airfoil as shown in Fig. 2 and express the contours of the given airfoils using parameter ϑ as the following equations in which $\vartheta'=\vartheta-\theta_T$ and the trailing and leading edges correspond to $\vartheta=\theta_T$ and $\vartheta=\theta_T+\pi$ respectively.

$$\frac{x}{c}=\frac{\lambda}{\pi}\left\{\cos\gamma\tanh^{-1}\frac{2\kappa\cos\vartheta}{1+\kappa^2}+\sin\gamma\tan^{-1}\frac{2\kappa\sin\vartheta}{1-\kappa^2}\right\}$$
$$+m\lambda\sin\gamma,\tag{4.2}$$

$$\frac{y}{c}=a_0+\sum_{n=1}^{\infty}a_n\cos n\vartheta'+\sum_{n=1}^{\infty}b_n\sin n\vartheta'+m\lambda\cos\gamma.\tag{4.3}$$

Putting $\varsigma=e^{i\theta}$ in eq. (4.1), we get the following relations assuming ε and $C_n=A_n+iB_n$ are very small.

$$\frac{x}{c}=\frac{\lambda}{\pi}\left\{\cos\gamma\tanh^{-1}\frac{2\kappa\cos\theta}{1+\kappa^2}+\sin\gamma\tan^{-1}\frac{2\kappa\sin\theta}{1-\kappa^2}\right\}$$
$$+A_0+\sum_{n=1}^{\infty}A_n\cos n\theta'+\sum_{n=1}^{\infty}B_n\sin n\theta'+m\lambda\sin\gamma,\tag{4.4}$$

$$\frac{y}{c}=\frac{2\lambda\varepsilon}{\pi}\sum_{n=0}^{\infty}\kappa^{2n+1}\left\{\sin[\gamma-(2n+1)\theta_T]\cos(2n+1)\theta'\right.$$
$$\left.-\cos[\gamma-(2n+1)\theta_T]\sin(2n+1)\theta'\right\}$$
$$+B_0+\sum_{n=1}^{\infty}B_n\cos n\theta'-\sum_{n=1}^{\infty}A_n\sin n\theta'+m\lambda\cos\gamma.\tag{4.5}$$

Now the angle θ corresponding to the trailing edge may differ slightly from θ_T and we express it by $\theta_T+\delta_T$ and also at the leading edge we express it by $\theta_T+\pi+\delta_L$. And the geometrical conditions at the trailing and leading edges are as follows,

$$\frac{x}{c}=\frac{1}{2} \text{ and } \frac{dx}{d\theta}=0 \text{ at } \theta=\theta_T+\delta_T,$$

$$\frac{x}{c}=-\frac{1}{2} \text{ and } \frac{dx}{d\theta}=0 \text{ at } \theta=\theta_T+\pi+\delta_L.$$

From these conditions and eq. (4.4), neglecting small quantities, we get,

$$\begin{aligned}
&A_0+\sum A_n=0, \\
&-\frac{2\lambda\kappa\sqrt{K^3}}{\pi(1-\kappa^4)^2}\delta_T+\sum nB_n=0, \\
&A_0+\sum(-1)^n A_n=0, \qquad (4.6)\\
&\frac{2\lambda\kappa\sqrt{K^3}}{\pi(1-\kappa^4)^2}\delta_L+\sum(-1)^n nB_n=0.
\end{aligned}$$

Comparing eq. (4.2) with eq. (4.4) we get $\vartheta\approx\theta$ and by further comparison of eqs. (4.3) and (4.5) together with eq. (4.6) and the following relations,

$$\frac{\kappa\cos^2\gamma\sqrt{K}}{1-\kappa^4}=\sum_{n=0}^{\infty}\kappa^{2n+1}\cos(2n+1)\theta_T\cos\gamma.$$

$$\frac{\kappa\sin^2\gamma\sqrt{K}}{1-\kappa^4}=\sum_{n=0}^{\infty}\kappa^{2n+1}\sin(2n+1)\theta_T\sin\gamma,$$

we get

$$\begin{aligned}
A_0 &=\sum b_{2n}, \\
A_{2n} &=-b_{2n}, \\
A_{2n+1} &=-b_{2n+1}+\frac{(1-\kappa^4)\kappa^{2n}}{\sqrt{K}}\cos[\gamma-(2n+1)\theta_T]\sum_{n=0}^{\infty}b_{2n+1}, \\
B_0 &=a_0, \\
B_{2n} &=a_{2n}, \qquad (4.7)\\
B_{2n+1} &=a_{2n+1}+\frac{(1-\kappa^4)\kappa^{2n}}{\sqrt{K}}\sin[\gamma-(2n+1)\theta_T]\sum_{n=0}^{\infty}b_{2n+1}, \\
\varepsilon &=-\frac{\pi(1-\kappa^4)}{2\lambda\kappa\sqrt{K}}\sum_{n=0}^{\infty}b_{2n+1}, \\
\delta_T &=\frac{\pi(1-\kappa^4)^2}{2\lambda\kappa\sqrt{K^3}}\sum na_n-\frac{\pi\kappa(1-\kappa^4)}{\lambda\sqrt{K^3}}\sin 2\gamma\sum_{n=0}^{\infty}b_{2n+1}, \\
\delta_L &=-\frac{\pi(1-\kappa^4)^2}{2\lambda\kappa\sqrt{K^3}}\sum(-1)^n na_n-\frac{\pi\kappa(1-\kappa^4)}{\lambda\sqrt{K^3}}\sin 2\gamma\sum_{n=0}^{\infty}b_{2n+1}.
\end{aligned}$$

2. Flow around Airfoils.

The complex velocity potential of the flow around the unit circle in the plane is as follows.

$$W=\frac{Vd}{2\pi}\left\{e^{-i(\gamma+\alpha)}\log\frac{\kappa\varsigma+(1+\varepsilon)}{\kappa\varsigma-(1+\varepsilon)}+e^{i(\gamma+\alpha)}\log\frac{(1+\varepsilon)\varsigma+\kappa}{(1+\varepsilon)\varsigma-\kappa}\right\}$$
$$+\frac{i\Gamma}{4\pi}\log\frac{\varsigma^2-\left(\frac{\kappa}{1+\varepsilon}\right)^2}{\varsigma^2-\left(\frac{1+\varepsilon}{\kappa}\right)^2}. \tag{4.8}$$

The circulation is determined by the condition $\frac{dW}{d\varsigma}=0$ at $\theta=\theta_T+\delta_T$ and the lift coefficient becomes as follows,

$$C_L=\frac{8\lambda\kappa}{\sqrt{K}}\left\{\sin\alpha\left(1+\delta_T\frac{2\kappa^2\sin 2\gamma}{1-\kappa^4}-\varepsilon\frac{\kappa^4-2\kappa^2\cos 2\gamma+1}{1-\kappa^4}\right)\right.$$
$$\left.-\cos\alpha\left(\delta_T\frac{K}{1-\kappa^4}-\varepsilon\frac{2\kappa^2\sin 2\gamma}{1-\kappa^4}\right)\right\}. \tag{4.9}$$

The velocity w on the surface of the airfoil is also given by the following relation,

$$w=\frac{w_\theta}{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{1}{2}}}, \tag{4.10}$$

where

$$w_\theta=\left|\frac{dW}{d\varsigma}\right|_{\varsigma=e^{i\theta}}$$
$$=-\frac{2V\kappa d}{\pi K'}\{(1+\kappa^2)\cos(\gamma+\alpha)\sin\theta-(1-\kappa^2)\sin(\gamma+\alpha)\cos\theta\}$$
$$-\frac{\Gamma(1-\kappa^4)}{2\pi K'}$$
$$+\frac{2\varepsilon\kappa}{\pi K'^2}\Big[Vd\{\kappa^2(\kappa^4+2\kappa^2\cos 2\theta-3)\sin(\theta+\gamma+\alpha)$$
$$+(3\kappa^4-2\kappa^2\cos 2\theta-1)\sin(\theta-\gamma-\alpha)\}$$
$$-\Gamma\kappa(\kappa^4\cos 2\theta-2\kappa^2+\cos 2\theta)\Big],$$

$$\frac{dx}{d\theta}=-\frac{2\kappa d}{\pi K'}\{(1+\kappa^2)\cos\gamma\sin\theta-(1-\kappa^2)\sin\gamma\cos\theta\}$$
$$-\sum n(A_n\sin n\theta'-B_n\cos n\theta')c,$$
$$\frac{dy}{d\theta}=-\frac{2\varepsilon\kappa d}{\pi K'^2}\{(1-\kappa^2)(\kappa^4+2\kappa^2\cos 2\theta+4\kappa^2+1)\sin\gamma\sin\theta$$
$$+(1+\kappa^2)(\kappa^4+2\kappa^2\cos 2\theta-4\kappa^2+1)\cos\gamma\cos\theta\}$$
$$-\sum n(B_n\sin n\theta'+A_n\cos n\theta')c.$$