On the Calculation of the Buckling Stress of a Rectangular Plate by the Slope Deflection Method.

By

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Synopsis.

The energy method is populary used in the calculation of the buckling stress of a rectangular plate, but, although this method is very convenient when the four sides are simply supported, it is not favourable in other cases. In the case when a rectangular plate is simply supported on the two sides perpendicular to the direction of normal forces and has various boundary conditions on the other two sides, there are R. Barbré's method based on the method of integration and K. Nölke's method based on the method of energy.

According to the former, however, the number of lines and columns of the determinant which represents the equation of buckling condition increases to such an extent that the calculation becomes very difficult. According to the latter, the calculation is possible, but extremely complicated and laboursome. In an attempt to simplify the calculation, the author has induced a formula by the slope deflection method to be applied to such cases and has obtained very satisfactory results.

1. Introduction.

Concerning the problem of the buckling of a plate, ever since G. H. Bryan [1] solved the buckling of a rectangular plate with four edges simply supported, attempts have been made to solve plates with various conditions of supported edges as the case of the buckling of a column. The solution for the plate, however, are far less in number than the solution for the column. This is due to the fact that, compared with the column, the boundary conditions in the case of a plate are in many cases very perplexing.

Most of the studies in the past were made on the buckling of a simple rectangular plate, but in order to resist a greater buckling load it is necessary to use a plate reinforced by stiffeners or a plate with various thickness such as the thickness is increased step by step or changes gradually.

S. Timoshenko has studied some of these problems, but it is only recently since scholars started to pay attention to most of them.

The method of solution of the buckling of a plate can be classified into the following threes.

a) Method of integration. This method is to solve the differential equation of the buckled plate, derive the conditional equation of buckling by using the boundary conditions and with this equation calculate the buckling load. But it cannot be said that the equation of the buckled plate is always solvable. The solution is possible in a particular case when the rigidity of the plate is constant and when the forces acting on the plate is uniformly distributed. Therefore, we cannot always derive the buckling equation of any sort of plate when the boundary conditions and forces applied are arbitrary.

b) Method of energy. In this case it is necessary to assume the equation of the deflected plate. If this assumed deflected plate coincides with the true deflected plate, the true value of the critical buckling load is obtained, but if the assumption is not true, the obtained result is no more than an approximate value. Also it is never easy to assume the deflected plate which always satisfies all boundary conditions.

c) Method of difference equation. The solution is always possible according to this method, but compared with the case of the column, the calculation is extremely laboursome.

In the buckling of a column longitudinally compressed, the slope deflection method has been used by R. v. Mises and J. Ratzersdorfer[2], S. Ban[3] and D. Hiura[4]. Considering the benefit of this method, the author derived the slope deflection method for the plate which is similar to that for the compressed column, and applying this method to the problems which have been studied by many scholars in the past, the author's method was ascertained to be very effective.

2. The fundamental formula by the slope deflection method of the uniformly compressed rectangular plate simply supported along two opposite sides perpendicular to the direction of compression.

In the discussion of buckling, both the method of energy and the method of integration of the differential equation for the deflected plate can be used. In applying the method of integration, we use the following equation, which is for the case of uniform compression along the x-axis (see Fig. 1), with q considered positive for compression,

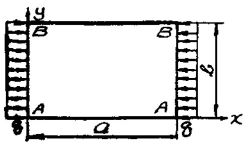

Fig. 1

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=-\frac{q}{N}\frac{\partial^2 w}{\partial x^2} \qquad (1)^{5)}$$

where N is the flexural rigidity of the plate.

Assuming that the plate under the action of compressive forces buckles in m sinusoidal half-waves, we shall presume the solution of eq. (1) in the following form

$$w=\sum Y(y)\sin\frac{m\pi x}{a} \qquad (2)$$

in which $Y(y)$ is a function of y alone, which is to be determined later. Expression (2) satisfies the boundary conditions along the simply supported sides $x=0$ and $x=a$ of the plate, since

$$w=0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}=0, \qquad \text{for} \qquad x=0 \quad \text{and} \quad x=a$$

Substituting eq. (2) in eq. (1), we obtain the following ordinary differential equation to determinate the function $Y(y)$:

$$Y''''-2\left(\frac{m\pi}{a}\right)^2 Y''+\left\{\left(\frac{m\pi}{a}\right)^4-\frac{q}{N}\left(\frac{m\pi}{a}\right)^2\right\}Y=0 \qquad (3)$$

Noting that, owing to some constraints along the sides $y=0$ and $y=b$, we always have $\frac{q}{N}>\left(\frac{m\pi}{a}\right)^2$ and, using the notations

$$\alpha^2=\left(\frac{m\pi}{a}\right)^2+\sqrt{\frac{q}{N}\left(\frac{m\pi}{a}\right)^2}, \qquad \beta^2=-\left(\frac{m\pi}{a}\right)^2+\sqrt{\frac{q}{N}\left(\frac{m\pi}{a}\right)^2} \qquad (4)$$

the general solution of eq. (3) can be represented in the following form;

$$Y(y)=C_1\cosh\alpha y+C_2\sinh\alpha y+C_3\cos\beta y+C_4\sin\beta y \qquad (5)$$

The constants of integration in this solution must be determined by the conditions of constraint along the sides $y=0$ and $y=b$.

Now let us represent the boundary conditions along the sides $y=0$ and $y=b$ as follows,

$$\left.\begin{aligned} &w=\sum\delta_A\sin\frac{m\pi x}{a}, \quad M=\sum M_A\sin\frac{m\pi x}{a}, \qquad \text{for} \qquad y=0 \\ &w=\sum\delta_B\sin\frac{m\pi x}{a}, \quad M=\sum M_B\sin\frac{m\pi x}{a}, \qquad \text{for} \qquad y=b \end{aligned}\right\} \qquad (6)$$

Then, the four constants of integration can be determined from eq. (6) as follows.

$$\left.\begin{aligned}
C_1&=\frac{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2}{\alpha^2+\beta^2}-\frac{M_A}{N}\frac{1}{\alpha^2+\beta^2}\\
C_2&=\delta_B\frac{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2}{(\alpha^2+\beta^2)\sinh\alpha b}-\delta_A\frac{\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b}{(\alpha^2+\beta^2)\sinh\alpha b}\\
&\quad+\frac{M_A}{N}\frac{\cosh\alpha b}{(\alpha^2+\beta^2)\sinh\alpha b}-\frac{M_B}{N}\frac{1}{(\alpha^2+\beta^2)\sinh\alpha b}\\
C_3&=\delta_A\frac{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2}{\alpha^2+\beta^2}+\frac{M_A}{N}\frac{1}{\alpha^2+\beta^2}\\
C_4&=\delta_B\frac{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2}{(\alpha^2+\beta^2)\sin\beta b}-\delta_A\frac{\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\cos\beta b}{(\alpha^2+\beta^2)\sin\beta b}\\
&\quad-\frac{M_A}{N}\frac{\cos\beta b}{(\alpha^2+\beta^2)\sin\beta b}-\frac{M_B}{N}\frac{1}{(\alpha^2+\beta^2)\sin\beta b}
\end{aligned}\right\}\quad(7)$$

Next, we shall assume that the other boundary conditions are represented as follows.

$$\left.\begin{aligned}
\theta&=\sum\theta_A\sin\frac{m\pi x}{a},\quad V=\sum V_A\sin\frac{m\pi x}{a},\quad \text{for}\quad y=0\\
\theta&=\sum\theta_B\sin\frac{m\pi x}{a},\quad V=\sum V_B\sin\frac{m\pi x}{a},\quad \text{for}\quad y=b
\end{aligned}\right\}\quad(8)$$

By using eq. (7), the slope and shearing force in eq. (8) can be written as follows.

$$\begin{aligned}
\Delta&=(\alpha^2+\beta^2)\sinh\alpha b\sin\beta b\\
\Delta\theta_A&=\frac{M_A}{N}(\alpha\cosh\alpha b\sin\beta b-\beta\sinh\alpha b\cos\beta b)+\frac{M_B}{N}(\beta\sinh\alpha b-\alpha\sin\beta b)\\
&\quad-\delta_A\left[\alpha\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b\sin\beta b+\beta\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\cos\beta b\right]\\
&\quad+\delta_B\left[\alpha\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\sin\beta b+\beta\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\right]\\
\Delta\theta_B&=-\frac{M_A}{N}(\beta\sinh\alpha b-\alpha\sin\beta b)-\frac{M_B}{N}(\alpha\cosh\alpha b\sin\beta b-\beta\sinh\alpha b\cos\beta b)\\
&\quad-\delta_A\left[\alpha\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\sin\beta b+\beta\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\right]
\end{aligned}$$

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\begin{aligned}
&+\delta_B\left[\alpha\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b\sin\beta b+\beta\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\cos\beta b\right].\\
-\varDelta V_A=&M_A\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b\sin\beta b+\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\right.\\
&\qquad\left.\times\sinh\alpha b\cos\beta b\right]\\
&-M_B\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\tau}{a}\right)^2\right\}\sin\beta b+\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\right]\\
&-\delta_A N\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b\sin\beta b\right.\\
&\qquad\left.-\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\cos\beta b\right]\\
&+\delta_B N\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\sin\beta b\right.\\
&\qquad\left.-\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\alpha^2-\nu\left(\frac{m\pi}{b}\right)^2\right\}\sinh\alpha b\right]\\
-\varDelta V_B=&M_A\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\sin\beta b+\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\right]\\
&-M_B\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b\sin\beta b+\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\right.\\
&\qquad\left.\times\sinh\alpha b\cos\beta b\right]\\
&-\delta_A N\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\sin\beta b\right.\\
&\qquad\left.-\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\right]\\
&+\delta_B N\left[\alpha\left\{\alpha^2-(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\beta^2+\nu\left(\frac{m\pi}{a}\right)^2\right\}\cosh\alpha b\sin\beta b\right.\\
&\qquad\left.-\beta\left\{\beta^2+(2-\nu)\left(\frac{m\pi}{a}\right)^2\right\}\left\{\alpha^2-\nu\left(\frac{m\pi}{a}\right)^2\right\}\sinh\alpha b\cos\beta b\right]
\end{aligned}
\right\}\quad(9)
$$

Taking into account the fact that the smallest value of q is obtained by taking $m=1$, *i.e.*, by assuming that the buckled plate has only one half-wave, we shall take $m=1$, and for the purpose of making the above equation resemble the equation of the slope deflection method used in the solution of a rigid frame and continuous beam, we shall write $M_A=M_{AB}$, $M_B=-M_{BA}$, $V_A=V_{AB}$, and also $V_B=V_{BA}$.

The sign of the slope expressed in eq. (8) must be determined in accordance with the rule adopted in the slope deflection method, (clockwise positive, counter-clockwise negative,) but in this case the sign appearing may be taken as it is.

Therefore, taking $m=1$,

$$\alpha=\frac{\pi}{a}A,\quad A=\sqrt{1+\sqrt{z\frac{a^2}{b^2}}},\quad \beta=\frac{\pi}{a}B,\quad B=\sqrt{-1+\sqrt{z\frac{a^2}{b^2}}},$$

where

$$z=\frac{qb^2}{\pi^2N},$$

thus the following equations are obtained.

$$\left.\begin{aligned}\frac{\pi}{a}\theta_A&=\frac{M_{AB}}{N}c(z)-\frac{M_{BA}}{N}s(z)+\frac{\pi^2}{a^2}\left\{\delta_B d(z)-\delta_A t(z)\right\}\\ \frac{\pi}{a}\theta_B&=-\frac{M_{AB}}{N}s(z)+\frac{M_{BA}}{N}c(z)+\frac{\pi^2}{a^2}\left\{\delta_B t(z)-\delta_A d(z)\right\}\end{aligned}\right\}\quad(10)$$

$$\left.\begin{aligned}-\frac{V_{AB}}{N}&=\frac{\pi}{a}\left[\frac{M_{BA}}{N}t(z)+\frac{M_{BA}}{N}d(z)+\frac{\pi^2}{a^2}\left\{\delta_B e(z)-\delta_A u(z)\right\}\right]\\ -\frac{V_{BA}}{N}&=\frac{\pi}{a}\left[\frac{M_{AB}}{N}d(z)+\frac{M_{BA}}{N}t(z)+\frac{\pi^2}{a^2}\left\{\delta_B u(z)-\delta_A e(z)\right\}\right]\end{aligned}\right\}\quad(11)$$

where

$$\left.\begin{aligned}&\Delta=(A^2+B^2)\sinh\alpha b\sin\beta b\\ &c(z)\Delta=A\cosh\alpha b\sin\beta b-B\sinh\alpha b\cos\beta b\\ &s(z)\Delta=B\sinh\alpha b-A\sin\beta b\\ &d(z)\Delta=A(B^2+\nu)\sin\beta b+B(A^2-\nu)\sinh\alpha b\\ &t(z)\Delta=A(B^2+\nu)\cosh\alpha b\sin\beta b+B(A^2-\nu)\sinh\alpha b\cos\beta b\\ &u(z)\Delta=A(B^2+\nu)^2\cosh\alpha b\sin\beta b-B(A^2-\nu)^2\sinh\alpha b\cos\beta b\\ &e(z)\Delta=A(B^2+\nu)^2\sin\beta b-B(A^2-\nu)^2\sinh\alpha b\end{aligned}\right\}\quad(12)$$

From eq. (10), we can represent the bending moment M in terms of slope θ and deflection δ, and also the shearing force V in a form similar to that of the moment. The result thus obtained is as follows.

$$\left.\begin{aligned}M_{AB}&=\frac{N}{c^2(z)-s^2(z)}\frac{\pi}{a}\left[c(z)\theta_A+s(z)\theta_B-\frac{\pi}{a}\left\{j(z)\delta_B-i(z)\delta_A\right\}\right]\\ M_{BA}&=\frac{N}{c^2(z)-s^2(z)}\frac{\pi}{a}\left[s(z)\theta_A+c(z)\theta_B-\frac{\pi}{a}\left\{i(z)\delta_B-j(z)\delta_A\right\}\right]\end{aligned}\right\}\quad(13)$$

$$\left.\begin{aligned}V_{AB}&=-\frac{N}{c^2(z)-s^2(z)}\frac{\pi^2}{a^2}\left[i(z)\theta_A+j(z)\theta_B-\frac{\pi}{a}\left\{l(z)\delta_B-h(z)\delta_A\right\}\right]\\ V_{BA}&=-\frac{N}{c^2(z)-s^2(z)}\frac{\pi^2}{a^2}\left[j(z)\theta_A+i(z)\theta_B-\frac{\pi}{a}\left\{h(z)\delta_B-l(z)\delta_A\right\}\right]\end{aligned}\right\}\quad(14)$$

where

$$\left.\begin{aligned}i(z)&=c(z)\,t(z)+s(z)d(z)\\ j(z)&=c(z)d(z)+s(z)\,t(z)\\ h(z)&=i(z)\,t(z)+j(z)d(z)-u(z)\{c^2(z)-s^2(z)\}\\ l(z)&=i(z)d(z)+j(z)\,t(z)-e(z)\{c^2(z)-s^2(z)\}\end{aligned}\right\}\quad(15)$$

This expression resembles the formula of a column subjected to compressive force which is written as follows,

$$M_{AB}=\frac{EJ}{l}\,\frac{1}{c^2(z)-s^2(z)}\left[c(z)\varphi_A+s(z)\varphi_B-\frac{\delta_B-\delta_A}{l}\{c(z)+s(z)\}\right]$$

$$M_{BA}=\frac{EJ}{l}\,\frac{1}{c^2(z)-s^2(z)}\left[s(z)\varphi_A+c(z)\varphi_B-\frac{\delta_B-\delta_A}{l}\{c(z)+s(z)\}\right]$$

$$z^2=\frac{Sl^2}{EJ},\qquad c(z)=\frac{1}{z^2}-\frac{\cot z}{z},\qquad s(z)=-\frac{1}{z^2}+\frac{\operatorname{cosec} z}{z}$$

Therefore eq. (13), (14) are called the formulae of the compressed plate based upon the slope deflection method.

For special cases when M_{AB} is equal to zero at the side $y=0$,

$$\left.\begin{aligned}M_{BA}&=\frac{N}{c^2(z)-s^2(z)}\,\frac{\pi}{a}\left[\frac{c^2(z)-s^2(z)}{c(z)}\,\theta_B-\frac{\pi}{a}\{i'(z)\delta_B-j'(z)\delta_A\}\right]\\ V_{BA}&=-\frac{N}{c^2(z)-s^2(z)}\,\frac{\pi^2}{a^2}\left[i'(z)\theta_B-\frac{\pi}{a}\{h'(z)\delta_B-l'(z)\delta_A\}\right]\end{aligned}\right\}\quad(16)$$

and when M_{BA} is equal to zero at the side $y=b$,

$$\left.\begin{aligned}M_{AB}&=\frac{N}{c^2(z)-s^2(z)}\,\frac{\pi}{a}\left[\frac{c^2(z)-s^2(z)}{c(z)}\,\theta_A-\frac{\pi}{a}\{j'(z)\delta_B-i'(z)\delta_A\}\right]\\ V_{AB}&=-\frac{N}{c^2(z)-s^2(z)}\,\frac{\pi^2}{a^2}\left[i'(z)\theta_A-\frac{\pi}{a}\{l'(z)\delta_B-h'(z)\delta_A\}\right]\end{aligned}\right\}\quad(17)$$

where

$$\left.\begin{aligned}i'(z)&=i(z)-s(z)/c(z)\cdot j(z), & h'(z)&=h(z)-j(z)/c(z)\cdot j(z)\\ j'(z)&=j(z)-s(z)/c(z)\cdot i(z), & l'(z)&=l(z)-j(z)/c(z)\cdot i(z)\end{aligned}\right\}\quad(18)$$

Among the functions included in the formulae derived above, those necessary in the calculation hereafter are the following 11, that is, $\frac{1}{c}$, $\frac{c}{c^2-s^2}$, $\frac{s}{c^2-s^2}$, $\frac{i}{c^2-s^2}$, $\frac{j}{c^2-s^2}$, $\frac{h}{c^2-s^2}$, $\frac{l}{c^2-s^2}$, $\frac{i'}{c^2-s^2}$, $\frac{j'}{c^2-s^2}$, $\frac{h'}{c^2-s^2}$ and $\frac{l'}{c^2-s^2}$.

By rewritting these functions in a form convenient for doing the calculation, we get

a/b =1	z	$\frac{1}{c}$	$\frac{c}{c^2-s^2}$	$\frac{s}{c^2-s^2}$	$\frac{i}{c^2-s^2}$	$\frac{j}{c^2-s^2}$	$\frac{h}{c^2-s^2}$	$\frac{l}{c^2-s^2}$	$\frac{i'}{c^2-s^2}$	$\frac{j'}{c^2-s^2}$	$\frac{h'}{c^2-s^2}$	$\frac{l'}{c^2-s^2}$
tension	5.0	2.62408	2.63099	0.13482	2.75196	0.11000	6.43624	-0.04115	2.74632	-0.03101	6.43164	-0.15621
	4.5	2.58375	2.59249	0.15048	2.64917	0.13650	6.07482	0.00009	2.64125	-0.01726	6.06763	-0.13940
	4.0	2.54094	2.55198	0.16778	2.54221	0.16590	5.69424	0.04716	2.53131	-0.00124	5.68345	-0.11810
	3.5	2.49534	2.51389	0.18694	2.43067	0.19893	5.30911	0.10077	2.41588	0.01781	5.29340	-0.09200
	3.0	2.44637	2.46397	0.20825	2.31389	0.23609	4.91299	0.16234	2.29393	0.04053	4.89037	-0.05938
	2.5	2.39368	2.41592	0.23195	2.19141	0.27783	4.50495	0.23320	2.16480	0.06737	4.47312	-0.01893
	2.0	2.33635	2.36461	0.25849	2.06229	0.32504	4.08310	0.31434	2.02675	0.09960	4.03842	0.03086
	1.5	2.27333	2.31027	0.28819	1.92588	0.37834	3.64618	0.40598	1.87885	0.13798	3.58451	0.09038
	1.0	2.20459	2.25058	0.32173	1.78080	0.43907	3.19141	0.51175	1.71804	0.18450	3.10575	0.16433
	0.5	2.13248	2.18185	0.35955	1.62617	0.50811	2.73783	0.63412	1.54398	0.23978	2.59921	0.25478
	0.	2.04879	2.10840	0.40272	1.45983	0.58754	2.21878	0.77675	1.34760	0.30870	2.05505	0.36994
compression	0.5	1.94609	2.03606	0.45186	1.28051	0.67860	1.69413	0.94183	1.11947	0.39367	1.47608	0.51368
	1.0	1.82433	1.95659	0.50872	1.08473	0.78473	1.13620	1.13620	0.86030	0.50270	0.82146	0.70114
	1.5	1.68735	1.86366	0.57427	0.87062	0.90777	0.54123	1.39473	0.58251	0.63696	0.10478	0.93471
	2.0	1.51599	1.75745	0.65142	0.63214	1.05371	-0.10270	1.63549	0.24156	0.81940	-0.73448	1.25648
	2.5	1.31051	1.63815	0.74129	0.36644	1.22534	-0.78042	2.02022	-0.17161	1.04762	-1.69007	1.66188
	3.0	1.01651	1.49940	0.85015	0.06207	1.43409	-1.57702	2.35272	-0.75105	1.39889	-2.94865	2.29335
	3.5	0.62096	1.33911	0.98065	-0.28514	1.68406	-2.43835	2.82861	-1.51839	1.89287	-4.55622	3.18720
	4.0	0.	1.14474	1.14474	-0.70000	2.00000	-3.43421	3.43421	-2.70000	2.70000	-6.92846	4.65720
	4.5	-1.09120	0.90937	1.34879	-1.19615	2.39551	-4.59159	4.19639	-4.74922	4.16966	-10.90199	7.34736
	5.0	-3.61935	0.61400	1.61223	-1.81114	2.90808	-5.98471	5.18919	-9.44711	7.66373	-19.75821	13.76727

a/b =2	z	$\frac{1}{c}$	$\frac{c}{c^2-s^2}$	$\frac{s}{c^2-s^2}$	$\frac{i}{c^2-s^2}$	$\frac{j}{c^2-s^2}$	$\frac{h}{c^2-s^2}$	$\frac{l}{c^2-s^2}$	$\frac{i'}{c^2-s^2}$	$\frac{j'}{c^2-s^2}$	$\frac{h'}{c^2-s^2}$	$\frac{l'}{c^2-s^2}$
tension	5.0	3.32062	3.50155	0.79596	5.11367	1.47356	15.68242	1.32827	4.77870	0.31114	15.06230	-0.82371
	4.5	3.25836	3.45565	0.82568	4.92988	1.56070	14.71059	1.57040	4.55697	0.38276	14.00573	-0.65611
	4.0	3.19309	3.40841	0.85667	4.74168	1.65182	13.72399	1.82503	4.32651	0.46004	12.92346	-0.47294
	3.5	3.12455	3.35976	0.88898	4.54884	1.74714	12.72185	2.09284	4.08658	0.54350	11.81340	-0.27272
	3.0	3.05237	3.30961	0.92269	4.35107	1.84693	11.70329	2.37472	3.83616	0.63388	10.67261	-0.05340
	2.5	2.97626	3.25789	0.95789	4.14814	1.95143	10.66753	2.67143	3.57443	0.73174	9.49879	0.18665
	2.0	2.89575	3.20450	0.99467	3.93972	2.06095	9.61350	2.98402	3.30000	0.83808	8.28800	0.45020
	1.5	2.81047	3.14936	1.03313	3.72554	2.17582	8.54031	3.31340	3.01184	0.95363	7.03728	0.73935
	1.0	2.71976	3.09234	1.07338	3.50520	2.29637	7.44669	3.66077	2.70811	1.07969	5.74140	1.05780
	0.5	2.62735	3.03334	1.11551	3.27838	2.42295	6.33159	4.02653	2.38745	1.21723	4.39653	1.40826
	0.	2.51975	2.97222	1.15967	3.04462	2.55596	5.19353	4.41402	2.04736	1.36804	2.99554	1.79580
compression	0.5	2.40894	2.90887	1.20598	2.80354	2.69584	4.03129	4.82259	1.68606	1.53341	1.53334	2.22402
	1.0	2.28947	2.84310	1.25460	2.55460	2.84310	2.84310	5.25460	1.30002	1.71582	0	2.70000
	1.5	2.16050	2.77480	1.30567	2.29734	2.99820	1.62749	5.71148	0.88716	1.97959	-1.61134	3.22862
	2.0	2.02025	2.70374	1.35940	2.03112	3.16177	0.38237	6.19531	0.44142	2.14055	-3.31501	3.82011
	2.5	1.86752	2.62977	1.42188	1.75539	3.33435	-0.89401	6.70781	-0.04256	2.38805	-5.12047	4.48121
	3.0	1.69969	2.55264	1.47556	1.46936	3.51673	-2.20419	7.25151	-0.56893	2.66590	-7.04913	5.22720
	3.5	1.51506	2.47216	1.53843	1.17240	3.70955	-3.55026	7.82846	-1.13210	2.97858	-9.11438	6.06766
	4.0	1.30948	2.38802	1.60486	0.86353	3.91377	-4.93533	8.44176	-1.76670	3.33343	-11.34967	7.02651
	4.5	1.07996	2.29997	1.67510	0.57194	4.13020	-6.36206	9.09405	-2.43615	3.71365	-13.82392	8.06698
	5.0	0.82128	2.20768	1.74949	0.20656	4.35990	-7.83387	9.78872	-3.24847	4.19621	-16.44416	9.38078

$a/b=3$	z	$\frac{1}{c}$	$\frac{c}{c^2-s^2}$	$\frac{s}{c^2-s^2}$	$\frac{i}{c^2-s^2}$	$\frac{j}{c^2-s^2}$	$\frac{h}{c^2-s^2}$	$\frac{l}{c^2-s^2}$	$\frac{i'}{c^2-s^2}$	$\frac{j'}{c^2-s^2}$	$\frac{h'}{c^2-s^2}$	$\frac{l'}{c^2-s^2}$
tension	5.0	4.00225	4.52220	1.53068	8.34942	4.33512	29.75350	7.26107	6.88207	1.50899	25.59773	-0.74294
	4.5	3.93961	4.48310	1.56092	8.13083	4.45148	28.15244	7.73500	6.58091	1.62049	23.73236	-0.33848
	4.0	3.87513	4.44339	1.58902	7.90937	4.57193	26.54655	8.22344	6.27439	1.74343	21.84236	0.08527
	3.5	3.80869	4.40305	1.61770	7.68493	4.69438	24.92212	8.72298	5.96019	1.87090	19.91713	0.52957
	3.0	3.74019	4.36207	1.64702	7.45743	4.81970	23.28328	9.23536	5.63761	2.00394	17.95794	0.99557
	2.5	3.66950	4.32042	1.67696	7.22688	4.94790	21.62956	9.76085	5.30636	2.14279	15.96305	1.48436
	2.0	3.59660	4.27813	1.70762	6.99322	5.07909	19.96082	10.30010	4.96590	2.28775	13.93081	1.99760
	1.5	3.52109	4.23504	1.73886	6.75593	5.21341	18.27596	10.85299	4.61542	2.43949	11.85817	2.53627
	1.0	3.44309	4.19127	1.77083	6.51545	5.35086	16.57513	11.42041	4.25469	2.59806	9.74386	3.10235
	0.5	3.36235	4.14674	1.80351	6.27138	5.49156	14.85762	12.00265	3.82806	2.76400	7.58509	3.69740
	0.	3.27873	4.10148	1.83698	6.02382	5.63579	13.12330	12.60053	3.49965	2.93783	5.37924	4.32328
compression	0.5	3.19203	4.05540	1.87118	5.77247	5.78342	11.37123	13.21408	3.10400	3.11995	3.12357	4.98187
	1.0	3.10205	4.00850	1.90617	5.51728	5.93465	9.60193	13.84400	2.69517	3.31100	0.81468	5.67556
	1.5	3.00861	3.96076	1.94198	5.25816	6.08961	7.81219	14.49082	2.27243	3.51149	-1.55039	6.40639
	2.0	2.91143	3.91215	1.97863	4.99496	6.24842	6.00410	15.15513	1.83473	3.72214	-3.97577	7.17726
	2.5	2.81030	3.86264	2.01608	4.72757	6.41119	4.17620	15.83747	1.38134	3.94363	-6.46491	7.99058
	3.0	2.70491	3.81220	2.05456	4.45583	6.57806	2.32784	16.53849	0.91063	4.17663	-9.02280	8.84983
	3.5	2.58309	3.76081	2.09450	4.17962	6.74918	0.45841	17.25881	0.42082	4.42144	-11.65356	9.75789
	4.0	2.48019	3.70842	2.13526	3.89879	6.92469	-1.43280	17.99912	-0.08836	4.67981	-14.36319	10.71895
	4.5	2.36015	3.65501	2.17548	3.61319	7.10473	-3.34647	18.76009	-0.61559	4.95414	-17.15689	11.73665
	5.0	2.23447	3.60055	2.21780	3.32266	7.28948	-5.28335	19.54246	-1.16737	5.24285	-20.04123	12.81558

$a/b=4$	z	$\frac{1}{c}$	$\frac{c}{c^2-s^2}$	$\frac{s}{c^2-s^2}$	$\frac{i}{c^2-s^2}$	$\frac{j}{c^2-s^2}$	$\frac{h}{c^2-s^2}$	$\frac{l}{c^2-s^2}$	$\frac{i'}{c^2-s^2}$	$\frac{j'}{c^2-s^2}$	$\frac{h'}{c^2-s^2}$	$\frac{l'}{c^2-s^2}$
tension	5.0	4.75142	5.64078	2.24524	12.68866	8.51346	50.76706	20.20712	9.87662	3.46291	37.91797	1.05652
	4.5	4.69085	5.06849	2.26860	12.45428	8.64446	48.55640	20.90918	8.95766	3.60678	35.23256	1.71319
	4.0	4.63353	5.57589	2.29225	12.21804	8.77723	46.33408	21.62169	8.60972	3.75438	32.31751	2.38880
	3.5	4.57485	5.54300	2.31623	11.97995	8.91185	44.10004	22.34496	8.25121	3.90584	29.77187	3.08398
	3.0	4.51556	5.50979	2.34051	11.73991	9.04828	41.85391	23.07897	7.89628	4.06128	26.99464	3.79946
	2.5	4.45481	5.47623	2.36506	11.49780	9.18644	39.59527	24.25254	7.53039	4.22080	24.18494	4.53593
	2.0	4.39284	5.44235	2.38993	11.25368	9.32647	37.32416	24.57941	7.15810	4.38458	21.34154	5.29417
	1.5	4.32961	5.40816	2.41514	11.00761	9.46852	35.04062	25.34684	6.77922	4.55280	18.46331	6.07490
	1.0	4.26508	5.37363	2.44069	10.75948	9.61253	32.74424	26.12590	6.39350	4.72561	15.54904	6.87900
	0.5	4.19919	5.33875	2.46656	10.50921	9.75849	30.43470	26.91663	6.00015	4.90314	12.59759	7.70727
	0.	4.13189	5.30353	2.49276	10.25680	9.90646	28.11184	27.71934	5.60059	5.08558	9.60757	8.56067
compression	0.5	4.06310	5.26796	2.51931	10.00223	10.05651	25.77549	28.53431	5.19289	5.27311	6.57770	9.44010
	1.0	3.99280	5.23203	2.54621	9.74543	10.20864	23.42534	29.36169	4.77730	5.46594	3.50644	10.34660
	1.5	3.92101	5.19571	2.57345	9.48632	10.36285	21.06109	30.20164	4.35356	5.66424	0.39238	11.28114
	2.0	3.84761	5.15900	2.60105	9.22490	10.51922	18.68256	31.05449	3.92135	5.86824	-2.76616	12.24489
	2.5	3.77248	5.12192	2.62903	8.96117	10.67782	16.28950	31.92055	3.48037	6.07816	-5.97072	13.23893
	3.0	3.69559	5.08446	2.65738	8.69505	10.83868	13.83189	32.80002	3.03026	6.29425	-9.22319	14.26456
	3.5	3.61689	5.04660	2.68611	8.42650	11.00182	11.45916	33.69312	2.57068	6.51673	-12.52528	15.32293
	4.0	3.53630	5.00833	2.71522	8.15548	11.16730	9.02116	34.60015	2.10123	6.74588	-15.87906	16.41552
	4.5	3.45374	4.96967	2.74475	7.88196	11.33518	6.56764	35.52144	1.62151	6.98194	-19.28647	17.54368
	5.0	3.36913	4.93058	2.77467	7.60588	11.50548	4.09829	36.45717	1.13120	7.22528	-22.74968	18.70888

$a/b=5$	z	$\frac{1}{c}$	$\frac{c}{c^2-s^2}$	$\frac{s}{c^2-s^2}$	$\frac{i}{c^2-s^2}$	$\frac{j}{c^2-s^2}$	$\frac{h}{c^2-s^2}$	$\frac{l}{c^2-s^2}$	$\frac{i'}{c^2-s^2}$	$\frac{j'}{c^2-s^2}$	$\frac{h'}{c^2-s^2}$	$\frac{l'}{c^2-s^2}$
tension	5.0	5.54894	6.81289	2.93448	18.20218	13.94590	81.04176	42.47842	12.19535	6.10579	52.49468	5.21879
	4.5	5.49960	6.78580	2.95429	17.95959	14.08407	78.22598	43.40010	11.82789	6.26511	48.99421	6.12461
	4.0	5.44962	6.75856	2.97430	17.71578	14.22355	75.40051	44.33101	11.45630	6.42721	45.46668	7.04770
	3.5	5.39898	6.73114	2.99447	17.47067	14.36425	72.56496	45.27090	11.08049	6.59212	41.91165	7.98849
	3.0	5.34768	6.70355	3.01480	17.22425	14.50621	69.71928	46.21990	10.70035	6.75992	38.32846	8.94740
	2.5	5.29570	6.67579	3.03531	16.97649	14.64942	66.86333	47.17820	10.31580	6.93065	34.71654	9.92452
	2.0	5.24301	6.64785	3.05598	16.72738	14.79388	63.99599	48.14576	9.92672	7.10439	31.07524	10.92126
	1.5	5.18959	6.61969	3.07679	16.47678	14.93943	61.11966	49.12224	9.53300	7.28118	27.40390	11.93990
	1.0	5.13546	6.59142	3.09784	16.22501	15.08661	58.23253	50.10909	9.13459	7.46117	23.70191	12.97789
	0.5	5.08130	6.56322	3.11932	15.97279	15.23599	55.33791	51.10885	8.73153	7.64456	19.96877	14.03212
	0.	5.02510	6.53480	3.14094	15.71901	15.38655	52.43192	52.11784	8.32350	7.83142	16.20342	15.10654
compression	0.5	4.96862	6.50580	3.16235	15.46242	15.53706	49.51014	53.13182	7.91017	8.02108	12.40480	16.20436
	1.0	4.91131	6.47654	3.18387	15.20415	15.68367	46.57645	54.15496	7.49161	8.21434	8.57247	17.32466
	1.5	4.85324	6.44728	3.20576	14.94508	15.84231	43.57141	55.19061	7.06785	8.41121	4.70603	18.46738
	2.0	4.79435	6.41786	3.22789	14.68470	15.99750	40.68061	56.23707	6.63868	8.61175	0.80438	19.63319
	2.5	4.73459	6.38822	3.25018	14.42275	16.15402	37.71558	57.29363	6.20394	8.81603	-3.13343	20.82250
	3.0	4.67395	6.35838	3.27267	14.15930	16.31197	34.73901	58.36072	5.76350	9.02415	-7.10820	22.03603
	3.5	4.61238	6.32835	3.29535	13.89435	16.47138	31.75080	59.43848	5.31723	9.23619	-11.12067	23.27445
	4.0	4.54988	6.29813	3.31824	13.62786	16.63225	28.75078	60.52702	4.86498	9.45226	-15.17192	24.52837
	4.5	4.48783	6.26979	3.34348	13.36887	16.80385	25.77802	61.66618	4.40792	9.67465	-19.25845	25.83588
	5.0	4.42230	6.23710	3.36463	13.09019	16.95844	22.71465	62.73689	3.94189	9.89689	-23.39472	27.14516

$a/b=6$	z	$\frac{1}{c}$	$\frac{c}{c^2-s^2}$	$\frac{s}{c^2-s^2}$	$\frac{i}{c^2-s^2}$	$\frac{j}{c^2-s^2}$	$\frac{h}{c^2-s^2}$	$\frac{l}{c^2-s^2}$	$\frac{i'}{c^2-s^2}$	$\frac{j'}{c^2-s^2}$	$\frac{h'}{c^2-s^2}$	$\frac{l'}{c^2-s^2}$
tension	5.0	6.39078	8.01562	3.60886	24.91300	20.61067	122.90722	76.38224	15.63347	9.39411	69.91068	12.32306
	4.5	6.34744	7.99245	3.62595	24.66578	20.75310	119.49266	77.51913	15.25069	9.56294	65.60539	13.47226
	4.0	6.30371	7.96921	3.64316	24.41777	20.89558	116.07008	78.66439	14.86482	9.73389	61.27581	14.63697
	3.5	6.25955	7.94583	3.66043	24.16863	21.04072	112.63814	79.81658	14.47575	9.90688	56.92191	15.81755
	3.0	6.21497	7.92236	3.67781	23.91858	21.18578	109.19759	80.97673	14.08348	10.08202	52.54304	17.01419
	2.5	6.16999	7.89881	3.69533	23.66779	21.33198	105.74896	82.14565	13.68798	10.25940	48.13860	18.22707
	2.0	6.12458	7.87516	3.71295	23.41604	21.47909	102.29156	83.32258	13.28916	10.43899	43.70847	19.45656
	1.5	6.07871	7.85137	3.73064	23.16313	21.62691	98.82468	84.50674	12.88694	10.62077	39.25257	20.70302
	1.0	6.03240	7.82748	3.74843	22.90924	21.77563	95.34884	85.69896	12.48130	10.80482	34.77019	21.96666
	0.5	5.98565	7.80350	3.76635	22.65451	21.92542	91.86443	86.89993	12.07221	10.99123	30.26070	23.24767
	0.	5.93845	7.77942	3.78439	22.39885	22.07621	88.37108	88.10931	11.65960	10.98659	25.72381	24.54646
comyression	0.5	5.89076	7.75527	3.80256	22.14223	22.22800	84.81245	89.32709	11.24339	11.37122	21.15922	25.86342
	1.0	5.84260	7.73099	3.82083	21.88455	22.38069	81.35686	90.55301	10.82352	11.56485	16.56636	27.19876
	1.5	5.79396	7.70653	3.83914	21.62564	22.53409	77.83490	91.78660	10.39989	11.76089	11.93908	28.55269
	2.0	5.74482	7.68194	3.85755	21.36566	22.68840	74.30338	93.02854	9.97248	11.95944	7.29393	29.92564
	2.5	5.69517	7.65728	3.87610	21.10476	22.84380	70.76279	94.27942	9.54125	12.16059	2.61331	31.31798
	3.0	5.64503	7.63250	3.89477	20.84284	23.00020	67.21271	95.53902	9.10612	12.36436	-2.09741	32.73003
	3.5	5.59444	7.60759	3.91354	20.57983	23.15756	63.65289	96.80724	8.66700	12.57079	-6.83889	34.16211
	4.0	5.54330	7.58257	3.93241	20.31576	23.31592	60.08337	98.08425	8.22384	12.77993	-11.61158	35.61461
	4.5	5.49145	7.55744	3.95140	20.05066	23.47529	56.50422	99.37018	7.77663	12.99182	-16.42583	37.08787
	5.0	5.43918	7.53219	3.97050	19.78445	23.63566	52.91514	100.66500	7.32519	13.20652	-21.25248	38.58230

$$
\left.
\begin{aligned}
&\Delta=(A^2-B^2)\sinh\alpha b\sin\beta b+2(1-\cosh\alpha b\cos\beta b)AB\\
&\frac{c(z)}{c^2(z)-s^2(z)}\Delta=(A^2+B^2)(A\cosh\alpha b\sin\beta b-B\sinh\alpha b\cos\beta b)\\
&\frac{s(z)}{c^2(z)-s^2(z)}\Delta=(A^2+B^2)(B\sinh\alpha b-A\sin\beta b)\\
&\frac{i(z)}{c^2(z)-s^2(z)}\Delta=2A^2B^2\sinh\alpha b\sinh\beta b-AB(A^2-B^2)(1-\cosh\alpha b\cos\beta b)\\
&\qquad+\nu\Delta\\
&\frac{j(z)}{c^2(z)-s^2(z)}\Delta=AB(A^2+B^2)(\cosh\alpha b-\cos\beta b)\\
&\frac{h(z)}{c^2(z)-s^2(z)}\Delta=AB(A^2+B^2)(B\cosh\alpha b\sin\beta b+A\sinh\alpha b\cos\beta b)\\
&\frac{l(z)}{c^2(z)-s^2(z)}\Delta=AB(A^2+B^2)(B\sin\beta b+A\sin\alpha b)\\
&\alpha=\frac{\pi}{a}A,\quad A=\sqrt{1+\sqrt{z\frac{a^2}{b^2}}},\qquad \beta=\frac{\pi}{a}B,\quad B=\sqrt{-1+\sqrt{z\frac{a^2}{b^2}}}\,.
\end{aligned}
\right\}\quad(19)
$$

As can be understood from the above 6 equations, those other than $\frac{i}{c^2-s^2}$ are the functions of z and $\frac{a}{b}$. The numerical values of these functions for various values of z and $\frac{a}{b}$ are given in Table, where Poisson's ratio ν is taken as 0.3.

3. The fundamental formula by the slope deflection method of the uniformly tensioned rectangular plate simply supported along two opposite sides perpendicular to the direction of tension.

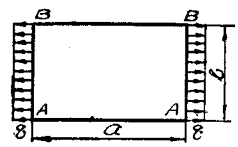

Fig. 2

In this case (Fig. 2), formulae similar to eq. (13)~(18) are obtained, and a form convenient for doing the calculation is as follows.

$$
\left.
\begin{aligned}
&\Delta=B^2\sinh^2\alpha b-A^2\sin^2\beta b\\
&\frac{c(z)}{c^2(z)-s^2(z)}\Delta=2AB\,(B\sinh\alpha b\cosh\alpha b-A\sin\beta b\cos\beta b)\\
&\frac{s(z)}{c^2(z)-s^2(z)}\Delta=2AB\,(A\cosh\alpha b\sin\beta b-B\sinh\alpha b\cos\beta b)\\
&\frac{i(z)}{c^2(z)-s^2(z)}\Delta=A^4\sin^2\beta b+B^4\sinh^2\alpha b+A^2B^2\,(\cosh^2\alpha b-\cos^2\beta b)+\nu\Delta\\
&\frac{j(z)}{c^2(z)-s^2(z)}\Delta=2AB\,(A^2+B^2)\sinh\alpha b\sin\beta b
\end{aligned}
\right\}\quad(20)
$$

$$\left.\begin{aligned}
&\frac{h(z)}{c^2(z)-s^2(z)}\Delta=2AB\,(A^2+B^2)(B\sinh \alpha b\cosh \alpha b+A\sin \beta b\cos \beta b)\\
&\frac{l(z)}{c^2(z)-s^2(z)}\Delta=2AB\,(A^2+B^2)(A\cosh \alpha b\sin \beta b+B\sinh \alpha b\cos \beta b)\\
&\qquad \alpha=\frac{\pi}{a}A,\ A^2=0.5\,(\sec\varphi+1),\ \beta=\frac{\pi}{a}B,\ B^2=0.5\,(\sec\varphi-1),\\
&\qquad \tan^2\varphi=z\frac{a^2}{b^2}
\end{aligned}\right|$$

The numerical values of these terms are given in the precedent tables.

4. Representation of the boundary conditions by the slope deflection method.

The boundary conditions in the rectangular plate are classified into the following two.

a. Conditions for end side, b. Conditions for continuity.

The latter are such as those which exist at the position of stiffeners of a rectangular plate with longitudinal stiffeners. In this chapter, assuming the thickness of the plate, so that the flexural rigidity is constant, the following notations are used.

$$\left.\begin{aligned}
&a=\varepsilon b,\quad a=\varepsilon_m b_m,\quad q=\frac{z_c\pi^2N}{b^2}=\frac{z_m\pi^2N}{b_m^2},\quad B_m=k_mB_c,\quad F_m=\bar{k}_mF_c,\\
&\gamma_m=\frac{\pi B_m}{Nb_m}=\frac{k_m\varepsilon_m\gamma_c}{\varepsilon},\quad \gamma_c=\frac{\pi B_c}{Nb},\quad \mu_m=\frac{\pi F_m}{b_mt}=\frac{\bar{k}_m\varepsilon_m\mu_c}{\varepsilon},\quad \mu_c=\frac{\pi F_c}{bt},\\
&\tau_0=\frac{\pi C_0}{Na},\quad \tau_{n-1}=\frac{\pi C_n}{Na},\quad \sigma=\frac{q}{t}=\frac{z_c\pi^2N}{b^2t}=\frac{z_m\pi^2N}{b_m^2t},\quad \therefore z_m=z_c\left(\frac{\varepsilon}{\varepsilon_m}\right)^2\\
&\quad c_m(z_m)=c_m,\quad s_m(z_m)=s_m,\quad i_m(z_m)=i_m,\quad j_m(z_m)=j_m,\\
&\quad h_m(z_m)=h_m,\quad l_m(z_m)=l_m,\quad i'_m(z_m)=i'_m,\quad j'_m(z_m)=j'_m\\
&\quad h'_m(z_m)=h'_m,\quad l'_m(z_m)=l'_m,
\end{aligned}\right\}\quad(21)$$

where B, F, C represent the flexural rigidity, cross sectional area and torsional rigidity respectively. Values with suffix c are considered to be the standard among those. t is the thickness of the plate.

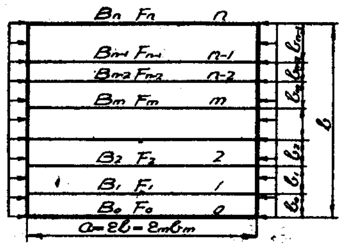

Fig. 3

a) Condition for end side. Several particular cases will now be considered.

1. The sides $y=0$ and $y=b$ are simply supported. In ordinary cases, these conditions are represented as

$$\left.\begin{array}{llll} \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}=0, & w=0, & \text{for} & y=0 \\ \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}=0, & w=0, & \text{for} & y=b \end{array}\right\} \quad (22)$$

The representation for these is greatly simplified as follows, by using the slope deflection method.

$$\left.\begin{array}{l} M_{0,1}=\dfrac{N}{c_0^2-s_0^2}\dfrac{\pi}{a}\left\{c_0\theta_0+s_0\theta_1-\dfrac{\pi}{a}j_0\delta_1\right\}=0 \\ M_{n,n-1}=\dfrac{N}{c^2_{n-1}-s^2_{n-1}}\dfrac{\pi}{a}\left\{s_{n-1}\theta_{n-1}+c_{n-1}\theta_n+\dfrac{\pi}{a}j_{n-1}\delta_{n-1}\right\}=0 \end{array}\right\} \quad (23)$$

In this case, for $y=0$,

$$\left.\begin{array}{l} M_{1,0}=\dfrac{N}{c_0^2-s_0^2}\dfrac{\pi^2}{a^2}\left\{\dfrac{c_0^2-s_0^2}{c_0}\theta_1-\dfrac{\pi}{a}i_0'\delta_1\right\} \\ V_{1,0}=-\dfrac{N}{c_0^2-s_0^2}\dfrac{\pi^2}{a^2}\left\{i'_0\theta_1-\dfrac{\pi}{a}h_0'\delta_1\right\} \end{array}\right\} \quad (24)$$

and for $y=b$,

$$\left.\begin{array}{l} M_{n-1,n}=\dfrac{N}{c^2_{n-1}-s^2_{n-1}}\dfrac{\pi}{a}\left\{\dfrac{c^2_{n-1}-s^2_{n-1}}{c_{n-1}}\theta_{n-1}-\dfrac{\pi}{a}i'_{n-1}\delta_{n-1}\right\} \\ V_{n-1,n}=-\dfrac{N}{c^2_{n-1}-s^2_{n-1}}\dfrac{\pi^2}{a^2}\left\{i'_{n-1}\theta_{n-1}-\dfrac{\pi}{a}h'_{n-1}\delta_{n-1}\right\} \end{array}\right\} \quad (25)$$

are favourably used.

2. The sides $y=0$ and $y=b$ are built-in. In this case,

$$\left.\begin{array}{llll} \theta_A=0, & \delta_A=0, & \text{for} & y=0 \\ \theta_B=0, & \delta_B=0, & \text{for} & y=b \end{array}\right\} \quad (26)$$

3. The sides $y=0$ and $y=b$ are free. In this case, the following representation is usually used.

$$\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}=0,\quad \frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial y\partial x^2}=0,\quad \text{for} \quad y=0 \text{ and } y=b$$

But, in our case these conditions are as follows.

$$\left.\begin{array}{lll} M_{0,1}=V_{0,1}=0, & \text{for} & y=0 \\ M_{n,n-1}=V_{n,n-1}=0, & \text{for} & y=b \end{array}\right\} \quad (26)$$

4. The sides $y=0$ and $y=b$ are elastically built-in[6]**.** In the previous discussions, two extreme assumptions for the constraint along the sides have been considered, namely, a simply supported edge and a built-in edge. In practical cases, we will usually have some intermediate condition of constraint. Take, for instance, the case of compression member of a T cross section. While the upper edge of the vertical web cannot be assumed to rotate freely during buckling, neither can it be considered as rigidly built in since during buckling of the web some rotation of the horizontal flange will take place. We consider in this case the upper edge of the plate as elastically built-in, since the bending moments that appear during buckling along this edge are proportional at each point to the angle of rotation of the edge.

The conditions of elastically built-in edge are represented by the following equations.

$$-N\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)=-C_0\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x\partial y}\right),\quad \text{for}\quad y=0$$

$$-N\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x}\right)=C_n\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x\partial y}\right),\quad \text{for}\quad y=b$$

These equations can be expressed as follows by the slope deflection method.

$$\left.\begin{aligned}&\left(\frac{c}{c_0^2-s_0^2}-\tau_0\right)\theta_0+\frac{s_0}{c_0^2-s_0^2}\theta_1+\frac{\pi}{a}\left(\frac{i_0}{c_0^2-s_0^2}\delta_0-\frac{j_0}{c_0^2-s_0^2}\delta_1\right)=0\\&\frac{s_{n-1}}{c^2_{n-1}-s^2_{n-1}}\theta_{n-1}+\left(\frac{c_{n-1}}{c^2_{n-1}-s^2_{n-1}}-\tau_{n-1}\right)\theta_n\\&\quad+\frac{\pi}{a}\left(\frac{j_{n-1}}{c^2_{n-1}-s^2_{n-1}}\delta_{n-1}-\frac{i_{n-1}}{c^2_{n-1}-s^2_{n-1}}\delta_n\right)=0\end{aligned}\right\}\qquad(27)$$

Furthermore, when we can neglect the small deflection due to the large flexural rigidity, δ_0 and δ_n in eq. (27) can be made into zero.

5. Both sides $y=0$ and $y=b$ are supported by elastic beams[7]. Along the sides $y=0$ and $y=b$, the plate is free to rotate during buckling, but deflections of the plate at these edges are resisted by two elastic supporting beams. The condition of freedom of rotation requires that

$$\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}=0,\qquad \text{for}\qquad y=0\quad\text{and}\quad y=b,$$

To get a second expression for this boundary, the deflection of supporting beams must be considered. Now, if we assume that these beams are simply supported at the ends and have the flexural rigidity of B_0 and B_n, and are compressed together with the plate so that the compressive forces on each side are equal to $F\sigma$, the differential equations for the deflection of the beams are as follows.

$$B_0\frac{\partial^4 w}{\partial x^4}=p_0-F_0\sigma\frac{\partial^2 w}{\partial x^2},\qquad B_n\frac{\partial^4 w}{\partial x^4}=p_n-F_n\sigma\frac{\partial^2 w}{\partial x^2},$$

where p is the intensity of the load transmitted from the plate to the beams. From the expression for shearing forces, this intensity p is

$$p_0=-N\left\{\frac{\partial^3 w}{\partial y^3}-(2-\nu)\frac{\partial^3 w}{\partial y\,\partial x^2}\right\},\quad p_n=N\left\{\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial y\,\partial x^2}\right\}$$

From these equations, the following equations are obtained by the slope deflection method.

$$\left.\begin{aligned}
&\frac{c_0}{c_0^2-s_0^2}\theta_0+\frac{s_0}{c_0^2-s_0^2}\theta_1+\frac{\pi}{a}\left(\frac{i_0}{c_0^2-s_0^2}\delta_0-\frac{j_0}{c_0^2-s_0^2}\delta_1\right)=0\\
&\frac{i_0}{c_0^2-s_0^2}\theta_0-\frac{j_0}{c_0^2-s_0^2}\theta_1+\frac{\pi}{a}\left\{\left(-\frac{l_0}{c_0^2-s_0^2}+z_0\mu_0\varepsilon_0-\frac{\gamma_0}{\varepsilon_0}\right)\delta_0+\frac{l_0}{c_0^2-s_0^2}\delta_1\right\}=0\\
&\frac{s_{n-1}}{c^2_{n-1}-s^2_{n-1}}\theta_{n-1}+\frac{c_n}{c^2_{n-1}-s^2_{n-1}}\theta_n-\frac{\pi}{a}\left(\frac{j_{n-1}}{c^2_{n-1}-s^2_{n-1}}\delta_{n-1}-\frac{i_{n-1}}{c^2_{n-1}-s^2_{n-1}}\delta_n\right)=0\\
&\frac{j_{n-1}}{c^2_{n-1}-s^2_{n-1}}\theta_{n-1}+\frac{i_{n-1}}{c^2_{n-1}-s^2_{n-1}}\theta_n+\frac{\pi}{a}\left\{\frac{l_{n-1}}{c^2_{n-1}-s^2_{n-1}}\delta_{n-1}\right.\\
&\qquad\left.+\left(\frac{h_{n-1}}{c^2_{n-1}-s^2_{n-1}}-z_{n-1}\mu_{n-1}\varepsilon_{n-1}+\frac{\gamma_{n-1}}{\varepsilon_{n-1}}\right)\delta_n\right\}=0
\end{aligned}\right\}\quad(28)$$

where it is assumed that $m=1$, *i.e.*, there is only one half-wave formed by the buckled plate.

b) Conditions for continuity[8]. In the case of a large number of equal and equidistant stiffeners parallel to one of the sides of a compressed rectangular plate, we usually consider the stiffened plate as a plate having two different flexural rigidities in two perpendicular directions, but in our treatment, we consider the plate as a plate having equal rigidity except for the lines reinforced by the stiffeners where, since the rib is rigidly connected with the plate, a portion of the plate must be taken in calculating the flexural rigidity of stiffeners.

As the conditions for continuity, several cases may be considered, but we will consider only the case where the plate is elastically supported by stiffeners, *i.e.*, deflective beams. In this case we have the following four conditions.

1. The two parts of the plate separated by a stiffener have the same deflection curve at the position of the stiffener.

2. The same can be said about the slope of the plate.

3. The bending moments of both parts of the plate and the torsional moment of the stiffener must be equivalent.

4. The shearing forces of both parts of the plate and the compressive force acting on the stiffener must be equal to the load intensity applied to the stiffener.

Of the above four conditions, the former two which are necessary in the case of the method of integration are not required in our slope deflection method, so that the latter two must be written as follows.

$$\left.\begin{aligned}
&\frac{s_{m-1}}{c^2_{m-1}-s^2_{m-1}}\theta_{m-1}+\left(\frac{c_{m-1}}{c^2_{m-1}-s^2_{m-1}}+\frac{c_m}{c^2_m-s^2_m}\right)\theta_m+\frac{s_m}{c^2_m-s^2_m}\theta_{m+1}\\
&+\frac{\pi}{a}\left\{\frac{j_{m-1}}{c^2_{m-1}-s^2_{m-1}}\delta_{m-1}-\left(\frac{i_{m-1}}{c^2_{m-1}-s^2_{m-1}}-\frac{i_m}{c^2_m-s^2_m}\right)\delta_m\right.\\
&\left.-\frac{j_m}{c^2_m-s^2_m}\delta_{m+1}\right\}=0\\
&\frac{j_{m-1}}{c^2_{m-1}-s^2_{m-1}}\theta_{m-1}+\left(\frac{i_{m-1}}{c^2_{m-1}-s^2_{m-1}}-\frac{i_m}{c^2_m-s^2_m}\right)\theta_m-\frac{j_m}{c^2_m-s^2_m}\theta_{m-1}\\
&+\frac{\pi}{a}\left\{\frac{l_{m-1}}{c^2_{m-1}-s^2_{m-1}}\delta_{m-1}\right.\\
&\left.-\left(\frac{h_{m-1}}{c^2_{m-1}-s^2_{m-1}}+\frac{h_m}{c^2_m-s^2_m}-z_m\mu_m\varepsilon_m+\frac{\gamma_m}{\varepsilon_m}\right)\delta_m+\frac{l_m}{c^2_m-s^2_m}\delta_{m+1}\right\}=0
\end{aligned}\right\}\quad(29)$$

In a plate girder, for example, L, I and Z sections are often used as stiffeners, but their torsional rigidity is so small and therefore "τ" is so negligible small that we can neglect them in our calculation[9]. If there is no supporting beam, we can put that the term $z_m\mu_m\varepsilon_m-\frac{\gamma_m}{\varepsilon_m}$ is equal to zero. If there is a rigidly supporting beam, the 4th. condition is not needed and in the 3rd. condition we can put $\delta_m=0$.

5. Buckling of a uniformly compressed rectangular plate simply supported along two opposite perpendicular to the direction of compression and having various edge conditions along the other two sides.

In the solution of this problem, both methods, the method of energy and the method of integration of the differential equation for the delected plate, can be used. When we use the method of integration, the following solution of the fundamental differential equation is obtained.

$$w=(C_1\cosh \alpha y+C_2\sinh \alpha y+C_3\cos\beta y+C_3\sin\beta y)\sin\frac{m\pi x}{a}$$

where $\alpha^2=\left(\frac{m\pi}{a}\right)^2+\sqrt{\frac{q}{N}\left(\frac{m\pi}{a}\right)^2}$, $\beta^2=-\left(\frac{m\pi}{a}\right)^2+\sqrt{\frac{q}{N}\left(\frac{m\pi}{a}\right)^2}$

Putting the boundary conditions into the above solution, we get the same number of equations as that of integration constants. Equating the determinant of these equations thus obtained to zero, the equation to determine the critical value of the compression is obtained. Whenever a different boundary condition is given, we are obliged to repeat the calculation from the beginning, and there seems to be no relation between the buckling equations thus obtained, but by the author's slope deflection method, the buckling equation hitherto studied can be represented very simply.

In the following several lines, many cases which give the clearer contrast with the method treated in S. Timoshenko's "The Theory of Elastic Stability" will be explained.

1. Both sides $y=0$ and $y=b$ are built-in[10]. In this case the boundary conditions are

$$\theta_A=\theta_B=0 \quad \text{and} \quad \delta_A=\delta_B=0$$

From eq. (10),

$$M_{AB}\cdot c(z)-M_{BA}\cdot s(z)=0, \quad -M_{AB}\cdot s(z)+M_{BA}\cdot c(z)=0$$

Therefore, we can derive the following equation to determine the critical value of the compressive force.

$$c^2(z)-s^2(z)=0$$

This equation coincides with the one reduced by S. Timoshenko as follows.

$$(\cos\beta b-\cosh\alpha b)^2=-(\sin\beta b-\frac{\beta}{\alpha}\sinh\alpha b)(\sin\beta b+\frac{\alpha}{\beta}\sinh\alpha b),$$

when m in α and β is taken equal to 1.

2. Side $y=0$ is simply supported and side $y=b$ is free[11]. In this case the boundary condition are

$$M_{AB}=\delta_A=0, \quad \text{for} \quad y=0; \qquad M_{BA}=V_{BA}=0, \quad \text{for} \quad y=b.$$

From eq. (16), we have

$$\frac{c^2(z)-s^2(z)}{c(z)}\theta_B-\frac{\pi}{a}i'(z)\delta_B=0, \qquad i'(z)\theta_B-\frac{\pi}{a}h'(z)\delta_B=0$$

Equating the determinant of these equations to zero, we get

$$\frac{c^2(z)-s^2(z)}{c(z)}\cdot h'(z)-i'^2(z)=0$$

This equation seems very complicated at first sight, but using eq. (15) and (18), we obtain

$$u(z) = 0$$

In this case, according to S. Timoshenko, the following equation is given

$$\beta\left(\alpha^2-\nu\frac{m^2\pi^2}{a^2}\right)\tanh\alpha b=\alpha\left(\beta^2+\nu\frac{m^2\pi^2}{a^2}\right)\tanh\beta b$$

By substituting $m=1$ into this equation, it becomes $u(z)=0$.

3. Side $y=0$ is built-in and side $y=b$ is free[12]. In this case the edge conditions are

$$\theta_A=\delta_A=0,\quad \text{for}\quad y=0;\qquad M_{BA}=V_{BA}=0,\quad \text{for}\quad y=b.$$

From eq. (13) and (14), we have

$$c(z)\theta_B-\frac{\pi}{a}i(z)\delta_B=0,\qquad i(z)\theta_B-\frac{\pi}{a}h(z)\delta_B=0$$

Equating the determinant of these equations to zero, we get

$$c(z)h(z)-i^2(z)=0$$

This coincides with

$$2ts+(t^2+s^2)\cos\beta b\cosh\alpha b=\frac{1}{\alpha^2\beta^2}(\alpha^2t^2-\beta^2s^2)\sin\beta b\sinh\alpha b,$$

$$t=\beta^2+\nu\frac{m^2\pi^2}{a^2},\qquad s=\alpha^2-\nu\frac{m^2\pi^2}{a^2},$$

when m is equal to 1.

From these results we can see that by the method of integration of the differential equation, the necessary buckling equation is generally complicated and to get the numerical value, we must calculate in each case. On the contrary, by the slope defection method, the buckling equation is represented in a very simplified form and the numerical calculation is very easy, because the coefficients necessary for the calculation are given in a table. The benefit of the slope deflection method will be displayed in the calculation of a rectangular plate with stiffeners which will be explained in the next chapter.

6. The buckling of a uniformly compressed rectangular plate with longitudinal stiffeners.

The stability of a rectangular plate reinforced by stiffeners has been solved by S. Timoshenko[13], E. Chwalla[14], R. Barbré[15] and other scholars.

S. Timoshenko solved by the method of energy the uniformly compressed rectangular plate with 1~3 longitudinal or transverse stiffening ribs when the plate is simply supported at four edges and also E. Chwalla used the same method to solve such problems. R. Barbré solved such problems by the method of integration, and the advantage of his method is that the problems with any edge condition such as simply supported, free, built in and etc. can be solved similarly. On the contrary, the case which can be solved by the method of energy is almost limited to the case of four simply supported edges, but the solution of other cases is also possible by the method of energy as can be understood from K. Nölke's solution. His solution, however, becomes very complicated, but resorting to our slope defection method we can remove the disadvantage as will be known later.

In the plate which we are now discussing, we shall number the stiffeners as 1, 2, ⋯, m, ⋯, $n-1$ and both edges as 0, n. In this chapter, we shall use eq. (22)~(28) as the condition for end sides and eq. (29) as the condition for continuity derived in chapter 4. Equating the determinant of these equations to zero, the buckling equation can be induced, but this determinant consists of many lines and columns. Therefore, a great labours is required in doing the calculation, so we shall resort to the means mentioned below.

If both sides are simply supported or fixed, the unknown terms are $\theta_m(m=1, 2, \cdots, m, \cdots, n-1)$ and $\delta_m(m=1, 2, \cdots, m, n-1)$. If τ_c is suitably assumed, θ_m can be represented by terms of δ_m only from the equilibrium equations of the bending moment. This calculation is easily done and the general solution is as follows.

$$\left.\begin{aligned}
\theta_1 &= \alpha_{1,1}\delta_1 + \alpha_{1,2}\delta_1 + \cdots\cdots + \alpha_{1,m}\delta_m + \cdots\cdots + \alpha_{1,n-1}\delta_{n-1} \\
\theta_2 &= \alpha_{2,1}\delta_1 + \alpha_{2,2}\delta_2 + \cdots\cdots + \alpha_{2,m}\delta_m + \cdots\cdots + \alpha_{2,n-1}\delta_{n-1} \\
&\vdots \\
\theta_m &= \alpha_{m,1}\delta_1 + \alpha_{m,2}\delta_2 + \cdots\cdots + \alpha_{m,m}\delta_m + \cdots\cdots + \alpha_{m,n-1}\delta_{n-1} \\
&\vdots \\
\theta_{n-1} &= \alpha_{n-1,1}\delta_1 + \alpha_{n-1,2}\delta_2 + \cdots\cdots + \alpha_{n-1,m}\delta_m + \cdots\cdots + \alpha_{n-1,n-1}\delta_{n-1}
\end{aligned}\right\} \quad (30)$$

where α represents numerical values.

Putting these equations into the other equilibrium equations of the shearing force, we obtain the next equations.

$$\left.\begin{aligned}
\beta_{1,1}\delta_1 + \beta_{1,2}\delta_2 + \cdots\cdots + \beta_{1,m}\delta_m + \cdots\cdots + \beta_{1,n-1}\delta_{n-1} &= 0 \\
\beta_{2,1}\delta_1 + \beta_{2,2}\delta_2 + \cdots\cdots + \beta_{2,m}\delta_m + \cdots\cdots + \beta_{2,n-1}\delta_{n-1} &= 0 \\
\vdots \\
\beta_{m,1}\delta_1 + \beta_{m,2}\delta_2 + \cdots\cdots + \beta_{m,m}\delta_m + \cdots\cdots + \beta_{m,n-1}\delta_{n-1} &= 0 \\
\vdots \\
\beta_{n-1,1}\delta_1 + \beta_{n-1,2}\delta_2 + \cdots\cdots + \beta_{n-1,m}\delta_m + \cdots\cdots + \beta_{n-1,n-1}\delta_{n-1} &= 0
\end{aligned}\right\} \quad (31)$$

where β represents numerical values.

The determinant consisting of β_{ij} which is expressed by $\Delta(\beta)$ is easily calculated. If the value of τ_c assumed first coincides with that of the buckling force, $\Delta(\beta)$ must be equal to zero, but generally $\Delta(\beta)$ is not equal to zero for the arbitrarily assumed value of τ_c. So when we calculate the value of $\Delta(\beta)$ for several values of z_c, we can determine the root of $\Delta(\beta)=0$ by means of interpolation.

Example.

Several examples dealt with by R. Barbré [16] will be solved by our slope delection method. In R. Barbré's treatise, the buckling equation generally represented is that of the case of the edge conditions of elastically supported, and the buckling equation of the case of both edges simply supported or built-in is derived as a special case of the edge condition. On the contrary, by the slope deflection method, the edge conditions can be represented very easily, so it is better to obtain the buckling equation respectively for all cases of the edge conditions.

As shown in Fig. 4, 5, 7, let the rectangular plate reinforced by stiffeners which is simply supported at the two edges $x=0$ and $x=a$ be subjected to uniformly distributed compressive forces on these sides.

Case I (Fig. 4). Two edges $y=0$ and $y=b$ are built-in and a stiffener is placed at $b_0=b_1=b/2$. In this case the boundary conditions are $\theta_0=\theta_2=0$, $\delta_0=\delta_2=0$. This plate being stiffened in the middle,

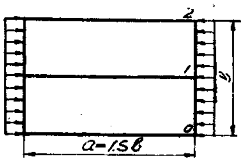

Fig. 4

$$c_m=c,\quad s_m=s,\quad i_m=i,\quad j_m=j\quad h_m=h,\quad l_m=l,\quad z_m=z,\quad (m=0,\ 1)$$

are obtained, and the equilibrium equations are as follows.

$$\frac{2c}{c^2-s^2}\theta_1+0\cdot\delta_1=0$$

$$0\cdot\theta_1+\frac{\pi}{a}\left\{-\left(\frac{2h}{c^2-s^2}-z\mu\varepsilon+\frac{\gamma}{\varepsilon}\right)\delta_1\right\}=0$$

Therefore, the following equation is obtained as the buckling equation.

$$\frac{2h}{c^2-s^2}-z\mu\varepsilon+\frac{\gamma}{\varepsilon}=0$$

In this case, $b_0=b_1=b/2$ and $\varepsilon_0=\varepsilon_1=3$. Taking $\gamma=10\pi$, $\left(\frac{B}{bN}=5\right)$; $\mu=0.2\pi$, $\left(\frac{F}{bt}=0.1\right)$,

$$f(z)=\frac{2h}{c^2-s^2}-0.6\pi z+\frac{10\pi}{3}=0$$

The root of the above equation is obtained by the trial method. By use of the table $\left(\frac{a}{b}=3\right)$, taking $z=4.0$, we get $f(4.0)=0.067$ and also $z=4.5$, $f(4.5)=-1.357$. Thus by the interpolation $z=4.02$ is obtained, and then we get $q=16.1\pi^2N/b^2$.

The above calculation is for the case when the plate buckles in the form of one half-wave. To compare the result with the case of two half-waves ($m=2$), we must slove the case of $\frac{a}{b}=0.75$ and $m=1$. This case is solved by either of the next means;

1. Using the table of the case of $\frac{a}{b}=1.5$ and $m=1$.

2. Assuming the two parallel lines $y=\frac{b}{4}$ and $y=\frac{3b}{4}$, and using the table of the case of $\frac{a}{b}=3$ and $m=1$.

If the value of z_c obtained by either of the two methods mentioned above is larger than the value obtained before, the value of z_c corresponding to the critical force is decided as equal to 16.1. For the simplicity, the calculation for the case $m=2$ is omitted here.

The other buckling equation is

$$\frac{2c}{c^2-s^2}=0$$

This equation corresponds to the case when the plate buckles without the deflection of the stiffener. This case is for the case when $\gamma=\infty$, that is, the plate is supported by a rigid beam in the middle.

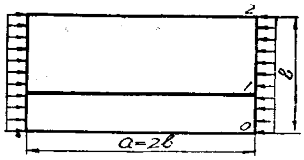

Fig. 5

Case II (Fig. 5). Two edges $y=0$ and $y=b$ are simply supported and a stiffener is placed at $b_0=b/3$ and $b_1=2b/3$. In this case the boundary conditions are $M_{01}=\delta_0=0$ and $M_{21}=\delta_2=0$. As $\varepsilon=2$, $\varepsilon_0=6$ and $\varepsilon_1=3$, we get $z_0=z_c/9$, $z_1=4z_c/9$ and $\gamma_1=3\gamma_c/2$, $\mu_1=3\mu_c/2$. The equilibrium equations are as follows.

$$\left(\frac{1}{c_0}+\frac{1}{c_1}\right)\theta_1-\frac{\pi}{a}\left(\frac{i'_0}{c_0^2-s_0^2}-\frac{i'_1}{c_1^2-s_1^2}\right)\delta_1=0$$

$$\left(\frac{i'_0}{c_0^2-s_0^2}-\frac{i'_1}{c_1^2-s_1^2}\right)\theta_1-\frac{\pi}{a}\left(\frac{h'_0}{c_0^2-s_0^2}+\frac{h'_1}{c_1^2-s_1^2}-z_1\mu_1\varepsilon_1+\frac{\gamma_1}{\varepsilon_1}\right)\delta_1=0$$

Therefore, the buckling equation is obtained as follows.

$$f(z_1)=\left(\frac{1}{c_0}+\frac{1}{c_1}\right)\left(\frac{h'_0}{c_0^2-s_0^2}+\frac{h'_1}{c_1^2-s_1^2}-z_1\mu_1\varepsilon_1+\frac{\gamma_1}{\varepsilon_1}\right)-\left(\frac{i'_0}{c_0^2-s_0^2}-\frac{i'_1}{c_1^2-s_1^2}\right)^2=0$$

Taking $\gamma_c=5.0\pi$, $\mu_c=0.1\pi$, the above equation becomes

$$f(z_1)=\left(\frac{1}{c_0}+\frac{1}{c_1}\right)\left(\frac{h'_1}{c_0^2-s_0^2}+\frac{h'_1}{c_1^2-s_1^2}-0.45\pi z_1+2.50\pi\right)$$
$$-\left(\frac{i'_0}{c_0^2-s_0^2}-\frac{i'_1}{c_1^2-s_1^2}\right)^2=0$$

Assuming $z_1=3.0$ and $z_0=0.75$, we get $f(3.0)=12.498$. Next assuming $z_1=3.5$ and $z_0=0.875$, $f(3.5)=-34.750$. Obtaining $z_1=3.16$ by the interpolation, we get

$q=7.11\pi^2N/b^2$, because $z_e=9z_1/4=7.11$.

The section of the possible buckled form of the plate is shown in Fig. 6. In this figure, I shows the buckled form having two nodal lines at $b_0=b_1=b_2=b/3$ and the value of z corresponding to this form is 36, $\left(q=\frac{4\pi^2N}{(b/3)^2}=36\pi^2N/b^2\right)$. IIa corresponds to the above calculation and IIb to the case of $\gamma_e=\infty$, and the buckling equation of the latter case is $f(z)=\frac{1}{c_0}+\frac{1}{c_1}=0$.

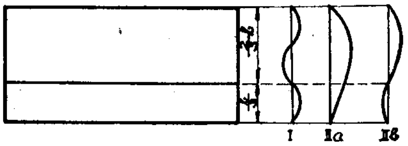

Fig. 6

The above calculation is the case of one half-wave. As explained in case I, we must consider the case of more than two half-waves, but such calculation will be omitted as in case I.

Case III (Fig. 7). Two edges $y=0$ and $y=b$ are simply supported and two equal stiffeners are placed at $b_0=b_1=b_2=b/3$.

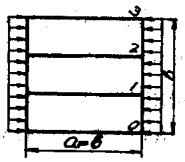

Fig. 7

In this case, the boundary conditions are $M_{01}=\delta_0=0$ and $M_{32}=\delta_3=0$, and the coefficients can be simplified as $c_m=c$, $s_m=s$, $i_m=i$, $j_m=j$, $h_m=h$, $l_m=l$, $z_m=z$, $\varepsilon_m=\varepsilon$, $\gamma_m=\gamma$ and $\mu_m=\mu$ $(m=0, 1, 2)$. Therefore, the equilibrium equations are as follows.

$$\left(\frac{1}{c}+\frac{c}{c^2-s^2}\right)\theta_1+\frac{s}{c^2-s^2}\theta_2+\frac{\pi}{a}\left\{-\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)\delta_1-\frac{j}{c^2-s^2}\delta_2\right\}=0$$

$$\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)\theta_1-\frac{j}{c^2-s^2}\theta_2+\frac{\pi}{a}\left\{-\left(\frac{h'}{c^2-s^2}+\frac{h}{c^2-s^2}-z\mu\varepsilon+\frac{\gamma}{\varepsilon}\right)\delta_1+\frac{l}{c^2-s^2}\delta_2\right\}=0$$

$$\frac{s}{c^2-s^2}\theta_1+\left(\frac{c}{c^2-s^2}+\frac{1}{c}\right)\theta_2+\frac{\pi}{a}\left\{\frac{j}{c^2-s^2}\delta_1-\left(\frac{i}{c^2-s^2}-\frac{i'}{c^2-s^2}\right)\delta_2\right\}=0$$

$$\frac{j}{c^2-s^2}\theta_1+\left(\frac{i}{c^2-s^2}-\frac{i'}{c^2-s^2}\right)\theta_2+\frac{\pi}{a}\left\{\frac{l}{c^2-s^2}\delta_1-\left(\frac{h}{c^2-s^2}+\frac{h'}{c^2-s^2}-z\mu\varepsilon+\frac{\gamma}{\varepsilon}\right)\delta_2\right\}=0$$

If we assume, at first, the symmetrical buckled form, we can put $\theta_1=-\theta_2$ and $\delta_1=\delta_2$. Therefore, the above equations can be simplified as follows.

$$\left\{\left(\frac{1}{c}+\frac{c}{c^2-s^2}\right)-\frac{s}{c^2-s^2}\right\}\theta_1-\frac{\pi}{a}\left\{\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)+\frac{j}{c^2-s^2}\right\}\delta_1=0$$

$$\left\{\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)+\frac{j}{c^2-s^2}\right\}\theta_1-\frac{\pi}{a}\left\{\left(\frac{h'}{c^2-s^2}+\frac{h}{c^2-s^2}-z\mu\varepsilon+\frac{\gamma}{\varepsilon}\right)-\frac{l}{c^2-s^2}\right\}\delta_1=0$$

Thus the following buckling equation is obtained.

$$f(z)=\left\{\left(\frac{1}{c}+\frac{c}{c^2-s^2}\right)-\frac{s}{c^2-s^2}\right\}\left\{\left(\frac{h'}{c^2-s^2}+\frac{h}{c^2-s^2}-z\mu\varepsilon+\frac{\gamma}{\varepsilon}\right)-\frac{l}{c^2-s^2}\right\}$$

$$-\left\{\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)+\frac{j}{c^2-s^2}\right\}^2=0$$

In this case, $\varepsilon=3$. Now taking $\gamma=15\pi$, $\left(\frac{B}{b\cdot N}=5\right)$ and $\mu=0.3\pi$, $\left(\frac{F}{b\cdot t}=0.1\right)$,

$$f(z)=\left\{\left(\frac{1}{c}+\frac{c}{c^2-s^2}\right)-\frac{s}{c^2-s^2}\right\}\left\{\left(\frac{h'}{c^2-s^2}+\frac{h}{c^2-s^2}-0.9\pi z+5.0\pi\right)-\frac{l}{c^2-s^2}\right\}$$

$$-\left\{\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)+\frac{j}{c^2-s^2}\right\}^2=0$$

The root of the above equation is obtained by the the trial method as follows.

$f(1.5)=6.646$, for $z=1.5$; $f(2.0)=-24.429$, for $z=2.0$.

Therefore, by the interpolation, we get $z=1.61$ and then $q=14.5\pi^2N/b^2$, because $z_e=1.61\times3^2=14.5$.

The possible buckled forms are shown in Fig. 8. The above calculation is for the case of the symmetrical buckled form, corresponding to Fig. 8-IIb. Fig. 8-IIa shows the symmetrical buckled form having the nodal lines at the position of stiffeners. In this case, we can consider this problem as that of a rectangular plate of $a/b=3$ with four simply supported edges. Therefore, the critical load is $q=\frac{4\pi^2N}{(b/3)^2}=\frac{36\pi^2N}{b^2}$. Fig. 8-I shows the unsymmetrical buckled forms, and I*b* corresponds to case II, whose critical load is four times as that of case II, that is, $z_e=4\times7.1=28.4$. The value of z_e corresponding to Fig. 8-Ia is four times as that of Fig. 8-IIa. The cases when there are more than two half-waves will be omitted as before.

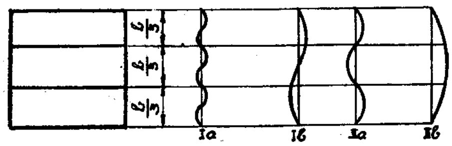

Fig. 8

R. Barbré's solution [17] based on the method of integration is as follows.

Obtaining the solution of the fundamental differential equation, we have four equilibrium equations of deflection, slope, bending moment and shearing force at the position of stiffener. Adding the boundary conditions to these, we have eight equations for case I and case II, and twelve equations for case III (Fig. 9). Equating the determinant of the coefficients of these equations to zero, we get the buckling equation. However, as can be understood from Fig. 9, it is hardly easy to develope the determinant which consists of eight lines and columns or of twelve lines and columns, and to obtain the buckling equations. On the contrary, by the slope deflection method, the determinant consists of two lines and columns for cases I and II, and also for case III, using the symmetrical relation.

	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2	=
1	—	—	—	—					0
2	—	—	—	—					0
3	—	—	—	—	—	—	—	—	0
4	—	—	—	—	—	—	—	—	0
5	—	—	—	—	—	—	—	—	0
6	—	—	—	—	—	—	—	—	0
7					—	—	—	—	0
8					—	—	—	—	0

	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2	A_3	B_3	C_3	D_3	=
1		—		—									0
2		—		—									0
3	—	—	—	—	—	—	—	—					0
4	—	—	—	—	—	—	—	—					0
5	—	—	—	—	—	—	—	—					0
6	—	—	—	—	—	—	—	—					0
7					—	—	—	—	—	—	—	—	0
8					—	—	—	—	—	—	—	—	0
9					—	—	—	—	—	—	—	—	0
10					—	—	—	—	—	—	—	—	0
11									—	—	—	—	0
12									—	—	—	—	0

Fig. 9

7. Buckling of a rectangular plate simply supported along two opposite sides perpendicular to the direction of compression and having various edge conditions along other two sides, when subjected to combined bending and compression.

In the discussion of this problem, which is necessary to design of plate girder, the method of energy is favourably used, because when distributed forces, acting in the middle plane of the plate, are applied along both simply supported sides $x=0$ and $x=a$, their intensity being given by $q_x=q_e\left(1-\alpha\frac{y}{b}\right)$, we can not solve the differential equation, so we are obliged to solve it by means of a different method such as the method of energy. For example, S. Timoshenko[18] and E. Chwalla[19] treated this problem about the case of four simply supported edges by the method of energy. This method is favourably used in such a case, but if the sides $y=0$ and $y=b$ are not simply supported edges, the method of energy is so complicated that the calculation is very hard as can be understood from K. Nölke's treatise[20]. In such a case the slope deflection method displays its merits. In solving this problem by the slope deflection method, we must adopt the following procedure.

Fig. 10 shows the rectangular plate of which the buckling forces shall be obtained. Now, the plate is divided into n strips, n being arbitrary but larger the number of division is, the more difficult the calculation becomes. It is necessary that the point of non-stress comes at the dividing line.

If the value of n is suitably chosen, the next process is to find the average force (tensile or compressive) of the varying force of each section. We shall consider the given rectangular plate as a plate in which the average normal force thus obtained is uniformly distributed in each section. Therefore, in the given plate the acting normal force varies step by step.

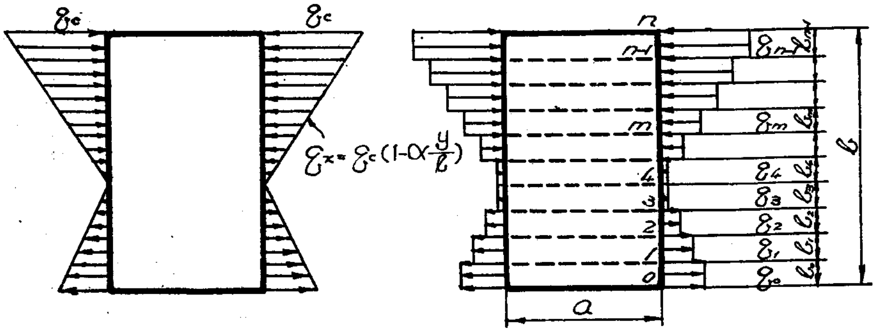

Fig. 10

In the calculation of a rectangular plate by the slope deflection method, the given plate is substituted by a rectangular plate which is subjected to normal force varying in a stepped form and the substituted plate is solved. As a matter of fact, the result obtained is not for the former subjected to a given normal force, but for the latter subjected to stepped varying normal force. But choosing n suitably, the result obtained is sufficiently correct in practical use and the larger the value of n is chosen, the more accurate the result becomes.

The dividing lines of the adopted substituted plate will be numbered as 1, 2, $\cdots$, m, $\cdots$, $n-1$, and the edges as 0 and n.

Notations used here are the same as those in chapter 4, except for the followings.

$$q_m = k_m' q_c\,,\quad q_c = \frac{z_c \pi^2 N}{b^2}\,;\qquad z_m = \frac{q_m b_m^2}{\pi^2 N} = k_m' z_c \left(\frac{\varepsilon}{\varepsilon_m}\right)^2 \tag{32}$$

The method of calculation is the same as that of chapter 6 and will be explained by the following examples.

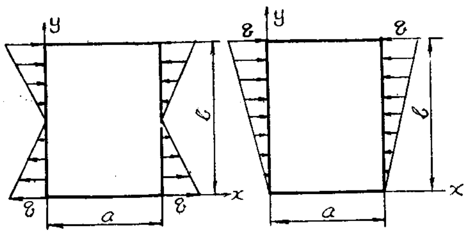

Fig. 11

Example a.

As shown in Fig. 11, let $a/b=0.75$ and the two edges $x=0$ and $x=a$ be simply supported. When these two edges are subjected to combined bending and compression, we will obtain the buckling force of the rectangular plate by changing the boundary conditions of the edges $y=0$ and $y=b$ variously.

Case I (Fig. 12). Pure bending is applied and two sides $y=0$ and $y=b$ are simply supported.

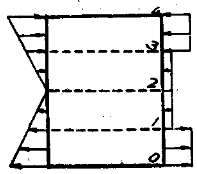

Fig. 12

In this case, the plate is divided into four equal plates by the lines parallel to the two sides $y=0$ and $y=b$. Therefore $\varepsilon_0=\varepsilon_1=\varepsilon_2=\varepsilon_3=3$ and as can be understood from Fig. 12, $k_0'=k_3'=3/4$ and $k_1'=k_2'=1/4$.

If we assume that $z_c=19.2$, that is, $z_0=z_3=0.9$ and $z_1=z_2=0.3$, the equilibrium equations of the bending moment and shearing force are as follows.

θ_1	θ_2	θ_3	δ_1	δ_2	δ_3	
7.55538	1.81690		2.00300	−5.54933		= 0
1.81690	8.20238	1.85750	5.54933	−0.29935	−5.72437	= 0
	1.85750	7.19379		5.72437	−3.09607	= 0
−2.00300	−5.54933		−23.47600	12.24180		= 0
5.54933	0.29935	−5.72437	12.24180	−26.23595	12.96866	= 0
	5.72437	3.09607		12.96866	−13.34857	= 0

By the method explained in chapter 6, the equations consisting of δ only become as follows.

δ_1	δ_2	δ_3	
−19.42014	10.46389	− 3.42548	= 0
10.46389	−17.58627	10.80453	= 0
−3.42548	10.80453	8.68717	= 0

In the above equations, the terms are symmetrical about a diagonal line. This is a essential fact to check the error of the calculation. The value of the determinant of above eveation is

$$\Delta(19.2)=\begin{vmatrix} -19.42013 & 10.46389 & -3.42548 \\ 19.46389 & -17.58627 & 10.80453 \\ -3.42548 & 10.80453 & -8.68717 \end{vmatrix} = -316.856$$

Next, if $z_c=25.6$ is assumed, the result obtained by the same method is $\Delta(25.6)=132.462$. Therefore, we can decide the value of z_c to be equal to 23.7 by interpolation. Then the critical value is $q=23.7\pi^2N/b^2$. According to the solution of S. Timoshenko, who solved this problem by means of the method of energy, z_c is equal to 24.1[21)], which is 1.3% larger than the value obtained by our slope deflection method. In this solution, it is considered that m is equal to 1 and that the plate buckles in the form of one half-wave. If it is necessary to know the critcal value when the plate buckles into two half-waves, we must solve the case of $a/b=0.375$ and $m=1$. This case is solved by either of the next means.

1. Dividing the plate into eight plates whose ratio of side length is 3.
2. Dividing the plate into four plates whose ratio of side length is 1.5.

If the value of z_e obtained by either of the two methods mentioned above is larger than the value obtained before, the value of z_e corresponding to the critical value is taken as equal to 23.7. The calculation of the case $m=2$ is omitted.

Case II (Fig. 12). Pure bending is applied, and the side $y=0$ is fixed and $y=b$ simply supported. The method of calculation is quite same as that of case I. If we assume $z_e=19.2$ and 25.6 respectively, $\Delta(19.2)=-461.66$ and $\Delta(25.6)=162.47$ are obtained. Therefore, we can decide the value of z_e as equal to 23.9 by interpolation. According to the solution of K. Nölke based on the method of energy, z_e is equal to 24.91[22] which is 3.9 % larger than the value obtained by the author's slope deflection method.

Comparing with case I, the error of case II is larger than that of case I. This is due to the fact that in the case of the clamped edge the number of division must be chosen larger than the case of the simply supported edge. In this case, $n=4$ is a little smaller than a suitable number to be adopted in such a case of clamped edge.

Case III. Bending and compression are applied, and stress diagram is triangular as shown in Fig. 13. Two sides $y=0$ and $y=b$ are simply supported. Dividing the plate into four plates, the following results are obtained.

$$\Delta(6.4)=-173.037, \quad \Delta(12.8)=316.26.$$

Therefore, we get $z_e=8.6$ which is 2.6% larger than value obtained by S. Timoshenko[23].

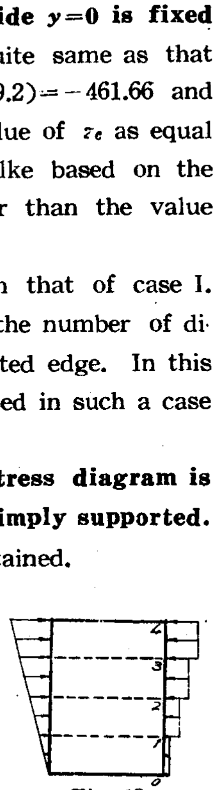

Fig. 13

Case IV (Fig. 13). The stress diagram is the same as that of case III, and two sides $y=0$ and $y=b$ are clamped. The result is as follows.

$$\Delta(12.8)=-88.943, \quad \Delta(19.2)=934.158.$$

Therefore, $z_e=13.4$, which is 4.0% smaller than K. Nölke's value $z_e=13.91$[24]. The reason is quite same as that of case II.

Case V (Fig. 14). Pure bending is applied and the plate is reinforced by a stiffener $\left(\frac{B}{Nb}=5,\ \frac{F}{bt}=0.12\right)$ in the middle point of the compressive side, that is, at $y=\frac{3}{4}b$. Two sides $y=0$ and $y=b$ are simply supported. The method of calculation is quite the same as the above examples, except that we must consider the " $-z\mu z+\frac{\gamma}{\varepsilon}$ " in the equlibrium equation of the shearing force at the dividing line 3. $\Delta(57.6)=-944.73$ and $\Delta(64.0)=254.22$, therefore we get $z_c=62.6$ by interpolation.

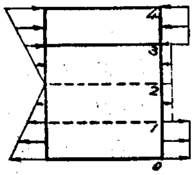

Fig. 14

E. Chwalla solved such a problem by the method of energy, and the value of z_c in the case of $a/b=0.8$ seems to be equal to 65 from his diagram[25]. According to S. Timoshenko's solution[26] in the case of non-stiffener, when $a/b=0.75$, we get $z_c=24.1$ and when $a/b=0.8$, $z_c=24.4$. Therefore, if the plate is reinforced by the same stiffener at the same position, we can obtain, without making a large mistake, the value of z_c of case V as $65.0\times24.1/24.4=64.2$. Thus the value 62.6 obtained by the author's slope deflection method is almost correct.

Example b.

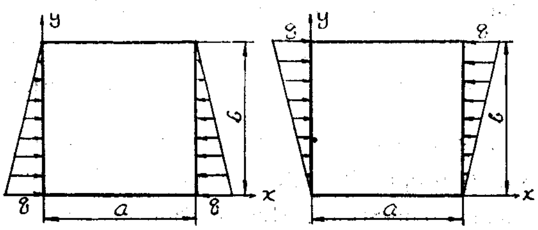

Fig. 15

As shown in Fig. 15, let $a/b=1.0$ and three edges $x=0$, $x=a$ and $y=0$ be simply supported, and the edge $y=b$ supported by elastic beam. When the edges $x=0$ and $x=a$ are subjected to combined bending and compression and the stress diagram is triangular as shown in Fig. 15, we shall obtain the buckling force[27]. In this case, the plate is divided into two equal plates by a line parallel to the two sides $y=0$ and $y=b$. Being $b_0=b_1=b/2$, the equilibrium equations are as follows.

$$\left(\frac{1}{c}+\frac{c}{c^2-s^2}\right)\theta_1+\frac{s}{c^2-s^2}\theta_2+\frac{\pi}{a}\left\{-\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)\delta_1-\frac{j}{c^2-s^2}\delta_2\right\}=0$$

$$\frac{s}{c^2-s^2}\theta_1+\frac{c}{c^2-s^2}\theta_2+\frac{\pi}{a}\left(\frac{j}{c^2-s^2}\delta_1-\frac{i}{c^2-s^2}\delta_2\right)=0$$

$$\left(\frac{i'}{c^2-s^2}-\frac{i}{c^2-s^2}\right)\theta_1-\frac{j}{c^2-s^2}\theta_2+\frac{\pi}{a}\left\{-\left(\frac{h'}{c^2-s^2}+\frac{h}{c^2-s^2}\right)\delta_1+\frac{l}{c^2-s^2}\delta_2\right\}=0$$

$$\frac{j}{c^2-s^2}\theta_1+\frac{i}{c^2-s^2}\theta_2+\frac{\pi}{a}\left\{\frac{l}{c^2-s^2}\delta_1-\left(\frac{h}{c^2-s^2}-z\mu_2+\frac{\gamma}{\varepsilon}\right)\delta_2\right\}=0$$

Equating the determinant of the coefficients of these equations to zero, the buckling equation will be obtained. When the flexural rigidity is given by $\frac{B_2}{Na}=4.0$, that is, $\gamma_c=4.0\pi$, the numerical examples will be calculated, assuming $\mu_c=0$.

Case I (Fig. 16a). In this case, $z_1=\frac{3}{4}\left(\frac{1}{2}\right)^2 z_c=\frac{3}{16}z_c$, $z_1=\frac{1}{16}z_c$ and $\gamma=8.0\pi$. Assuming $z_c=4.0$, that is, $z_0=0.75$ and $z_1=0.25$, the determinant $\Delta(4.0)$ which consists of four lines and columns is 36.167. Next, for $z_c=8.0$, $\Delta(8.0)=-6.745$ is obtained. Therefore, we can decide that the value of z_c is equal to 7.4

by interpolation.

Case II (Fig. 16b). In this case, $z_0 = \frac{1}{16} z_c$, $z_1 = \frac{3}{16} z_c$ and $\gamma = 8.0\pi$, and the result is as follows.

$\Delta(4.0) = 445.767$,

$\Delta(8.0) = -192.954$,

$\therefore \; z_c = 6.7$

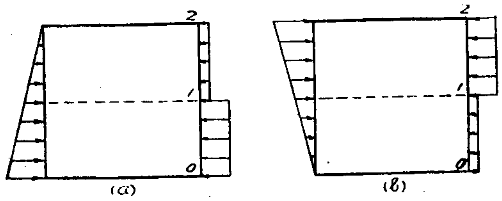

Fig. 16

Therefore, the critical force is $q = 7.4\pi^2 N/a^2$ for case I and $q = 6.7\pi^2 N/a^2$ for case II. These values are quite same as that calculated by S. Ban[27].

8. Conculsion.

In the calculation of the critical force of a rectangular plate simply supported along two opposite sides perpendicular to the direction of force and having various boundary conditions, the relation between the author's slope deflection method and the other two methods can be understood from Fig. 17. That is, by the slope deflection method, for a rectangular plate with the given ratio a/b, the critical force can always be easily obtained for any kind of boundary condition. On the contrary, by the method of energy and the method of integration, for a rectangular plate with the given boundary conditions, the critical force can be calculated for any kind of ratio a/b after inducing the buckling equation. But, the induction of the buckling equation and the numerical calculation for any given ratio a/b are hardly easy as can be understood from many treatises in the past.

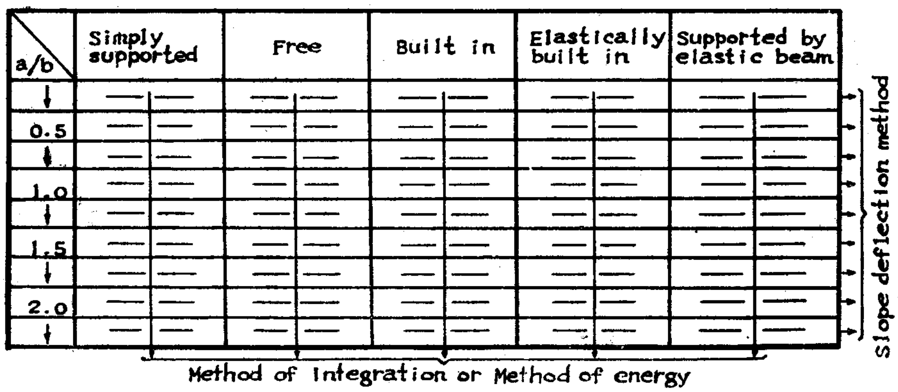

Fig. 17

From the above several examples, we can ascertain the following points.

1) By the auther's slope deflection methed, the boundary conditions can be

so easily expressed for any kind of conditions that the solution is applicable to any kind of edge conditions. But on the contrary, the application of the method of energy is limited to a great extent by the boundary conditions.

2) By the slope deflection method, the number of equations necessary to determine the critical value is decreased to half as compared with the method of integration. Therefore, the number of lines and columns of the determinant is decreased, making the calculation far easier.

3) The values of functions necessaray to the calculation are given in the table beforehand. This results that the root of the buckling equation can be easily found by the trial method, using the table.

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16) See foot-note of 15).
17) See foot-note of 15).
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23) See foot-note of 18).
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