Simple Methods of Calculating Airfoil Section with Given Pressure Distribution

By

Busuke HUDIMOTO, Goro KAMIMOTO
and
Kozi HIROSE

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Simple methods of calculating the form of airfoil section with a given pressure distribution have been developed. In the design of axial flow type pumps and blowers, suitable airfoil profiles which revealed good characteristics in the wind tunnel tests are applied, but it is desirable to modify the section form because interference effect due to cascade arrangement changes the characteristics. In such case the two methods explained in this paper are useful as the profile form can be calculated in a simple way taking the pressure distribution of a suitable airfoil section in a uniform flow. One method is that developed applying the thin airfoil theory of Glauert and the other that based on the theory of wing lattice developed by the present authors.[(1)] These methods are explained in the following two paragraphs.

Part I

The effects of angle of incidence, camber and thickness upon the velocity on the surface of the airfoil are approximately independent of each other. Let V be the velocity of a uniform flow and w_u and w_l be velocities on the upper and lower surfaces of the airfoil respectively, then $w_u - w_l = \gamma$ gives the distribution of circulation and $\frac{1}{2}(w_u + w_l) - V = w_a$ gives the velocity increment due to the thickness.

Take x-axis in the direction of velocity V, the abscissa of the leading edge at $x=-1$ and that of the trailing edge at $x=1$ and substitute x by $x=\cos\theta$, and develope γ into the following series,

$$\frac{\gamma}{2V} = A_0 \tan\frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\theta. \tag{1.1}$$

(1) Hudimoto-Hirose, Theory of the Wing Lattice composed of Arbitrary Airfoils, Vol. XII, No.1 p. 20 of this memoirs.

Then the induced velocity v_0 due to the circulation γ on the surface of the airfoil which is perpendicular to the x-axis is as follows,

$$\frac{v_0}{V}=-A_0-\sum_{n=1}^{\infty}A_n\cos n\theta. \tag{1.2}$$

The ordinate of the mean camber y_c is determined by the following relation

$$\frac{dy_c}{dx}=\frac{v_0}{V}. \tag{1.3}$$

Integrating the above equation with the condition $y_c=0$ at $\theta=0$, the following is obtained,

$$y_c=A_0(1-\cos\theta)+\sum_{n=1}^{\infty}\int_0^{\theta}A_n\cos n\theta\sin\theta\,d\theta. \tag{1.4}$$

The half thickness of the airfoil is expressed as follows,

$$y_a=\sum_{n=1}^{\infty}b_n\sin n\theta,$$

and to satisfy the boundary condition sources and sinks are distributed along the camber line, whose strength q is approximately given by the following relation,

$$q=2V\frac{dy_a}{dx}=-\left(\frac{2V}{\sin\theta}\right)\sum_{n=1}^{\infty}nb_n\cos n\theta. \tag{1.5}$$

The induced velocity u_0 due to this distribution is as follows,

$$\frac{u_0}{V}=\frac{\sum_{n=1}^{\infty}nb_n\sin n\theta}{\sin\theta}. \tag{1.6}$$

Developing the velocity w_a into the following series

$$\frac{w_a}{V}\sin\theta=\sum_{n=1}^{\infty}B_n\sin n\theta, \tag{1.7}$$

and comparing eqs. (1.6) and (1.7), $nb_n=B_n$ is obtained and the half thickness is determined as follows,

$$y_a=\sum_{n=1}^{\infty}\frac{B_n}{n}\sin n\theta. \tag{1.8}$$

To show the accuracy of this method, an example is shown in Fig. 1. The velocity distribution of N. A. C. A. M. 6 airfoil at an angle of incidence 0.95° is given, which is shown at the top of the figure with γ and w_a. The calculated airfoil shown at the bottom of the figure by the full line must coincide with the contour line of the N. A. C. A. M. 6 airfoil which is shown by the dotted line if the proposed method is accurate. The contour lines coincide fairly well with each other although the calculation is very simple.

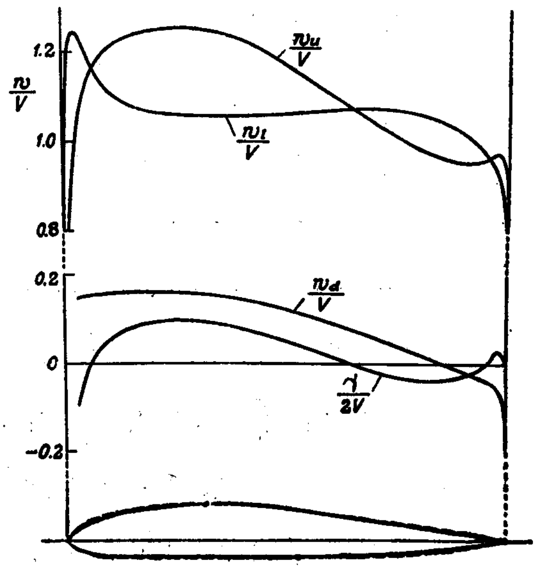

Fig. 1

In the case of a wing lattice the induced velocities due to distributions of vortices and source and sink are as follows,

$$u_\gamma(x)-iv_\gamma(x)=\frac{ie^{-i\beta}}{2d}\int_{-1}^{1}\gamma(\xi)\cot\left[\frac{\pi}{d}e^{-i\beta}(x-\xi)\right]d\xi, \tag{1.9}$$

$$u_q(x)-iv_q(x)=\frac{e^{-i\beta}}{2d}\int_{-1}^{1}q(\xi)\cot\left[\frac{\pi}{d}e^{-i\beta}(x-\xi)\right]d\xi, \tag{1.10}$$

where u_γ and u_q are x-components and v_γ and v_q are y-components of the induced velocities, d is the pitch of airfoil and β is the angle between the wing chord and the axis of the lattice as shown in Fig. 2.

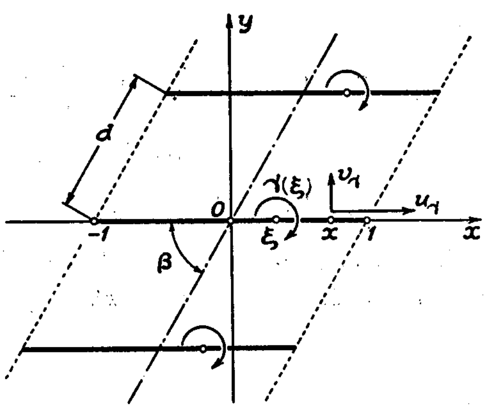

Fig. 2

The procedure of numerical calculation is quite the same as that of the single airfoil, except for the interference effect. The induced velocity v_0 of eq. (1.2) contains the effect of the thickness and the induced velocty u_0 of eq. (1.6) contains the effect of the camber and the condition determining the strength q of the source and sink distribution must be modified to the effect of the velocity gradient along the airfoil. Expanding eqs. (1.9) and (1.10) into series and neglecting small quantities, formulas are obtained.* The thickness is determined by the following eqs.,

$$\left.\begin{aligned}
&(1+\lambda_0)b_1+\lambda_2 b_2=-B_1',\\
&(1+\lambda_0)b_2+\lambda_1(b_1+b_3)=-\frac{B_2'}{2},\\
&(1+\lambda_0)b_3+\lambda_1(b_2+b_4)=-\frac{B_3}{3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}\right\} \tag{1.11}$$

where
$$\left(1-\frac{k^2}{6}\cos 2\beta\right)B_1'=B_1-\frac{k^2}{6}\sin 2\beta\left(A_0-\frac{1}{2}A_2\right),$$
$$B_2'=B_2-\frac{k^2}{6}\sin 2\beta\left(A_0+\frac{1}{2}A_1\right),$$
$$\lambda_0=-\frac{k^2}{6}\sin 2\beta\left(A_0-\frac{1}{2}A_2\right)+\frac{k^2}{6}\cos 2\beta\, B_1',$$
$$\lambda_1=-\frac{k^2}{6}\sin 2\beta\left(A_0+\frac{1}{2}A_1\right),$$

and
$$k=\frac{\pi}{d}.$$

The mean camber line is determined by the following eq.

$$\frac{dy}{dx}=-\sum_{n=0}^{\infty}C_n\cos n\theta, \tag{1.12}$$

where

$$C_0=A_0-\frac{k^2}{6}\cos 2\beta\left(A_0-\frac{1}{2}A_2\right)-\frac{k^2}{6}\sin 2\beta\, B_1',$$
$$C_1=A_1-\frac{k^2}{6}\cos 2\beta\,(2A_0+A_1),$$
$$C_2=A_2,\quad C_3=A_3,\quad \cdots\cdots\cdots\cdots\cdots\cdots$$

Eqs. (1.11) and (1.12) are obtained assuming the pitch-chord ratio $\pi/2k$ is very large and retaining only terms of k^2.

Part II

The wing lattice given in the z-plane can be transformed conformally nearly into a unit circle in the ζ-plane by the following well-known relation,

$$\frac{z}{c}=\frac{\lambda}{2\pi}\left\{e^{-i\gamma}\log\frac{\kappa\zeta+1}{\kappa\zeta-1}+e^{i\gamma}\log\frac{\zeta+\kappa}{\zeta-\kappa}\right\}, \tag{2.1}$$

* Only the most approximate formulas are shown here.

where c is the length of chord, λ is the pitch-chord ratio, γ is the angle of stagger and κ is a constant determined by λ and γ. If angle β is defined as shown in Fig. 2, then $\gamma=\frac{\pi}{2}-\beta$.

Taking θ as parameter the contour of the airfoil is expressed as follows,

$$\left.\begin{aligned}
&\frac{x}{c}=\frac{\lambda}{\pi}\left\{\cos\gamma\tanh^{-1}\frac{2\kappa\cos\theta}{1+\kappa^2}+\sin\gamma\tan^{-1}\frac{2\kappa\sin\theta}{1-\kappa^2}\right\}+m\lambda\cos\gamma,\\
&\frac{y}{c}=a_0+\sum_{n=1}^{\infty}a_n\cos n\theta'+\sum_{n=1}^{\infty}b_n\sin n\theta'+m\lambda\sin\gamma,\\
&m=0,\quad \pm1,\quad \pm2,\quad \cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}\right\}\tag{2.2}$$

where $\qquad \theta'=\theta-\theta_T$ and $\theta_T=\tan^{-1}\left(\frac{1-\kappa^2}{1+\kappa^2}\tan\gamma\right)$.

The velocity v_θ on the circle in the ζ-plane is, [2]

$$\begin{aligned}
-\frac{v_\theta}{Vc}=&\cos\alpha\,\frac{2\lambda\kappa}{\pi}\cdot\frac{\sqrt{K}\sin\theta'}{\kappa^4-2\kappa^2\cos 2(\theta_T+\theta')+1}\\
&-\sin\alpha\,\frac{2\lambda\kappa}{\pi}\cdot\frac{(1-\kappa^4)\cos\theta'+2\kappa^2\sin 2\gamma\sin\theta'}{\sqrt{K}\left\{\kappa^4-2\kappa^2\cos 2(\theta_T+\theta')+1\right\}}\\
&-\sin\alpha\left(\sum_{n=1}^{\infty}nb_n\cos n\theta'-\sum_{n=1}^{\infty}na_n\sin n\theta'\right)\\
&+\cos\alpha\left(\sum_{n=1}^{\infty}na_n\cos n\theta'+\sum_{n=1}^{\infty}nb_n\sin n\theta'\right)\\
&+\frac{\Gamma}{2\pi Vc}\cdot\frac{1-\kappa^4}{\kappa^4-2\kappa^2\cos 2(\theta_T+\theta')+1}\\
=&\cos\alpha\,\frac{2\lambda\kappa}{\pi}\sum_{n=0}^{\infty}\kappa^{2n}\left\{\sin\left[(2n+1)\theta_T-\gamma\right]\cos(2n+1)\theta'\right.\\
&\qquad\left.+\cos\left[(2n+1)\theta_T-\gamma\right]\sin(2n+1)\theta'\right\}\\
&-\sin\alpha\,\frac{2\lambda\kappa}{\pi}\sum_{n=0}^{\infty}\kappa^{2n}\left\{\cos\left[(2n+1)\theta_T-\gamma\right]\cos(2n+1)\theta'\right.\\
&\qquad\left.-\sin\left[(2n+1)\theta_T-\gamma\right]\sin(2n+1)\theta'\right\}\\
&-\sin\alpha\left(\sum_{n=1}^{\infty}nb_n\cos n\theta'-\sum_{n=1}^{\infty}na_n\sin n\theta'\right)
\end{aligned}\tag{2.3}$$

(2) loc. cit.

$$+\cos\alpha\left(\sum_{n=1}^{\infty} na_n \cos n\theta' + \sum_{n=1}^{\infty} nb_n \sin n\theta'\right)$$

$$+\frac{1}{2\pi}\left(\frac{\Gamma}{Vc}\right)\left\{1+2\sum_{n=1}^{\infty}\kappa^{2n}(\cos 2n\theta_T \cos 2n\theta' - \sin 2n\theta_T \sin 2n\theta')\right\},$$

where V is the velocity of mean uniform flow, α is the angle of incidence, Γ is the circulation around each airfoil and $K=\kappa^4+2\kappa^2\cos 2\gamma+1$.

Now from the given pressure distribution, the velocity w_x on the airfoil can be calculated and the problem is reduced to determining the coefficients a_n, b_n and α from a given value of w_x.

From the given velocity w_x, the corresponding velocity on the circle in the ζ–plane is determined by the following eq.,

$$\frac{v_\theta}{Vc}=\left(\frac{w_x}{V}\right)\frac{d\left(\frac{x}{c}\right)}{d\theta}=-\frac{2\lambda\kappa}{\pi}\cdot\frac{\sqrt{K}\sin\theta'}{\kappa^4-2\kappa^2\cos 2(\theta_T+\theta')+1}\left(\frac{w_x}{V}\right), \tag{2.4}$$

and expanding this velocity into Fourier series it is expressed as follows,

$$-\frac{v_\theta}{Vc}=A_0+\sum_{n=1}^{\infty}A_n\cos n\theta'+\sum_{n=1}^{\infty}B_n\sin n\theta'. \tag{2.5}$$

The coefficients of A_0, A_n and B_n are calculated from the numerical values given by eq. (2.4).

Then comparing eqs. (2.3) and (2.5), the following results are obtained,

$$\begin{aligned}
a_{2n+1}&=\frac{A_{2n+1}}{2n+1}\cos\alpha+\frac{B_{2n+1}}{2n+1}\sin\alpha-\frac{2\lambda}{\pi}\frac{\kappa^{2n+1}}{2n+1}\sin\left[(2n+1)\theta_T-\gamma\right],\\
b_{2n+1}&=\frac{B_{2n+1}}{2n+1}\cos\alpha-\frac{A_{2n+1}}{2n+1}\sin\alpha-\frac{2\lambda}{\pi}\frac{\kappa^{2n+1}}{2n+1}\cos\left[(2n+1)\theta_T-\gamma\right],\\
a_{2n}&=\frac{A_{2n}}{2n}\cos\alpha+\frac{B_{2n}}{2n}\sin\alpha-2A_0\frac{\kappa^{2n}}{2n}\cos(2n\theta_T+\alpha),
\end{aligned} \tag{2.6}$$

and $$b_{2n}=\frac{B_{2n}}{2n}\cos\alpha-\frac{A_{2n}}{2n}\sin\alpha+2A_0\frac{\kappa^{2n}}{2n}\sin(2n\theta_T+\alpha).$$

Making use of the condition that the leading and trailing edges of one airfoil are on the x–axis,

$$a_0+\sum_{n=1}^{\infty}(-1)^n a_n=0 \qquad \text{at the leading edge, where } \theta'=\pi,$$

and $$a_0+\sum_{n=1}^{\infty}a_n=0 \qquad \text{at the trailing edge, where } \theta'=0.$$

Hence,

$$\sum_{n=0}^{\infty} a_{2n+1}=0, \tag{2.7}$$

and

$$a_0=-\sum_{n=1}^{\infty} a_{2n}. \tag{2.8}$$

From eqs. (2.6) and (2.7),

$$\cos\alpha\sum_{n=0}^{\infty}\frac{A_{2n+1}}{2n+1}+\sin\alpha\sum_{n=0}^{\infty}\frac{B_{2n+1}}{2n+1}=\frac{2\lambda}{\pi}\sum_{n=0}^{\infty}\frac{\kappa^{2n+1}}{2n+1}\sin\left[(2n+1)\theta_T-\gamma\right], \tag{2.9}$$

and the angle of incidence α is determined by this relation and a_n and b_n are determined by eqs. (2.6).

The isolated airfoil is a special case of the above-mentioned theory and, in this case, $\kappa=0$, $\lambda\kappa=\frac{\pi}{4}$ and $\theta_T=\gamma$.

As an example, the theoretical pressure distribution of a wing lattice composed of N. A. C. A. 4412 airfoil section with $\lambda=1.284$, $\gamma=50^0$ and $\alpha=3^0$ was given as shown in Fig. 3 and the form of the airfoil was calculated from this pressure distribution. The points are those calculated and it is seen that they coincide satisfactorily with the contour line of N. A. C. A. 4412 airfoil section.

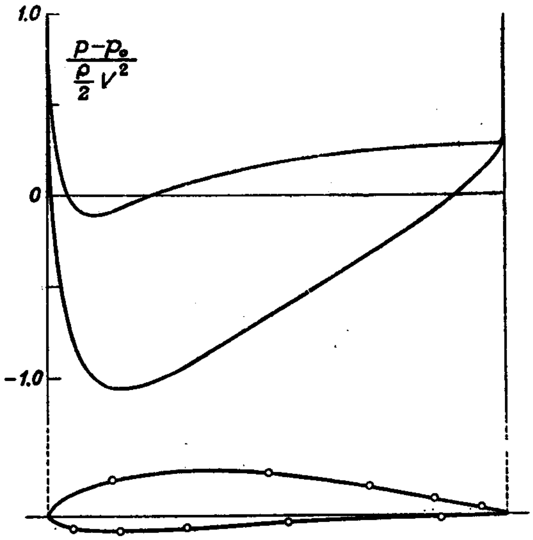

Fig. 3