# The Theory of Non-Linear Elasticity

By

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## 1. Introduction

The present paper named the theory of non-linear elasticity may, if simply considered, express a theory of elasticity of such materials as cast iron, concrete or rubbers which do not follow the Hooke's law, but the author believes from the point of view in connection with intermolecular forces, that it will be valid in the wider sense to many other materials.

It is a great advantage for us that steels and many other engineering metals follow the Hooke's law and the relations between stress and strain which are produced in these materials are expressed by the linear equations. In such a case, we may apply the law of superposition without any difficulty for combination of any two or more strains, but these relations are not actually linear in cast iron or concrete. For uniaxial tension, C. Bach expressed them in the form $\varepsilon=\alpha\sigma^n$ and Cox and Lang used a parabolic expression $\varepsilon=\sigma/(a-b\sigma)$ where $\varepsilon$ represents strain and $\sigma$ denotes stress, while we have even now no reasonable formula applicable to the state of biaxial tension or compression. This problem was once studied by Prof. M. Kakuzen [1]. He put

$$\left.\begin{aligned} &\varepsilon_i=\frac{\sigma_i}{a-b\sigma_i}-\frac{1}{m}\cdot\frac{\sigma_j}{a-b\sigma_j}, \\ &(i,\ j=x, y, z) \end{aligned}\right\} \qquad (a)$$

where $1/m$ means Poisson's ratio.

As, strictly speaking, any two non-linear expressions should not be superposed, the above equation is, as he already recognized in his report, not compatible. It my be only used as an approximate relation.

The author has established a new theory relating to stresses and strains in such materials and discussed it with respect to intermolecular forces.

(1) Professor of Doshisya University, Kyoto, Dissertation, 1938.

**2. Author's expressions for the relation between stresses and strains**

Now, we represent strains in the direction of the co-ordinate axes by $\varepsilon_i$, $\varepsilon_j$ and $\varepsilon_k$ and the corresponding normal stresses by $\sigma_i$, $\sigma_j$ and $\sigma_k$. As a normal strain be determined by the present state of normal stresses, we may write generally

$$\left.\begin{array}{l}\varepsilon_i=f_i(\sigma_i, \sigma_j, \sigma_k), \\ \quad (i, j, k=x, y, z)\end{array}\right\} \qquad (1)$$

$f_i$ represents a function of stresses, in which all shearing stresses are not contained as they do not give influence upon any normal strains. By taking the total differential of eq. (1), we have

$$d\varepsilon_i=\frac{\partial f_i}{\partial \sigma_i}d\sigma_i+\frac{\partial f_i}{\partial \sigma_j}d\sigma_j+\frac{\partial f_i}{\partial \sigma_k}d\sigma_k \qquad (2)$$

or simply

$$d\varepsilon_i=\frac{\partial f_i}{\partial \sigma_n}d\sigma_n, \ (n=i, j, k) \qquad (2')$$

As a special case, we assume that all the differential coefficients in eq. (2) would be constant. Then, for an isotropic body, we may put

$$\frac{\partial f_i}{\partial \sigma_n}=\begin{cases}C_1, & (i=n) \\ C_2, & (i\neq n)\end{cases}$$

Eq. (2) becomes

$$d\varepsilon_i=C_1 d\sigma_i+C_2(d\sigma_j+d\sigma_k) \qquad (3)$$

This is the ordinary expression in the elasticity based on the Hooke's law. It we put $C_1=1/E$ and $C_2=1/mE$, $E$ corresponds to the modulus of elasticity and $m$ to the Poisson's constant.

Next, if we assume that any of $\partial f_i/\partial \sigma_n$ is a function of $\sigma_n$ only, or we express $\partial f_i/\partial \sigma_i=\varphi_1(\sigma_i)$, $\partial f_i/\partial \sigma_j=\varphi_2(\sigma_j)$ and $\partial f_i/\partial \sigma_k=\varphi_2(\sigma_k)$, then we get

$$\varepsilon_i=F_1(\sigma_i)+F_2(\sigma_j)+F_2(\sigma_k)$$

where $F_1$ and $F_2$ are functions of each stress only. The foregoing expression (a) can be comprised in the above equation.

According to this equation, the effect which $\sigma_i$ has on a normal strain in any direction is perfectly independent of other stresses and this shows that the law of superposition would hold in this case. Therefore, the above mentioned assumption will be considered to be incompatible for materials which have non-linear relationship between stresses and strains.

From this reason, we generally must take the differential coefficients in eq. (1) as follows:

$$\left.\begin{aligned}\frac{\partial f_i}{\partial \sigma_i}&=\frac{1}{\Phi_i(\sigma_i, \sigma_j, \sigma_k)},\\ \frac{\partial f_i}{\partial \sigma_i}&=\frac{\partial f_i}{\partial \sigma_k}=\frac{1}{\Psi_i(\sigma_i, \sigma_j, \sigma_k)}\end{aligned}\right\} \tag{4}$$

where $\Phi_i$ and $\Psi_i$ denote functions of stresses. And we put

$$\frac{1}{\Psi_i(\sigma_i, \sigma_j, \sigma_k)}=-\frac{1}{m\cdot\Phi_i(\sigma_i, \sigma_j, \sigma_k)} \tag{5}$$

wherein $m$ may be, in general, considered to be a function of stresses $\sigma_i, \sigma_j$ and $\sigma_k$. Though it is different from ordinary Poisson's constant, it concides to the latter in the special case in which $\partial f_i/\partial \sigma_n$ takes constant value. By introdncing eq. (5), eq. (2) becomes

$$d\varepsilon_i=\frac{1}{\Phi_i}\left\{d\sigma_i-\frac{d\sigma_j+d\sigma_k}{m}\right\} \tag{6}$$

Now, the author put $m$ constant in the above equation according to the reason that is, he considers, approximately appropriate by the theory which he will later explain.

Moreover, we write

$$\sigma_{ei}=\sigma_i-\frac{1}{m}(\sigma_j+\sigma_k) \tag{7}$$

and let us call it an effective stress. Then, eq. (6) becomes

$$d\varepsilon_i=\frac{1}{\Phi_i}d\sigma_{ei} \tag{8}$$

Now, he assumes that $\Phi_i$ be a function of $\sigma_{ei}$ only; i.e. the normal strain $\varepsilon_i$ is determined by the value of $\sigma_{ei}$.

This assumption will be assertained by the theory, which he intends to explain later.

By using this assumption, we may be able to integrate the above equation and obtain $\varepsilon_i$ in the form

$$\varepsilon_i=\int\frac{1}{\Phi_i}d\sigma_{ei}=f_i(\sigma_{ei}) \tag{9}$$

## 3. Relation between the shearing stresses and the shearing strains

As the shearing strains are caused by the shearing stresses, they are generally expressed by

$$\gamma_i=f(\tau_i),\ (i=x, y, z) \tag{10}$$

where $\gamma_i$ means a shearing strain and $\tau_i$ a shearing stress.

Its differential is

$$d\gamma_i=\frac{df}{d\tau_i}d\tau_i \tag{11}$$

Hereupon, if we denote

$$\frac{df}{d\tau_i}=\frac{1}{\chi(\tau_i)}$$

eq. (11) becomes

$$d\gamma_i=\frac{1}{\chi(\tau_i)}d\tau_i \tag{12}$$

where $\chi$ is another function.

In order to obtain the relation between $\chi$ and $\Phi_i$, we consider the condition of pure shear, in which $\sigma_i=-\sigma_j=\sigma, \sigma_k=0$ and $\sigma$ is equal to the value of the shearing stress $\tau$. Then we have

$$\sigma_{ei}=\sigma_i-\frac{1}{m}(\sigma_j+\sigma_k)=\frac{m+1}{m}\sigma$$

$$\therefore \quad d\varepsilon_i=\frac{1}{\Phi_i\left(\frac{m+1}{m}\sigma\right)}d\left(\frac{m+1}{m}\sigma\right) \tag{13}$$

If both stress-strain curves of a material in tension and in compression have the same form, the following relation holds:

$$\chi(\tau)=\frac{m}{2(m+1)}\Phi_i\left(\frac{m+1}{m}\tau\right) \tag{14}$$

## 4. On the modulus of elasticity and Poisson's constant

The ratio of stress of strain $\sigma/\varepsilon$ obtained from a tensile test for such material as cast iron, which deviates from the Hooke's law, does not exactly denote an increasing rate of stress for given strain, but it is called occasionally the second modulus of elasticity owing to its meaning of the mean rate of stress-increase between the stresses 0 and $\sigma$.

On the contrary, the reciprocal of the coefficient $\partial f_i/\partial\sigma_i$ denotes the actual rate of stress-increase at the given stressed state. Therefore, it should be called, in general, the coefficient or modulus of elasticity in this case. While, as each strain is caused by three normal stresses acting in the directions of the co-ordinate axes in the complex state of deformation, $\Phi_i$ should be taken up as the modulus of elasticity and $\chi_i$ the modulus of shearing elasticity. Poisson's ratio is defined ordinarily by the ratio of the lateral strain to the longitudinal one produced in an uniaxial tension in the ordinary theory of elasticity and be written as

$$\left|\frac{\varepsilon'}{\varepsilon}\right|=\nu=\frac{1}{m'}$$

where $\varepsilon'$ means the lateral, $\varepsilon$ the longitudinal strain and $m'$ Poisson's constant.

In the materials, which follows the Hooke's law, $m'$ has the constant value $m$, but it can not be decided in cast iron or other materials of non-linear elasticity that $m'$ takes a constant value through all the slates of extension.

The author names the ratio $m$ defined as eq. (5) an auxiliary coefficient of elasticity.

## 5. Various types of stress-strain curves

Generally speaking, the index $n$ in Bach's equation is ordinarily greater than 1 and the constant $b$ in Cox and Lang's expression takes a positive value for comparatively more brittle materials, while $n$ takes a smaller value than 1 and $b$ is negative for many organic materials as rubber or leather.

It is clear that the above mentioned assumption which takes $\Phi_i$ for granted to be a function of $\sigma_{ei}$ coincides in the case of uniaxial stress to these expressions. Now, he enumerates the varions types of stress-strain curves.

(I) Let $\Phi_i$ be expressed by a linear equation of $\sigma_{ei}$ as follows:

$$\Phi_i = E - A\sigma_{ei} \tag{15}$$

in which $E$ and $A$ are both constant; the former corresponds to the modulus of elasticity when the stress-strain diagram is straight.

From eq. (9) and eq. (15), we obtain the relation

$$\left.\begin{aligned} \varepsilon_i &= \frac{1}{A} \log \frac{E}{E - A\sigma_{ei}} \\ \therefore \quad \sigma_{ei} &= \frac{E}{A} \cdot \frac{e^{A\varepsilon_i} - 1}{e^{A\varepsilon_i}} \end{aligned}\right\} \tag{16}$$

(II) Next, let us take

$$\Phi_i = \frac{(E - A\sigma_{ei})^n}{E^{n-1}} \tag{17}$$

then, the corresponding relation between stress and strain will be expressed by

$$\varepsilon_i = \frac{E^{n-1} - (E - A\sigma_{ei})^{n-1}}{(n-1)A \cdot (E - A\sigma_{ei})^{n-1}} \tag{18}$$

In the special cases of $n=2, 3, 4$, eq. (18) becomes as follows:

(a) When $n=2$, then

$$\Phi_i = E\left(1 - \frac{A}{E}\sigma_{ei}\right)^2 \tag{19}$$

and

$$\varepsilon_i = \frac{\sigma_{ei}}{E - A\sigma_{ei}} \tag{20}$$

This equation coincides exactly to the foregoing expression of Cox and Lang for a pure tension.

(b) Let us take $n=3$, then we have

$$\Phi_t=E\left(1-\frac{A}{E}\sigma_{et}\right)^3 \tag{21}$$

and

$$\varepsilon_t=\frac{E-\dfrac{A}{2}\sigma_{et}}{(E-A\sigma_{et})^2}\cdot\sigma_{et} \tag{22}$$

(c) For $n=4$, $\Phi_t$ and $\varepsilon_t$ are expressed by the equations

$$\Phi_t=E\left(1-\frac{A}{E}\sigma_{et}\right)^4 \tag{23}$$

and

$$\varepsilon_t=\frac{E^2-\dfrac{2}{3}AE\sigma_{et}+\dfrac{1}{3}A^2\sigma_{et}^2}{(E-A\sigma_{et})^3}\cdot\sigma_{et} \tag{24}$$

These curves are shown in Fig. (1). The curvature of stress-strain diagram increases with the value of $n$.

(III) (a) As the second term of the numerator in eq. (23) is very small compared to the first term, by neglecting it we have the following relations:

$$\varepsilon_t=\frac{E\sigma_{et}}{(E-A\sigma_{et})^2} \tag{25}$$

for which the corresponding $\Phi_t$ is

$$\Phi_t=\frac{(E-A\sigma_{et})^3}{E(E+A\sigma_{et})} \tag{26}$$

(b) By neglecting the smaller term in the numerator in eq. (24), we obtain

$$\varepsilon_t=\frac{E^2\varepsilon_{et}}{(E-A\sigma_{et})^3} \tag{27}$$

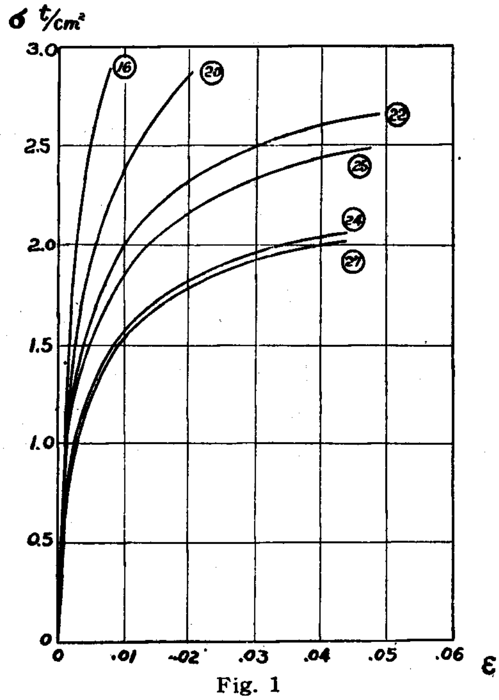

Fig. 1

We see that the curve of eq. (22) draws very near that of eq. (25) and similarly eq. (24) near eq. (27).

## 6. Foundamental equations of non-linear elasticity

As the relation between an effective stress and the stresses acting in the directions of co-ordinate axes is shown in eq. (7), we can easily derive the equation

$$\sigma_i=\frac{m(m-1)}{(m+1)(m-2)}\left\{\sigma_{ei}+\frac{\sigma_{ej}+\sigma_{ek}}{m-1}\right\} \tag{28}$$

from which we get the relation

$$\sigma_{ei}+\sigma_{ej}+\sigma_{ek}=\frac{m-2}{m}(\sigma_i+\sigma_j+\sigma_k)$$

This shows that the sum of effective stresses is also an invariant.

If we let $S_e$ stands for it, eq. (28) becomes

$$\sigma_i=\frac{m}{m+1}\left\{\sigma_{ei}+\frac{S_e}{m-2}\right\} \tag{29}$$

Then, the equations of equilibrium of forces of elasticity without body forces will be expressed by

$$\frac{m}{m+1}\,\frac{\partial\sigma_{ei}}{\partial i}+\frac{m}{(m+1)(m-2)}\,\frac{\partial S_e}{\partial i}+\frac{\partial\tau_k}{\partial j}+\frac{\partial\tau_j}{\partial k}=0 \tag{30}$$

$$(i,\ j,\ k=x,\ y,\ z)$$

By inserting the relations (8) into the above equation, we obtain

$$\frac{m}{m+1}\Phi_i\frac{\partial\varepsilon_i}{\partial i}+\frac{m}{(m+1)(m-2)}\,\frac{\partial S_e}{\partial i}+\frac{\partial\tau_k}{\partial j}+\frac{\partial\tau_j}{\partial k}=0$$

or

$$\frac{m}{(m+1)(m-2)}\left[(m-1)\Phi_i\frac{\partial\varepsilon_i}{\partial i}+\Phi_j\frac{\partial\varepsilon_j}{\partial j}+\Phi_k\frac{\partial\varepsilon_k}{\partial k}\right]+\chi_k\frac{\partial\gamma_k}{\partial j}+\chi_j\frac{\partial\gamma_j}{\partial k}=0 \tag{31}$$

in which $\gamma_k$ and $\gamma_j$ stand for shearing strains.

Now, if we denote the displacements in the direction of coordinate axes $i$, $j$, $k$ by $\lambda$, $\mu$, and $\nu$, the normal strains are expressed by

$$\varepsilon_i=\frac{\partial\lambda}{\partial i},\quad \varepsilon_j=\frac{\partial\mu}{\partial j},\quad \varepsilon_k=\frac{\partial\nu}{\partial k}$$

and the shearing strains

$$\gamma_k=\frac{\partial\lambda}{\partial j}+\frac{\partial\mu}{\partial i},\quad \gamma_i=\frac{\partial\mu}{\partial k}+\frac{\partial\nu}{\partial j},\quad \gamma_j=\frac{\partial\nu}{\partial i}+\frac{\partial\lambda}{\partial k}.$$

By introducing these expressions into eq. (31), we obtain

$$\frac{m}{(m+1)(m-2)}\left[(m-1)\Phi_i\frac{\partial^2\lambda}{\partial i^2}+\Phi_i\frac{\partial^2\mu}{\partial i\partial j}+\Phi_k\frac{\partial^2\nu}{\partial i\partial k}\right]$$

$$+\chi_k\left(\frac{\partial^2\lambda}{\partial j^2}+\frac{\partial^2\mu}{\partial j\partial i}\right)+\chi_j\left(\frac{\partial^2\nu}{\partial k\partial i}+\frac{\partial^2\lambda}{\partial k^2}\right)=0 \tag{32}$$

## 7. Elasticity of cast iron

Both stress-strain diagrams of tension and compression of cast iron do not take the same form and the elastic constants obtained from those curves have different values in each.

If we examine, for example, the experimental results of C. Bach, we see that the index $n$ in eq. $\varepsilon=\alpha\sigma^n$ for tension is somewhat greater than that for compression.

According to Cox and Lang's expressions,

$$\varepsilon=\sigma/(E-A\sigma) \quad \text{for tension}$$

and

$$\varepsilon=\sigma/(E+A'\sigma) \quad \text{for compression}$$

where $A>A'$ generally.

In order to get a relation between shearing stress and shearing strain in the case where both stress-strain curves do not take the same form, we may put $\sigma_i=\sigma=-\sigma_j$, and $\sigma_k=0$ considering the slate of pure shear. Then, we have

$$\sigma_{ei}=\sigma_i-\frac{\sigma_j}{m}=\frac{m+1}{m}\sigma$$

$$\sigma_{ej}=\sigma_j-\frac{\sigma_i}{m}=-\frac{m+1}{m}\sigma$$

Therefore, the author's theoretical eq. (6) is written as follows:

$$\left.\begin{aligned} d\varepsilon_i&=\frac{1}{\Phi_i(\sigma_{ei})}d\sigma_{ei}=\frac{1}{\Phi_i\left(\frac{m+1}{m}\sigma\right)}d\left(\frac{m+1}{m}\sigma\right)\\ d\varepsilon_j&=\frac{1}{\Phi_j'(\sigma_{ej})}d\sigma_{ej}=\frac{1}{\Phi_j'\left(-\frac{m+1}{m}\sigma\right)}d\left(-\frac{m+1}{m}\sigma\right) \end{aligned}\right\} \tag{33}$$

in which $\Phi_i$ and $\Phi_j'$ are both the functions to be determined by tension or compression curves.

This state of stress is no other than the state of shear in the direction inclined $45^0$ to the former and the shearing stress $\tau$ is equal to $\sigma$.

The increase of shearing strain due to it is expressed by

$$d\gamma=\frac{1}{\chi(\tau)}d\tau$$

The relation between this and the foregoing $d\varepsilon_i$ and $d\varepsilon_j$ is shown as follows:

$$\left.\frac{1+\frac{d\gamma}{2}}{1-\frac{d\gamma}{2}}=\frac{1+d\varepsilon_i}{1+d\varepsilon_j}, \quad \begin{aligned} d\varepsilon_i&>0\\ d\varepsilon_j&<0 \end{aligned}\right\} \tag{34}$$

or

$$d\gamma=\frac{2(d\varepsilon_i-d\varepsilon_j)}{2+d\varepsilon_i+d\varepsilon_j} \tag{35}$$

As the second and third terms of denominator in the right hand side of eq. (35) are both very small compared to the first term, we may neglect the sum of them, then

$$d\gamma=d\varepsilon_i-d\varepsilon_j=|d\varepsilon_i|+|d\varepsilon_j| \tag{36}$$

By putting this value into eq. (33), we get

$$\frac{d\tau}{\chi(\tau)}=\left\{\frac{1}{\Phi_i\left(\frac{m+1}{m}\tau\right)}+\frac{1}{\Phi_j'\left(-\frac{m+1}{m}\tau\right)}\right\}d\left(\frac{m+1}{m}\tau\right)$$

$$\therefore \quad \frac{1}{\chi(\tau)}=\frac{m+1}{m}\left\{\frac{1}{\Phi_i\left(\frac{m+1}{m}\tau\right)}+\frac{1}{\Phi_j'\left(-\frac{m+1}{m}\tau\right)}\right\} \tag{37}$$

For example, corresponding to Cox and Lang's expression, we put

$$\varepsilon_i=\frac{\sigma_{ei}}{E-A\sigma_{ei}}, \quad \varepsilon_j=\frac{\sigma_{ej}}{E+A'\sigma_{ej}}$$

then

$$\frac{1}{\Phi_i}=\frac{E}{(E-A\sigma_{ei})^2}, \quad \frac{1}{\Phi_j'}=\frac{E}{(E+A'\sigma_{ej})^2}$$

By inserting these into eq. (37), we obtain

$$\frac{1}{\chi(\tau)}=\frac{m+1}{m}\left\{\frac{E}{\left(E-A\frac{m+1}{m}\tau\right)^2}+\frac{E}{\left(E-A'\frac{m+1}{m}\tau\right)^2}\right\} \tag{38}$$

Hereupon, using the relation $G=mE/2(m+1)$ which exists between the modulus of longitudinal elasticity $E$ and that of shearing elasticity $G$, we get

$$\frac{1}{\chi(\tau)}=\frac{G}{2}\left\{\frac{1}{\left(G-\frac{A}{2}\tau\right)^2}+\frac{1}{\left(G-\frac{A'}{2}\tau\right)^2}\right\} \tag{38'}$$

Then, the relation between shearing stress and shearing strain will be expressed as follows:

$$\gamma=\frac{\tau}{2}\left\{\frac{1}{\left(G-\frac{A}{2}\tau\right)}+\frac{1}{\left(G-\frac{A'}{2}\tau\right)}\right\} \tag{39}$$

In the special case where $A=A'$ or both stress-strain curves of tension and compression have the same form,

$$\frac{1}{\chi(\tau)}=\frac{G}{\left(G-\frac{A}{2}\tau\right)^2} \tag{40}$$

and

$$\gamma=\frac{\tau}{\left(G-\frac{A}{2}\tau\right)} \tag{41}$$

It is very inconvenient that we must apply the different equations to tension and to compression as descibed above.

Therefore, the author presented a new equation as the relation applicable through both tension and compression as shown in Fig. 2

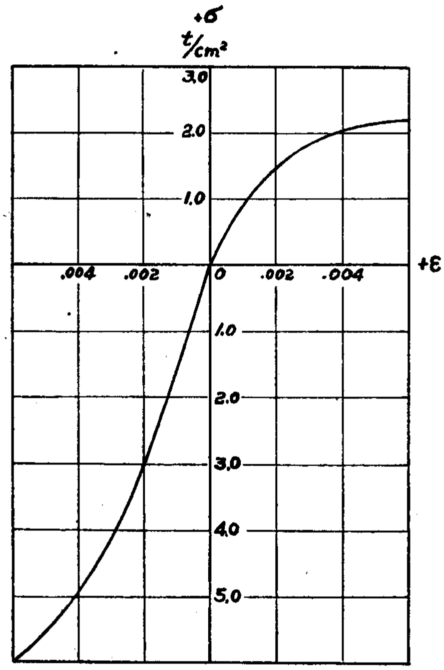

Fig: 2

$$\varepsilon_t=\frac{E\sigma_t}{(E-A\sigma_t)(E+A'\sigma_t)} \tag{42}$$

which can be simplified as

$$\varepsilon_t=\frac{1}{A+A'}\left\{\frac{A\sigma_t}{E-A\sigma_t}+\frac{A'\sigma_t}{E+A'\sigma_t}\right\}$$

Corresponding to it, we have

$$\frac{1}{\Phi_t}=\frac{E}{A+A'}\left\{\frac{A}{(E-A\sigma_t)^2}+\frac{A'}{(E+A'\sigma_t)^2}\right\} \tag{43}$$

and

$$\frac{1}{\chi(\tau)}=\frac{G}{2(A+A')}\left\{\frac{A}{\left(G-\frac{A}{2}\tau\right)^2}+\frac{A'}{\left(G+\frac{A'}{2}\tau\right)^2}+\frac{A}{\left(G+\frac{A}{2}\tau\right)^2}+\frac{A'}{\left(G-\frac{A'}{2}\tau\right)^2}\right\} \tag{44}$$

Then, the shearing strain is expressed

$$\gamma=\frac{G}{2}\left\{\frac{\tau}{\left(G-\frac{A}{2}\tau\right)\left(G+\frac{A'}{2}\tau\right)}+\frac{\tau}{\left(G+\frac{A}{2}\tau\right)\left(G-\frac{A'\tau}{2}\right)}\right\}$$

or

$$=\frac{G}{A+A'}\left\{\frac{A\tau}{G^2-\frac{A^2}{4}\tau^2}+\frac{A'\tau}{G^2-\frac{A'^2}{4}\tau^2}\right\} \tag{45}$$

When $A=A'$, eq. (42) becomes

$$\varepsilon_t=\frac{E\sigma_t}{E^2-A^2\sigma_t^2} \tag{46}$$

and eq. (45) is transformed to

$$\gamma=\frac{G\tau}{G^2-\frac{A^2}{4}\tau^2} \tag{47}$$

The normal stress is expressed from eq. (46) as a function of strain as

$$\sigma_t=\frac{E\varepsilon}{1+A^2\varepsilon^2} \tag{48}$$

and

$$\tau=\frac{G\gamma}{1+\frac{A^2}{4}\gamma^2} \tag{49}$$

By suitable approximation, eq. (42) can be transformed to an expression

$$\sigma_t=\frac{E\varepsilon}{(1+A\varepsilon+A^2\varepsilon^2)(1-A'\varepsilon+A'^2\varepsilon^2)} \tag{50}$$

applicable in the region of small strains.

## 8. Bending and torsion of a cast iron bar

In the bending of a cast iron bar, the neutral axis does not pass through the center of figure in its cross section, as the stress-strain curves of tension and compression are different each other, but its deviation is so small that we may neglect it for calculation of bending moment.

The position of the neutral axis of a rectangular cross section can be obtained approximately from the equation

$$\frac{2\eta_0}{h}=\frac{2M}{Ebh^2}\,\frac{A\left(1+\frac{2\eta_0}{h}\right)^3-A'\left(1-\frac{2\eta_0}{h}\right)^3}{\left(1+\frac{2\eta_0}{h}\right)^3+\left(1-\frac{2\eta_0}{h}\right)^3}$$

where $\eta_0$ means the distance between the neutral axis and the center line of the rod and $b$ is the breadth and $h$ the height of the cross section.

The value of $2\eta_0/h$ is plotted in Fig. 3 for the case where $A=400$ and $A'=150$.

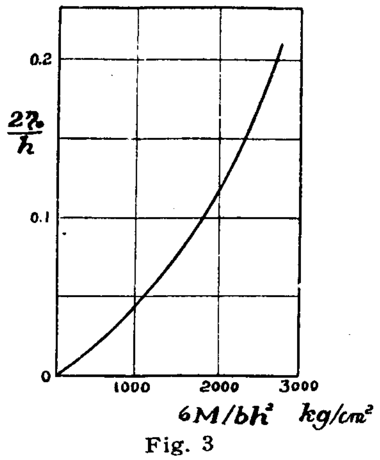

Fig. 3

Noticing that $\eta_0$ is very small, we may use the expression (46) as the mean stress-strain curve of tension and compression for calculation of the bending moment.

Let the fibre strain $\varepsilon$ be proportional to the distance from the center line, i. e. $\varepsilon=c\eta$, then the bending moment is expressed by

$$M=\int_F \sigma\eta dF=\int_F \frac{Ec\eta^2}{1+A^2c^2\eta^2}dF$$

or

$$M=\frac{Eb}{A^2c}\left[h-\frac{1}{Ac}\left\{\tan^{-1}\left(\frac{Ach}{2}\right)-\tan^{-1}\left(-\frac{Ach}{2}\right)\right\}\right] \tag{52}$$

Developing this into series and neglecting the higher terms of infinitesimals, we obtain

$$M=\frac{1}{2}bh^2E\varepsilon_1\left\{\frac{1}{3}-\frac{1}{5}A^2\varepsilon_1^2+\frac{1}{7}A^4\varepsilon_1^4-\right\} \tag{53}$$

where

$$\frac{1}{2}ch=ce_1=\varepsilon_1$$

Neglecting the higher terms of $\varepsilon_1$, we obtain

$$\sigma_1=\frac{(6M/bh^2)}{1+\frac{2}{5}\left(\frac{A}{E}\right)^2\left(\frac{6M}{bh^2}\right)^2} \tag{54}$$

This is a relation of the maximum stress expressed as a function of the bending moment.

After these equations, we see that the bending moment is not proportional to any of strain $\varepsilon_1$ and stress $\sigma_1$.

Let us compare this eq. (54) to the experimental data of H. Herbert [2]. The curve $M-\sigma_1$, and $M-\sigma_2$ shown in Fig. 4 are both the bending moment-stress diagrams in which $\sigma_1$ denotes the maximum tensile and $\sigma_2$ the maximum compressive stress. Though the amount of deviation of $M-\sigma_2$ curve from the straight line $\sigma_1=6M/bh^2$ is not so large, it is evident that either of them has a tendency of going upwards with the stress-increase. The value of $\sigma_1$ calculated from eq. (54) is plotted in this figure. This curve is concave upwards and shows a mean value of $M-\sigma_1$ and $M-\sigma_2$. While the relation between the bending moment and the maximum stress calculated from equation $\varepsilon=a\sigma^n$ in place of a hyperbolic one is expressed by

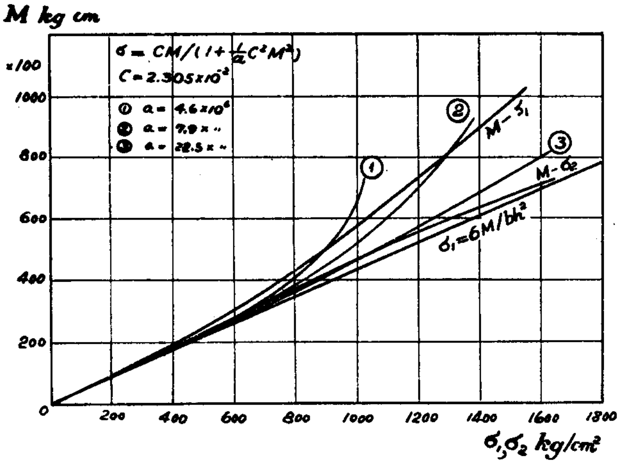


Fig. 4

$$M=\frac{bh^2}{2(2+1/n)}\sigma_1$$

2) H. Herbert, Z. V. D. I. 1910, p. 1387.

or

$$M \infty \sigma_1$$

whichshows a shraight line.

Therefore, this equation does not coincide to the actual data and should not be used for bending of cast iron.

Similarly, in the case of bending of a bar with a circular cross section whose radius is $r$, the moment is shown in the form

$$M=\int_F \sigma\eta dF=\frac{\alpha\pi}{\beta^2}\left\{1-\sqrt{1-\beta^2}\right\}\simeq\frac{\alpha\pi}{2}\left(1+\frac{1}{4}\beta^2\right)$$

where

$$\alpha=\frac{Ecr^4}{2+A^2c^2r^2},$$

and

$$\beta=\frac{A^2c^2r^2}{2+A^2c^2r^2}$$

This is transformed to

$$\sigma_1=\frac{(4M/\pi r^3)}{1+\frac{5}{8}\left(\frac{A}{E}\right)^2\left(\frac{4M}{\pi r^3}\right)^2} \tag{55}$$

For the Torsion of a cast iron bar with the circular cross section, let us use eq. (47) and derive the relation between the twisting moment and the maximum shearing stress in the cross section.

Now, we put $A/2=B$ in eq. (47) and $\gamma=r\theta$ in which $\theta$ stands for the specific angle of torsion. Then, we have

$$M_t=\int_0^{r_1}\tau\cdot 2\pi r^2 dr=\frac{\pi G}{B^2\theta}\left\{r_1^2-\frac{1}{B^2\theta^2}\log\left(1+B^2\theta^2 r_1^2\right)\right\}$$

in which $r_1$ is the radius of the bar.

By developing it in series, we get

$$M_t=\frac{\pi G}{B^2\theta}\left[r_1^2-\frac{1}{B^2\theta^2}\left\{B^2\theta^2 r_1^2-\frac{1}{2}(B^2\theta^2 r_1^2)+\frac{1}{3}(\,..\,)^4-\,..\right\}\right]$$

$$\simeq\pi G r_1^3\gamma_1\left\{\frac{1}{2}-\frac{1}{3}B^2\gamma_1^2\right\} \tag{56}$$

where $\gamma_1=r_1\theta$.

The maximum shearing stress in the cross section is expressed as a function of the twisting moment as follows:

$$\tau_1=\frac{(2M_t/\pi r^3)}{1+\frac{1}{3}\left(\frac{B^2}{G^2}\right)\left(\frac{2M_t}{\pi r_1^3}\right)^2} \tag{57}$$

All these equations denoting the relation between stress and moment have commonly a expression of hyperbolic type.

## 9. Explanation of the relation between stresses and molecular forces by using a model

Internal forces which are produced in a body are attributed in the end to the effect of atomic forces or molecular attraction in the micrographic point of view, but if we compare them to the stress which is based on the macrographic conception, we will find that their relation would not be easily deduced because there are remarkable difference between the strength of materials calculated from the point of view of molecular attraction and that obtained from the actual material testing. We must recognize that, in actual materials, there are many gaps or dislocations besides inhomogeneity.

While, the stress is no other than a force due to molecular attraction and denotes a mean value of forces acting on a small area of a plane passing through a given point in a body.

Therefore, he considers, it is to be determined in connection with molecular forces.

The Poisson's constant is defined ordinarily as a constant which specifies the relation between the lateral strain to the longitudinal one in the uniaxial tension, but this statement describes only a character of material that a lateral strain is associated with its corresponding longitudinal strain and does not give any explanation as to the mechanism with which the former is connected to the latter and then to the internal forces.

In order to make it clear, the present author intends to use a model shown in the following and to explain the mechanism of connexion of strains.

Firstly, let us take a small element arbitrarily cut from a stressed body and call it a particle, though it is not those which are arranged in regular order according to molecular orientation like crystal.

It may be considered that it is a particle of any size and that it is in equilibrium being pulled in all directions by the others surrounding it. Though mutual attraction is of course caused by molecular forces, it is not identical to the latter, but it may be considered to be a resultant of forces. Secondly, we take these paraticles regularly arranged in the directions of principal stresses or let them lie on each of lines of principal stress. Then, from the condition of symmetry, they would situate themselves on all corners of the octahedron shown in Fig. 5.

We specify its corners by denoting signs $i, j$ and $k$. Therefore, $ii$, $jj$ and $kk$ indicate the directions of principal stresses. Let us take the distance between opposite particles as unit of length; i. e. $ii=jj=kk=1$.

As an example, we consider the case of simple tension in which the tensile stress $\sigma_i$ acts in the direction of $ii$ and let the strain in this direction be $\varepsilon$ and the lateral strains in the directions of $jj$ and $kk$ be $\varepsilon'$, then the forces between particles may be generally expressed as the resultant of attractive and repulsive forces by

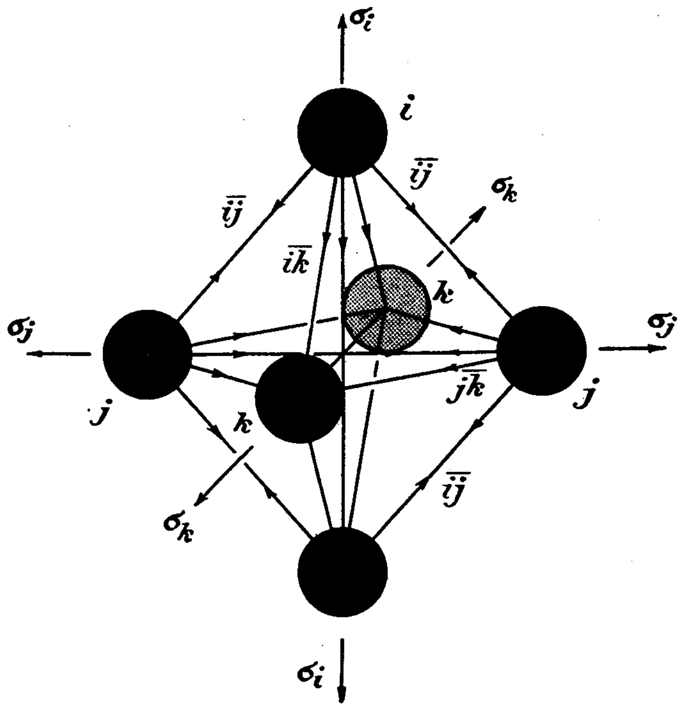


Fig. 5

$$\overline{ii}=\frac{a}{(1+\varepsilon)^n}-\frac{b}{(1+\varepsilon)^m} \tag{58}$$

where the first term in the right hand side is the resultant of attractive forces and the second means that of repulsive forces. If we consider it a molecular force, the index $m$ is generally greater than $n$ and they are presumed for example of a halogen salt $n=2$ and $m \simeq 10 \sim 12$.

As $ii$ is, however, a resultant, but not a molecular force itself, the values of these indices should be considered to differ from them.

From eq. (58), we have

$$\overline{ii}=\frac{a(1+\varepsilon)^{m-n}-b}{(1+\varepsilon)^m}\simeq\frac{(a-b)+c_1\varepsilon+c_2\varepsilon^2+\ldots}{(1+\varepsilon)^m} \tag{59}$$

Because $ii$ must be proportional to $\varepsilon$ in the range of small $\varepsilon$, we may take $a=b$, then the above equation is approximately expressed by

$$\overline{ii}=\frac{c_1\varepsilon}{(1+\varepsilon)^m}\simeq\frac{c_1\varepsilon}{1+m\varepsilon}\ . \tag{60}$$

Owing to the fact that there are many gaps or dislocations in addition to inhomogeneity in the actually existing materials and that thereby the constants $C_1$ and $m$ in this equation should be considered to be of different orders from those for molecules, we rewrite it as follows:

$$\overline{ii}=\frac{C\varepsilon}{1+D\varepsilon} \tag{61}$$

by putting new constants $C$ and $D$ into eq. (60) and consider that these constants should be determined from the results of material-testing.

For the stress-curve of compression which bends toword the direction of negative strain as in cast iron, the sign before $D$ is taken negative in the case of compression.

Similarly, internal forces $\overline{jj}$ and $\overline{kk}$ are described as follows:

$$\overline{jj}=\overline{kk}=\frac{C\varepsilon'}{1+D\varepsilon'} \tag{62}$$

The particle $i$ is not only pulled by the opposite particle $i$, but it is also attracted by all particles surrounding it, though the effect of remote particles upon it may be very slight and negligible. For such case, it will be generally very tremendous that we express it by a simple mathematical equation.

He intends to use a conventional method in order to treat this problem and to determine the magnitude and direction of resultant forces.

As there enter in general two elastic constants into elasticity of an isotropic body, it is convenient to conceive two kinds of resultant forces, which are assumed from the condition of symmetry to be one the normal force $\overline{ii}$, $\overline{jj}$ and $\overline{kk}$ and the other the inclined forces $\overline{ij}$, $\overline{jk}$ and $\overline{ki}$ acting in the direction dividing the intersecting angle of prinapal axes. The distance $ij$ changes its value with lateral strain $\varepsilon'$ as well as longitudinal strain $\varepsilon$ and accordingly the internal force $\overline{ij}$ changes. Let us assume it is of the form similar to eq. (61).

As a part of elongation of $ij$ is, as seen from Fig. 5, $\varepsilon/2\sqrt{2}$ due to the elongation of $ii$ and the length $ij$ is $1/\sqrt{2}$, the strain in this direction becomes $\varepsilon/2$, while the other part of it

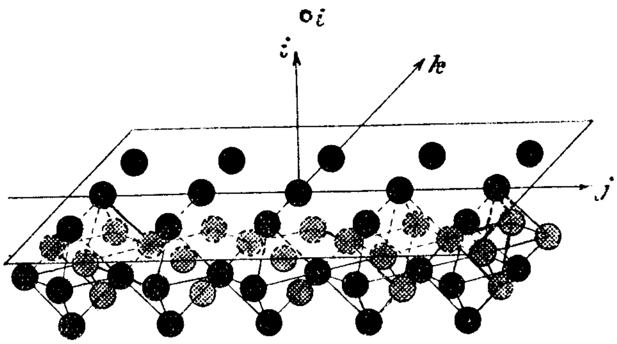

Fig. 6

is similarly caused $\varepsilon'/2$ due to the elongation $\varepsilon'$ of $jj$. Therefore, the internal force between particles $i$ and $j$ is expressed by

$$\overline{ij}=\overline{ik}\simeq\frac{C'(\varepsilon+\varepsilon')/2}{1+D(\varepsilon+\varepsilon')/2} \tag{63}$$

Now, let us assume that they are arranged regularly $n$ particles on a plane whose area is, for example, 1 cm² and consider the condition of equilibrium of forces in the direction normal to the plane, then we have.

$$\overline{ii}+2\overline{ij}\cos\frac{\pi}{4}+2\overline{ik}\cos\frac{\pi}{4}=\frac{\sigma_i}{n} \tag{64}$$

$$\therefore\quad \frac{C\varepsilon}{1+D\varepsilon}+2\sqrt{2}\cdot\frac{C'(\varepsilon+\varepsilon')/2}{1+D(\varepsilon+\varepsilon')/2}=\frac{\sigma_i}{n} \tag{64'}$$

The internal force between $jj$ particles is written similarly as

$$\overline{jj}=\frac{C\varepsilon'}{1+D\varepsilon'} \tag{65}$$

and that of $jk$ is obtained from Fig. 5 as follows:

$$\overline{jk}=\frac{C'\varepsilon'}{1+D\varepsilon'} \tag{66}$$

therefore, the equation of equilibrium of forces in the direction $jj$ is

$$\overline{jj}+2\overline{ji}\cos\frac{\pi}{4}+2\overline{jk}\cos\frac{\pi}{4}=0 \tag{67}$$

since $\sigma_j=0$

$$\therefore\quad \frac{C\varepsilon'}{1+D\varepsilon'}+\sqrt{2}\,\frac{C'(\varepsilon+\varepsilon')/2}{1+D(\varepsilon+\varepsilon')/2}+\sqrt{2}\,\frac{C'\varepsilon'}{1+D\varepsilon'}=0 \tag{67'}$$

For materials which follow the Hooke's law it becomes

$$C\varepsilon'+\sqrt{2}\,C'\cdot\frac{\varepsilon+\varepsilon'}{2}+\sqrt{2}\,C'\varepsilon'=0$$

$$\therefore\quad \left|\frac{\varepsilon'}{\varepsilon}\right|=\frac{C'}{\sqrt{2}\,C+3C'}=\nu=\frac{1}{m}$$

then

$$m=\sqrt{2}\,\frac{C}{C'}+3 \tag{68}$$

If we take $C'=4C$ as a special case, $m$ is equal to 3.354 and $\nu\cong0.3$ which correspond to the value for mild steel.

Next, let us consider the state of pure shear where we can put $\sigma_i=\sigma=-\sigma_j$, $\sigma_k=0$, then

$$\left.\begin{aligned}&\overline{ii}+\sqrt{2}\,\overline{ij}+\sqrt{2}\,\overline{ik}=\frac{\sigma}{n}\\&\overline{jj}+\sqrt{2}\,\overline{jk}+\sqrt{2}\,\overline{ji}=-\frac{\sigma}{n}\\&\overline{kk}+\sqrt{2}\,\overline{ki}+\sqrt{2}\,\overline{kj}=0\end{aligned}\right\}\qquad(69)$$

From the first and second equations of the above (69), we obtain the same relation

$$\varepsilon\left(C+\frac{C'}{\sqrt{2}}\right)=\frac{\sigma}{n}$$

while $\varepsilon=(m+1)\sigma/mE$

$$\therefore\quad E=\frac{n(m+1)}{m}\left(C+\frac{C'}{\sqrt{2}}\right)$$

Using the relation (68), we get

$$\left.\begin{aligned}&E=\frac{n(m+1)(m-2)}{m(m-3)}C,\\&G=\frac{n(m-2)}{2(m-3)}C,\end{aligned}\right\}\qquad(70)$$

Finally, we deal with the general state of stress and let three principal stresses be $\sigma_i, \sigma_j$ and $\sigma_k$, then we have

$$\left.\begin{aligned}&\overline{ii}+\sqrt{2}\,\overline{ij}+\sqrt{2}\,\overline{ik}=\frac{\sigma_i}{n}\\&\sqrt{2}\,\overline{ji}+\overline{jj}+\sqrt{2}\,\overline{jk}=\frac{\sigma_j}{n}\\&\sqrt{2}\,\overline{ki}+\sqrt{2}\,\overline{kj}+\overline{kk}=\frac{\sigma_k}{n}\end{aligned}\right\}\qquad(71)$$

where internal forces between particles are shown

$$\overline{ii}=\frac{C\varepsilon_i}{1+D\varepsilon_i},\quad \overline{jj}=\frac{C\varepsilon_i}{1+D\varepsilon_j},\quad \overline{kk}=\frac{C\varepsilon_k}{1+D\varepsilon_k}$$

and

$$\overline{ij}\simeq\frac{C'(\varepsilon_i+\varepsilon_j)/2}{1+D(\varepsilon_i+\varepsilon_j)/2},\quad \overline{jk}\simeq\frac{C'(\varepsilon_j+\varepsilon_k)/2}{1+D(\varepsilon_j+\varepsilon_k)/2},\quad \overline{ki}\simeq\frac{C'(\varepsilon_k+\varepsilon_i)/2}{1+D(\varepsilon_k+\varepsilon_i)/2}$$

From these equations, we get the following:

$$\left.\begin{aligned}&\varepsilon_i=\frac{\overline{ii}}{C-D\overline{ii}},\quad \varepsilon_j=\frac{\overline{jj}}{C-D\overline{jj}},\quad \varepsilon_k=\frac{\overline{kk}}{C-D\overline{kk}},\\&\text{and}\\&\frac{\varepsilon_i+\varepsilon_j}{2}\simeq\frac{\overline{ij}}{C'-D\overline{ij}},\quad \frac{\varepsilon_j+{}_k\varepsilon}{2}\simeq\frac{\overline{jk}}{C'-D\overline{jk}},\quad \frac{\varepsilon_k+\varepsilon_i}{2}\simeq\frac{\overline{ki}}{C'-D\overline{ki}}\end{aligned}\right\}\qquad(72)$$

or

$$\left.\begin{aligned}\frac{\overline{ii}}{C-D\overline{ii}}+\frac{\overline{jj}}{C-D\overline{jj}}&\simeq 2\frac{\overline{ij}}{G'-D\overline{ij}}\\ \frac{\overline{jj}}{C-D\overline{jj}}+\frac{\overline{kk}}{C-D\overline{kk}}&\simeq 2\frac{\overline{jk}}{C'-D\overline{jk}}\\ \frac{\overline{kk}}{C-D\overline{kk}}+\frac{\overline{ii}}{C-D\overline{ii}}&\simeq 2\frac{\overline{ki}}{C'-D\overline{ki}}\end{aligned}\right\} \tag{73}$$

Six internal forces of particles are expected to be obtained from the above equations (71) and (73), but it is not inevitable that their solution will be of much complicated expression. While, the second term in the denominator of each fraction in eq. (73) is very small and moreover rigorously considered eq. (72) will not strictly hold, because the resisting force due to the change of the direction occurs in ***ij***, ***jk*** and ***ki***.

Considering these effects, the author neglects the insignificant terms in eq. (73) and put simply as follows:

$$\left.\begin{aligned}\frac{1}{C}(\overline{ii}+\overline{jj})&=\frac{2}{C'}\overline{ij}\\ \frac{1}{C}(\overline{jj}+\overline{kk})&=\frac{2}{C'}\overline{jk}\\ \frac{1}{C}(\overline{kk}+\overline{kk})&=\frac{2}{C'}\overline{ki}\end{aligned}\right\} \tag{74}$$

Then, from these simultaneous equations of first order, we can obtain the solution

$$\overline{ii}=\frac{L}{n}\left(\sigma_i-\frac{\sigma_j}{m}-\frac{\sigma_k}{m}\right) \tag{75}$$

in which

$$L=\frac{C(2C+3\sqrt{2}C')}{(C+2\sqrt{2}C')(2C+\sqrt{2}C')} \tag{76}$$

and

$$m=\sqrt{2}\frac{C}{C'}+3 \tag{77}$$

And then,

$$L=\frac{m(m-3)}{(m-2)(m-1)} \tag{78}$$

Accordingly, eq. (75) is written as follows:

$$\overline{ii}=\frac{m(m-3)}{n(m-2)(m+1)}\left\{\sigma_i-\frac{\sigma_j+\sigma_k}{m}\right\} \tag{79}$$

We see that eq. (77) coincides to the foregoing eq. (68).

By using the relations (70) and rewriting $D=A$, we have

$$\left.\begin{aligned}\frac{E\varepsilon_i}{1+A\varepsilon_i}&=\sigma_i-\frac{\sigma_j+\sigma_k}{m}=\sigma_{ei}\\ \therefore\quad \varepsilon_i&=\frac{\sigma_{ei}}{E-A\sigma_{ei}}\end{aligned}\right\}\tag{80}$$

This equation is the same as eq. (20) in which the coefficient $\Phi_i$ is expressible in an equation of second order of $\sigma_{ei}$ and is a generalized form of Cox and Lang's equation.

In order to treat this problem referring to any rectangular coordinate axes, we consider a similar model in a body which is submitted to normal stresses $\sigma_x$, $\sigma_y$ and $\sigma_z$ acting on the planes perpendicular to each of the co-ordinate axes. Then we can express the equations for equilibrium as follows:

$$\left.\begin{aligned}\overline{xx}+\sqrt{2}\,\overline{xy}+\sqrt{2}\,\overline{xz}&=\frac{\sigma_x}{n}\\ \sqrt{2}\,\overline{yx}+\overline{yy}+\sqrt{2}\,\overline{yz}&=\frac{\sigma_y}{n}\\ \sqrt{2}\,\overline{zx}+\sqrt{2}\,\overline{zy}+\overline{zz}&=\frac{\sigma_z}{n}\end{aligned}\right\}\tag{81}$$

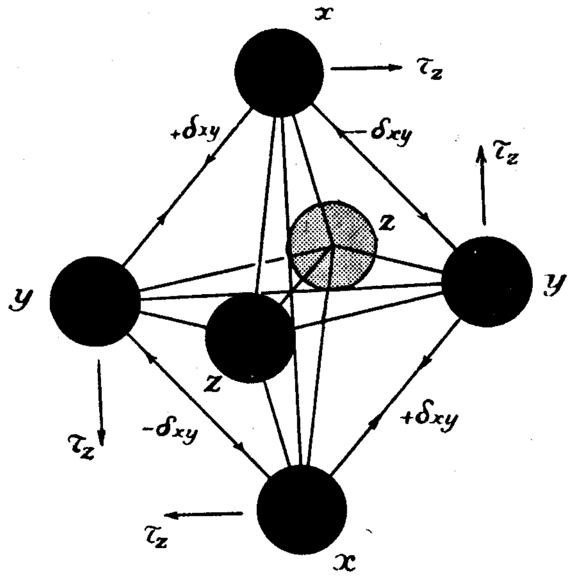

Fig. 7

In addition to these normal stresses, shearing stresses act also on those planes in the general case. They distort the lattice of particles as shown in Fig. 7.

Accordingly, the internal forces in the inclined direction $xy$, $yz$ and $zx$ change their values by the magnitude $\delta_{xy}$, $\delta_{yz}$ and $\delta_{zx}$.

Therefore, the left force in Fig. 7 becomes

and the right

$$\left.\begin{aligned}\overset{\diagup}{xy}&=\overline{xy}+\delta_{xy}\\ \overset{\diagdown}{xy}&=\overline{xy}-\delta_{xy}\end{aligned}\right\}\tag{82}$$

The sum of these forces in $xx$ direction is

$$\frac{1}{\sqrt{2}}(\overset{\diagup}{xy}+\overset{\diagdown}{xy})=\frac{1}{\sqrt{2}}\cdot 2\overline{xy}=\sqrt{2}\,\overline{xy}$$

which does not give no effect on eq. (81). The relation between the shearing stress $\tau_z$ and $\delta_{xy}$ is written as

$$\sqrt{2}\,\delta_{xy}=\frac{\tau_z}{n}$$

$\delta_{xy}$ is the resistance due to change of inclination of $xy$ line in addition to the elongation of it. It may be considered that the part of this resistance due to change of angle is very small when $m$ is equal to 4.

## 10. Conclusion

Regarding to the elasticity of such materials as cast iron in which the relation between stresses and strains can not be expressed by a linear equation, the author has pointed out that the law of superposition should not be applied and established a new theory and thereby derived a relation between stress and strain

$$\varepsilon_i=f_i(\sigma_{ei})$$

$$\sigma_{ei}=\sigma_i-\frac{1}{m}(\sigma_j+\sigma_k)$$

by suitable assumption which he verified from the point of view of intermolecular forces.