On a Graphical Solution for the Forced Vibration of a System with Non-linear Restoring Force

By

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Synopsis

This paper deals with the forced vibration of a single-degree-of-freedom system under simple harmonic force. The author discussed this problem in two cases separately; namely, the one of symmetric restoring fo.ce and the other of unsymmetric one. As the result of studying, the author expounded a graphical solution which is an approximate one but will be very conveniently used to get the steady vibration of this system. In this paper, the author proves by adequate calculations that the graphical solution is practically available for obtaining resonance curves and phase difference curves in the system with any non-linear restoring force.

Generally, the equation of motion in this case is expressed as

$$m\ddot{x}+c\dot{x}+f(x)=P_0 \sin \omega t, \tag{1}$$

where m is the vibrating mass, c the viscous damping coefficient, $f(x)$ the restoring force with any non-linear characteristic and $P_0 \sin \omega t$ the simple harmonic force acting on the vibrating mass, and x means displacement of the mass, t time, $\dot{x}$ and $\ddot{x}$ velocity and accelaration respectively. Dealing with this problem, it is conventional to discuss the solutions in the two cases; that is, the case when $f(x)$ is symmetric as to $x=0$ and the other case $f(x)$ is non-symmetric.

I. Forced Vibration of a System with Symmetric Restoring Force

1. First, we will calculate the case when the characteristic of restoring force is expressed as $f(x)=\alpha x-\beta x^3$. Equation of motion is, then,

$$m\ddot{x}+c\dot{x}+\alpha x-\beta x^3=P_0 \sin \omega t, \tag{2}$$

We assume that displacement is approximately expressed as the following,

$$x=a_1 \sin \omega t+b_1 \cos \omega t, \tag{3}$$

where a_1 and b_1 are unknown constants. Now if we apply (3) to (2) and compare the coefficients of $\sin\omega t$ and $\cos\omega t$ in both sides of equation (2), the following relations are obtained;

$$\left.\begin{aligned}\left\{\alpha - m\omega^2 - \frac{3}{4}\beta(a_1{}^2+b_1{}^2)\right\}a_1 - c\omega b_1 = P_0 \\ \left\{\alpha - m\omega^2 - \frac{3}{4}\beta(a_1{}^2+b_1{}^2)\right\}b_1 - c\omega a_1 = 0\end{aligned}\right\} \quad (4)$$

If we put the displacement $x = x_{max}\sin(\omega t - \varphi)$, then amplitude x_{max} is equal to $\sqrt{a_1{}^2+b_1{}^2}$ and phase difference φ is $\sin^{-1}\frac{-b_1}{\sqrt{a_1{}^2+b_1{}^2}}$. From eq. (4) we get,

$$\alpha\cdot x_{max} - \beta'\cdot x_{max}^3 = m\omega^2\cdot x_{max} \pm\sqrt{P_0{}^2-(c\omega x_{max})^2}\,, \quad (5)$$

$$\sin\varphi = \frac{c\omega\cdot x_{max}}{P_0}\,, \quad (6)$$

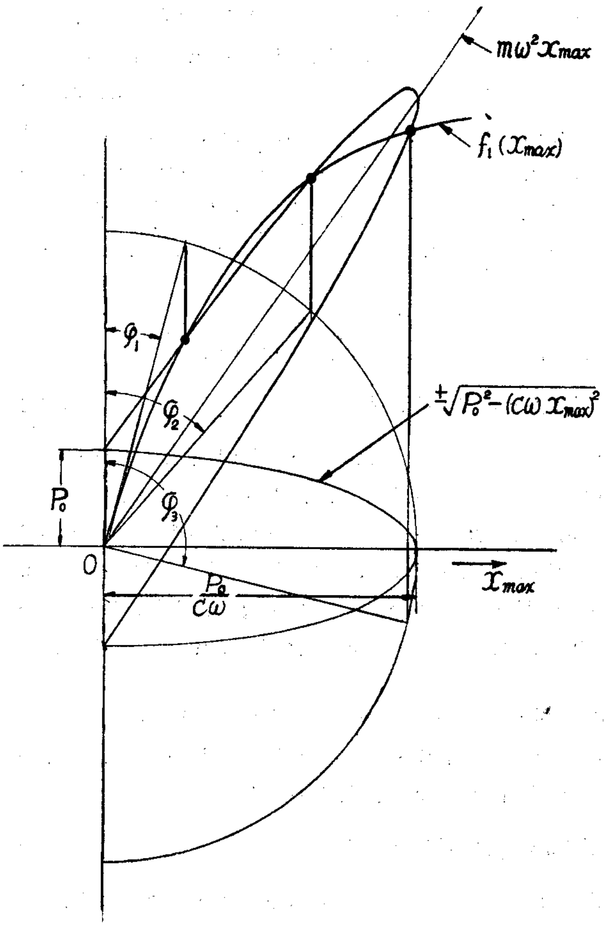

Fig. 1

where $\beta' = \frac{3}{4}\beta$. From eq. (5) and (6) we can calculate x_{max} and φ. A survey of the eq. (5) shows in Fig. 1 on left hand side represents the corrected curve of restoring force $f_1(x_{max})$ which is corrected by $\frac{3}{4}\beta$ from original curve $f(x)$ in this case, and the first term on right hand side represents the straight line passing the origin and making the angle $\tan^{-1} m\omega^2$ with the abscissa; the second term an ellipse with $2P_0$ and $\frac{2P_0}{c\omega}$ as the length of two axis. The required amplitude will be obtained by drawing the composed curve of the latter two, superposing the former on it and finding the cross point of the both curves as shown in Fig. 1. Study of Eq. (6) will tell that the graphical solution as to the value of φ is possible too as shown in the same figure. Thus we obtain a group of the resonance curves of a vibrating system with non-linear restoring force of the form $\alpha x-\beta x^3$ under various values of P_0 as shown Fig. 2. We will now proceed to evaluate the x_{max} and ω, corresponding the point P on each one of those resonance curves where maximum amplitude occurs. If we calculate $\frac{dx_{max}}{d\omega}$ from eq. (5) and put it to zero, we have

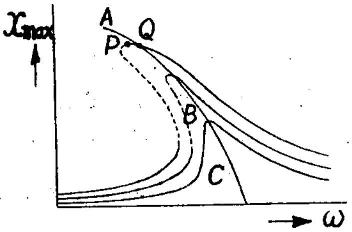

Fig. 2

$$\omega^2=\frac{1}{c^2\cdot x_{max}^2}\left(P_0{}^2-\frac{c^4}{4m^2}x_{max}^2\right). \tag{7}$$

Applying (7) to (5) and thus eliminating ω, we get,

$$x_{max}^2=\frac{\left(\alpha-\frac{c^2}{4m}\right)\pm\sqrt{\left(\alpha-\frac{c^2}{4m}\right)^2-\frac{4mP_0{}^2}{c^2}\beta'}}{2\beta'}. \tag{8}$$

The position of point will be determined by (7) and (8). Next, we will determine the position of point Q where phase difference is $\varphi=\frac{\pi}{2}$. From eq. (6) we get,

$$\omega^2=\frac{P_0{}^2}{c^2\cdot x_{max}^2}. \tag{9}$$

Eliminating ω from (9) and (5), we get,

$$x_{max}^2=\frac{\alpha\pm\sqrt{\alpha^2-\frac{4mP_0{}^2}{c^2}\beta'}}{2\beta'}. \tag{10}$$

Equation (9) and (10) will determine the position of point Q. From eqs. (8) and (10) we see that x_{max} at P is slightly larger than that of Q because $\frac{c^2}{4m}$ is very small compared to α. Similarly from eq. (7) and (9), we see that ω corresponding to point P is smaller, though slightly, than ω corresponding to point Q. Anyhow, the distance between P and Q is so small in either direction that it can actually be considered not to exist, or both points coincide. Let the locus of the point Q be a curve ABC in Fig. 2. At the point $Q, P_0=c\omega x_{max}$, then from eq. (5) the curve ABC is represented by the following formula,

$$\alpha x_{max}-\beta'\cdot x_{max}^3=m\omega^2 x_{max}. \tag{11}$$

Now in the free-oscillation of this system the equation of motion is expressed as,

$$m\ddot{x}_n+\alpha x_n-\beta x_n^3=0. \tag{12}$$

The suffix n of x in the above equation means the variable is the one in the case of free-oscillation and not of forced vibration. We assume that the displacement x_n is approximately expressed as,

$$x_n=a_{1n}\sin\omega_n t+b_{1n}\cos\omega_n t\equiv x_{nmax}\sin(\omega_n t-\varphi_n). \tag{13}$$

Then the relation between x_{nmax} and ω_n becomes,

$$\alpha\cdot x_{nmax}-\beta' x_{nmax}^3=m\omega_n^2\cdot x_{nmax}. \tag{11a}$$

This equation identically equals the equation (11).

2. Next we will discuss the general case of any non-linear restoring force $f(x)$. If the left hand side of eq. (5) could be obtained in any way, we shall be able to apply the above mentioned graphical solution in this case. We put the left hand side $f_1(x_{max})$, then equations (5), (11) and (11a) become,

$$f_1(x_{max})=m\omega^2 x_{max}\pm\sqrt{P_0{}^2-(c\omega x_{max})^2}, \tag{5a}$$

$$f_1(x_{max})=m\omega^2 x_{max}, \tag{14}$$

$$f_1(x_{nmax})=m\omega_n^2 x_{nmax}. \tag{14a}$$

In the free-oscillation of a system with any non-linear restoring force $f(x)$, the relation between maximum amplitude x_{nmax} and circular frequency ω_n $\left(\equiv\frac{2\pi}{T}\right)$ can be obtained by the formula [1],

$$T=\sqrt{8m}\int_0^{x_{nmax}}\frac{dx_n}{\sqrt{\int_{x_n}^{x_{nmax}}f(x_n)dx_n}} \tag{15}$$

We can calculate exactly the double integration of (15) in some cases, and in other cases we can calculate approximately at least. From this relation we can obtain graphically $f_1(x_{max})$ as shown in Fig. 4, then the graphical solution of eq. (5) and (5a) can be operated. We shall now discuss the actual procedure of the graphical solution. Firstly we find the relation between ω_n and x_{nmax} by integrating (15) in some way, then draw a curve of this relation as illustrated in Fig. 3. Take a point P arbitrary on the curve and let its coordinate be ω_{n0} and x_{n0max}. Then, on a diagram like the one shown in Fig. 4, locate the cross point P of the perpendicular line $x_{max}=x_{n0max}$ and the straight line $m\omega_{n0}^2 x_{max}$

(1) Timoshenko, Vibration Problems in Engineering, 1937, 120.

which pass the origin. Carry out the same process on each point on the curve and the one of $f_1(x_{max})$ as shown in Fig. 4 is acquired. We shall call this curve "corrected characteristic curve of the restoring force". Making use of this $f_1(x_{max})$ we can get resonance curve and phase difference curve by the graphical solution of (5) and (5a). This is what is called the method by free-oscillation and is the one devised by the author. This method provides closer solution to real value than the Appleton's[(2)], because the harmonic terms of higher order of the displacement x_n are taken into calculation in the course of obtaining the curve ABC. Besides it does not necessitate any restrictions on the given curve of restoring force, extending the category of the problems to which the method is to be applied. Only difficult point in this method is the procedure to acquire the relation between the frequency of free-oscillation and the displacement. But then, the following three steps will help considerably to surmount the difficulty.

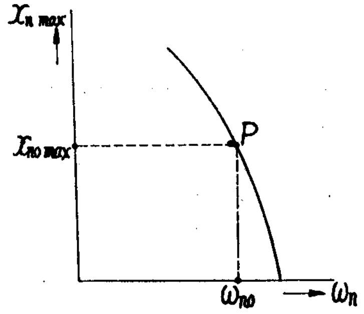

Fig. 3

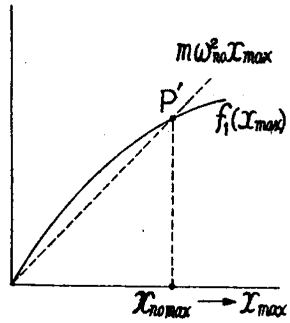

Fig. 4

(1) If $f(x)$ is given in a simple formula, integrate twice the equation (15). In some cases the elliptic integral is useful for the purpose.

(2) If the curve of $f(x)$ is decomposable into several straight lines, use Klotter's[(3)] method which is handy for such cases.

(3) If $f(x)$ is given in complicated curve that could not be expressed in analytical form, integrate graphically the equation (15). Far more trouble is inevitable in such cases than the above two.

II. Forced Vibration of a System with Non-symmetric Restoring Force

1. First, we will calculate the case when the characteristic of restoring force is expressed as $f(x)=\alpha x+\beta x^2$. Equation of motion is then,

$$m\ddot{x}+c\dot{x}+\alpha x+\beta x^2=P_0 \sin \omega t. \tag{16}$$

We assume that displacement is approximately expressed as the following,

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(2) E. V. Appleton, Phil. Mag. 1924, 47, 609.

(3) K. Klotter, Ingenieur-Archiv, 1936, April, 87.

$$x=b_0+a_1\sin\omega t+b_1\cos\omega t, \tag{17}$$

where b_0, a_1 and b_1 are unknown constants. Now if we apply (17) to (16) and compare the coefficients of $\sin\omega t$, $\cos\omega t$ and constant coefficients in both side of eq. (16), the following relations are obtained;

$$\beta b_0{}^2+\alpha b_0+\frac{\beta}{2}(a_1{}^2+b_1{}^2)=0, \tag{18}$$

$$(\alpha-m\omega^2+2\beta b_0)a_1-c\omega b_1=P_0, \tag{19}$$

$$c\omega a_1+(\alpha-m\omega^2+2\beta b_0)b_1=0. \tag{20}$$

From these equations b_0, a_1, b_1 can be evaluated.
If we put the displacement to

$$x=b_0+x_{max}\sin(\omega t-\varphi), \tag{17a}$$

from eq. (19) and (20), we get,

$$x_{max}(\alpha-m\omega^2+2\beta b_0)=\pm\sqrt{P_0{}^2-(c\omega x_{max})^2}\,. \tag{21}$$

While from (18), it follows,

$$b_0=\frac{-\alpha\pm\sqrt{\alpha^2-2\beta^2(a_1{}^2+b_1{}^2)}}{2\beta}.$$

Double sign in the above must be plus (+), because $x=-\dfrac{\alpha}{2\beta}$ means the point where the change of restoring force relative to x is zero. Therefore,

$$\begin{aligned} b_0&=\frac{-\alpha+\sqrt{\alpha^2-2\beta^2\cdot x_{max}}}{2\beta}\\ &=-\frac{\beta}{2\alpha}\cdot x_{max}^2-\frac{\beta^3}{4\alpha^3}x_{max}^4-\frac{\beta^5}{4\alpha^5}x_{max}^6-\cdots\cdots. \end{aligned} \tag{22}$$

Replacing the b_0 in (21) by (22),

$$\begin{aligned} &\alpha\cdot x_{max}-\frac{\beta^2}{\alpha}\cdot x_{max}^3-\frac{\beta^4}{2\alpha^3}\cdot x_{max}^5-\frac{\beta^6}{2\alpha^5}\cdot x_{max}-\cdots\cdots\\ &\quad=m\omega^2x_{max}\pm\sqrt{P_0{}^2-(c\omega x_{max})^2}\,. \end{aligned} \tag{23}$$

This equation makes it possible to evaluate x_{max} just like eqs. (5) and (5a). That is, the graphical solution shown in Fig. 1 will be similarly used. Corrected characteristic curve of the restoring force therefore given by the following equation,

$$\begin{aligned} f_1(x_{max})&=\alpha x_{max}-\frac{\beta^2}{\alpha}x_{max}^3-\frac{\beta^4}{2\alpha^3}x_{max}^5-\frac{\beta^6}{2\alpha^5}x_{max}^7-\cdots\cdots\\ &=\alpha x_{max}\left(1-\frac{2\beta^2}{\alpha^2}x_{max}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{24}$$

b_0 is evaluated by the following equation if x_{max} is known,

$$b_0 = -\frac{\alpha}{2\beta} + \frac{f_1(x_{max})}{x_{max}}. \tag{25}$$

Lastly the phase difference φ will be calculated by the following,

$$\sin\varphi = -\frac{b_1}{\sqrt{a_1^2 + b_1^2}} = \frac{c\omega x_{max}}{P_0}. \tag{26}$$

This corresponds to equation (6), and also φ is obtained by graphical solution shown in Fig. 1.

2. Next we will discuss the general case of any non-symmetric restoring force $f(x)$. In the treatment of a forced vibration of a system with non-symmetric restoring force, major problem is how the maximum displacement on positive side x_1 and negative side x_2 vary according to the change of ω of the external force $P_0 \sin \omega t$. Generally x_1 and x_2 do not assume the same value and naturally the center of the vibration-displacement is different from the origin. As we have pointed out in above discussion, x can be expressed as (17a) for practical use, and this is represented in Fig. 5. Therefore,

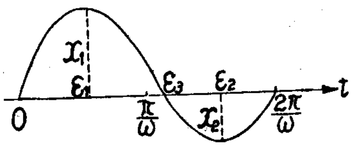

Fig. 5

$$\left.\begin{aligned} b_0 &= \frac{1}{2}(x_1 - x_2) \\ x_{max} &= \frac{1}{2}(x_1 + x_2) \end{aligned}\right\}. \tag{27}$$

These relations hold not only in forced vibrations but also in free vibrations. As stated in above discussion, the amplitude x_{max} in eq. (17a) is calculated by (23) and with this x_{max}, b_0 is obtained from eq. (25). As the generalization this procedure, the author suggests the following method of calculating x_{max} and b_0 in the problem of the forced vibration with any non-symmetric restoring force.

As the first step, replace the non-symmetric restoring force by a symmetric one by some way or other and calculate x_{max} in this supposed system of vibration. In the second step, calculate b_0 as a factor arising from non-symmetric characteristics. The resonance curve of the vibrating system with the symmetric restoring force will be represented as in Fig. 2, according to the magnitude of the external force. The curve enveloping the tip of each curve coincide with the curve ABC, or the free-oscillation curve, as was already explained in I. This fact suggests that the forced vibration of the system with symmetric restoring force, after all, depends upon the free-oscillation. It is easily seen, too, that in the system with non-symmetric restoring force, the situation is the same as above.

Let us explain more in detail. From a given characteristics of non-symmetric force, the value of x_{nmax} for free-oscillation will be acquired as shown in Fig. 3. From the figure we obtain the $f_1(x_{max})$ as shown in Fig. 4 in the same way as in the symmetric vibration. We adopt this new curve $f_1(x_{max})$ as the hypothetical symmetric corrected restoring force. Application of the graphical solution as shown in Fig. 1 will bring about the resonance curve as shown in Fig. 6. In this figure the curve DGH is the one corresponding to ABC in Fig. 2.

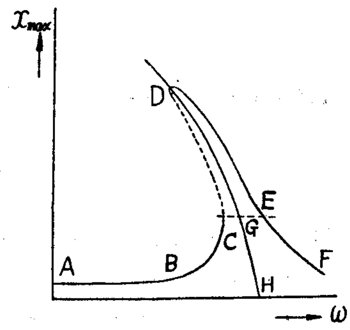

Fig. 6

Next step to take is to evaluate b_0. Now suppose $f(x)$ is given by $\alpha x+\beta x^2$. Then the equation of motion will be eq. (16). Assume that,

$$x=b_0+a_1\sin\omega t+b_1\cos\omega t+a_2\sin 2\omega t+b_2\cos 2\omega t+\cdots\cdots. \tag{17b}$$

Applying (17b) to (16) and carrying out the same step as we did before, we have the following;

$$\left.\begin{aligned}&\beta b_0^2+\alpha b_0+\frac{1}{2}\beta S=0,\\ \text{where}\quad &S=\sum_{i=1,2,\ldots}^{\infty}(a_i^2+b_i^2).\end{aligned}\right\} \tag{28}$$

Therefore,

$$b_0=\frac{-\alpha\pm\sqrt{\alpha^2-2\beta^2 S}}{2\beta}. \tag{29}$$

In the double sign plus must be taken because $x=-\frac{\alpha}{2\beta}$ represents the point where the restoring force's relative change as to x is zero.

In the equation (16), we now consider the case when $P_0=0$ and $c=0$; that is, the case of free-oscillation. In order to prevent confusion, we shall put suffix n to the variables for this case. The equation of motion will be taken, neglecting damping force, as

$$m\ddot{x}_n+\alpha x_n+\beta x_n^2=0. \tag{30}$$

Let,

$$x_n=b_{n0}+a_{n1}\sin\omega_n t+b_{n1}\cos\omega_n t+a_{n2}\sin 2\omega_n t+b_{n2}\cos 2\omega_u t+\cdots\cdots, \tag{31}$$

where ω_n is the circular frequency of fundamental natural oscillation. Replacing x_n in (30) by (31) as before, we have,

$$\left.\begin{aligned} b_{n0} &= \frac{-\alpha \pm \sqrt{\alpha^2 - 2\beta^2 S_n}}{2\beta}, \\ \text{where} \qquad S_n &= \sum_{i=1}^{\infty} (a_{ni}^2 + b_{ni}^2). \end{aligned}\right\} \tag{32}$$

Double sign must be plus. Therefore if,

$$S = S_n, \tag{33}$$

we have from (29) and (32),

$$b_0 = b_{n0}.$$

As it is already clear, $a_2, a_3, a_4,$ ...... and $b_2, b_3, b_4,$ ...... are smaller than a_1 and b_1. As $a_{n2}, a_{n3},$ ...... and $b_{n2}, b_{n3},$ ...... are smaller than a_{n1} and b_{n1} in the same way. Therefore the condition (33) is actually the same as the following,

$$a_1{}^2 + b_1{}^2 = a_{n1}^2 + b_{n1}^2 .$$

This means that if the amplitude x_{max} is the same in both the free and forced vibrations, the values of b_0, or the deviation of the center of the vibration from the origin, will also be the same in both vibrations. In Fig. 6, b_0 at point E and G are the same. Therefore the relation between x_{max} and b_0 in free-oscillation remains as it is in the forced vibration. Thus the value of x_{max} and b_0 of the forced vibration of the system with non-symmetric restoring force will be determined. In using the above method, the two curves of the free-oscillation under non-symmetric restoring force, that is, the one is the curve in Fig. 3 which shows the relation between x_{nmax} and ω_n, and the other is one in Fig. 7 which shows the relation between b_{n0} and x_{nmax}. We shall now discuss for a while about the method to acquire the required curues.

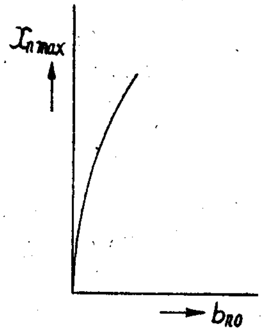

Fig. 7

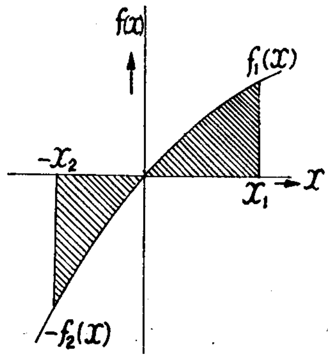

Fig. 8

Let the non-symmetric restoring force be $f_1(x)$ when $x>0$, and $-f_2(x)$ when $x<0$, as illustrated in Fig. 8. x_1 and x_2 indicate the maximum amplitudes on positive and negative sides respectively. If we assume the displacement curve to be as shown in Fig. 5, then $\dot{x}=0$ at $t=\varepsilon_1$, and $t=\varepsilon_2$. Therefore the kinetic energy is zero here. Potential energy assumes the following two forms respectively,

$$\int_0^{x_1} f_1(x)dx, \qquad \int_0^{x_2} f_2(x)dx.$$

At $t=\varepsilon_3$ the potential energy is zero and the kinetic energy is $\frac{m}{2}(\dot{x})^2_{x=0}$. Therefore if the system is without damping force, from the law of energy conservation,

$$\int_0^{x_1} f_1(x)dx=\frac{1}{2}m(\dot{x})^2_{x=0}=\int_0^{x_2} f_2(x)dx. \tag{34}$$

The above relation is illustrated in Fig. 8. That is, the hatched parts for $x>0$ and $x<0$ have the same area. Thus given $f_1(x)$ and $f_2(x)$, the relation between x_1 and x_2 is decided by equation (34) and the relation between x_{max} and b_0 by eq. (27).

Next, at $x>0$ and $x<0$ the following equations of motion are established respectively,

$$m\ddot{x}+f_1(x)=0, \tag{35}$$

$$m\ddot{x}-f_2(x)=0. \tag{36}$$

Now let the time required for x to attain $x=x_1$ and $x=-x_2$ from $x=0$ be respectively t_1 and t_2.

These t's can be calculated easily by the following two double integrals.

$$t_1=\int_0^{x_1}\frac{dx}{\sqrt{\frac{2}{m}\int_x^{x_1} f_1(x)dx}}, \tag{37}$$

$$t_2=\int_0^{x_2}\frac{dx}{\sqrt{\frac{2}{m}\int_x^{x_2} f_2(x)dx}}. \tag{38}$$

Fundamental circular frequency ω_n of the free-oscillation will be then,

$$\omega_n=\frac{\pi}{t_1+t_2}. \tag{39}$$

Relation between x_{max} and ω_n will be obtained by (37) to (39). ω_n is generally a function of amplitude except when $f_1(x)$ and $f_2(x)$ are linear. Integration of (37) and (38) is not generally possible unless $f_1(x)$ and $f_2(x)$ have some particular forms. However, graphical integration will give the same result as the numerical calculation when the latter is not possible. And if the curve of $f(x)$ is decomposable into several straight lines, Klotter's method is useful.

The above mentioned method is what is called the method by free-oscillation and is the one devised by the author.