New Graphical Solution of Stress in the Ground under Foundations.

By

Shinichiro MATSUO

(Received September, 1950)

Synopsis

In this article the author first discusses the graphical solutions of obtaining stresses in the ground under foundations proposed hitherto and then proposes 3 new simplified graphical methods by improving the methods of Burmister and Newmark and by transforming the scales in compliance with the coefficient of concentration according to Fröhlich, making practical application easier.

1. Introduction

In designing a structure, the stresses of the structure above the foundation can generally be obtained by statical computation. The stresses in the ground under foundations, however, can neither be obtained with ease nor with great accuracy because the mechanical properties of the ground are very complicated.

It can be said that the design of a structure is complete only after the stresses in the ground under the foundation are solved. Therefore, the computation of stresses in the ground is of great importance.

Stresses in the ground under foundations are generally complicated and are not yet thoroughly comprehendede. The analytical method proposed hitherto is derived from the theory of elasticity, but the integration is not always easy for any shape of the loading surface. Therefore graphical methods such as methods of D. M. Burmister [1] and N. M. Newmark [2] have been attempted for the solution. Recognizing the fact that the measured values are larger than the computed values near the vertical axis as natural ground foundations do not always show

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1) Burmister, D. M., "Graphical distribution of vertical pressure beneath foundations", Proc. A. S. C. E. Vol. 63, 1937.

2) Newmark, N. M., "Graphical procedure for computing vertical pressures", Univ. Illionis Eng. Exp. Sta., circ, 24, 1935.

"graphical procedure for computing vertical pressure (mimeographed)", Univ. Illinois, Urbana, III, 1937.

perfect elasticity, O. K. Fröhlich[3] modified the stress distribution and endeavoured to propose theory adaptable to natural ground foundations. Integration involved in this analytical method, however, is not always simple for all possible cases. Therefore, as a method of solution in this case, a graphical method has been proposed by D. P. Krynine.[4]

In the following, these praphical methods are discussed and 3 new simplified methods are proposed.

In this article only the 3-dimensional case is explained, the 2-dimensional case being included as a special case.

2. Discussion on the methods proposed hitherto.

(1) D. M. Burmister's Method.

A turbulent zone, so called by Kögler and Scheidig,[5] is formed just below the surface of the ground under a load. It has been stressed in the past that the stress in this turbulent zone must be considered strictly different from the stress occurring in deeper zone and that it cannot be obtained accurately by considering this zone as an elastic body. Therefore, Boussinesq's theory, and consequently, Burmister's method cannot be applied in this zone.

With the opinion, however, that the elastic theory holds good at a depth greater than 2/3 the maximum width of the foundation in the ground, D. M. Burmister has proposed a graphical solution based upon the elastic theory.

The defects of this method are that diagrams expressing the stresses at various depths must be prepared beforehand, or the stresses must be obtained by interpolation from the diagram of two points of approximately equal depth. It is generally difficult to read the value directly from the plan of the load on the diagram and a simple supplementary construction must be done. However, when the scale of the plan is usually constant, this method can be said to be more convenient than Newmakr's method which will be explained later. In this case it would be convenient if the diagram were constructed on tracing paper.

(2) N. M. Newmark's Method.

The range of application is the same as (1), being limited to the elastic zone.

In Burmister's Method, a number of concentric circles, the radii changing a unit length, are drawn on the plan with the point at which the stress is to be obtained as the center and very complicated curves are obtained by connecting

3) Fröhlich, O. K., "Druckverteilung im Baugrunde", Wien, 1934.

4) Krynine, D. P., "Pressures beneath a spread foundation", Proc. A. S. C. E., Vol. 63, 1937.

5) Kögler, F. u. Scheidig, A., "Druckverteilung im Baugrund" Bautechnik, Bd. 6 Heft 17, 1928.

the points which influences equally to the respective circular strips. Moreover, the calculation required for drawing the diagram must be done at each depth. With Newmark's Method, however, the diagram is simple because the sectors obtained by dividing the circumference equally into 100 parts are so divided that they all have the same degree of influence. Furthermore, as the calculation and the construction are made with the depth of the point at which the stress is to be obtained taken as a standard length, the calculation is required only once and just one diagram is necessary. These points can be considered as being more convenient in comparison with the former method.

There is the inconvenience of changing the scale of the plan of load with the change in the depth. It is somewhat difficult to judge the percentage of the influenece plane when the load comes partially on the influence plane, but this is not such a great defect because great accuracy is not required in view of the nature of the stress.

Although based upon the same theory, the former method obtained the change in the degree of influence on the circumferences of the concentric circles and the latter method, in the radial direction with the result that the above merits and defects on the construction and mumber of diagrams and the calculation of the two methods were revealed. In comparing the two methods it can be said that the latter is the beter method.

Newmark's Method is related in the publications of Krynine[6] (1941) and Terzaghi[7] (1948).

(3) D. P. Krynine's Method.

The above two methods being graphical methods based upon the elastic theory, there is an certain limit to the range of application as has already been stated.

Krynine's Method is a graphical method developed from Fröhlich's theory in order to improve this point. This method has a great advantage over the former two in that the results obtained are more rational than these obtained from the theory of elasticity because the coefficient of concentration is considered in compliance with the characteristics of the foundation. With this method, however, the construction must be repeated at every point where the stress is to be obtained which means a great deal of labour.

Judging from the present situation of soil mechanics, it is difficult to compute the stresses in the foundation with great accuracy. If the possible maximum and minimum stresses are to be taken, then it is not very practical to repeat the

6) Krynine, D. P., "Soil Mechanics", N. Y. 1941.
7) Terzaghi, K. & Peck, R. B., "Soil Mechanics", N. Y., 1948.

above construction at all the points. Therefore it is desirable to obthain stress at any point in the foundation in any case under any loading easily by means of a simplified method. If this were possible, the interest of designers towards stresses in the foundation would be roused, resulting in studies in this field making a great progress.

With the above opinion the author proposes the following method.

3. New graphical Solutions.

(1) Method 1.

The following solution is obtained when D. M. Burmister's graphical solution is improved to O. K. Fröhlich's theory.

Radial stress σ_r at any point (expressed as ρ, φ by the polar coordinate, and x, z by the rectangular coordinate) in the foundation by a concentrated load P acting on the surface is as follows.

$$\sigma_r = \frac{\nu D}{2\pi\rho^2} \cos^{\nu-2} \varphi \qquad (1)$$

Vertical stress at this point is

$$\sigma_z = \sigma_r \cdot \cos^2\varphi = \frac{\nu P}{2\pi\rho^2} \cos^{\nu} \varphi \qquad (2)$$

$$= \frac{\nu P}{2\pi\rho^2\left[1+\left(\frac{x}{z}\right)^2\right]^{\nu/2}} \qquad (3)$$

where ν is the coef. of concentration.

Next, when the load is a uniform circular load (intensity: p; radius: r_0) and the reaction at the bottom surface is uniform, then the vertical stress $\bar{\sigma}_z$ appearing at pt. A located at a depth Z along the vertical axis passing through the center of the load can be obtained by integrating eq. (3)

$$\bar{\sigma}_z = \int_0^{\varphi_0} \frac{\nu(2\pi r)dr}{2\pi\rho^2}\, p\cdot\cos^{\nu}\varphi = p[1-\cos^{\nu}\varphi] \qquad (4)$$

$$= p\left\{1 - \frac{1}{\left[1+\left(\frac{r_0}{z}\right)^2\right]^{\nu/2}}\right\} \qquad (1)$$

Where φ_0 is the angle between the line connecting pt. A at which the stress is to be obtained and the end of the load and the vertical axis passing through the center.

Stress $\Delta\bar{\sigma}_z$ due to load p acting between two concentric circles with radii of r_1 and r_2 is

$$\Delta\bar{\sigma}_z = p\left\{\frac{1}{\left[1+\left(\frac{r_1}{z}\right)^2\right]^{\nu/2}} - \frac{1}{\left[1+\left(\frac{r_2}{z}\right)^2\right]^{\nu/2}}\right\} = p(K_1-K_2) = p\cdot\Delta K \qquad (6)$$

The results of computing ΔK taking $z=10$ m, $r_1=0, 1, 2, \cdots 24$ m and $r_2=1, 2, \cdots 25$ m are shown in Table 1 in which α^0 the arcangle receiving 0.001 the influence of the stress in given

$$\alpha_0=\frac{360^0\times 0.001}{\Delta K} \qquad (7)$$

The curve corresponding to $\sigma_z=0.001$ t/m^2 can be determined when the points of intersections between the arcs and their respective α_0 are obtained and connected. Curves corresponding to $\sigma_z=0.01$, 0.1 t/m^2 can be obtained by taking $10\,\alpha^0$ and $100\,\alpha_0$. In this manner the equal stress line for the case when a load of 1 t/m^2 acts can be obtained.

In Fig 1, the real line represents the case for $z=10$ m, $\nu=3$ and the dotted line

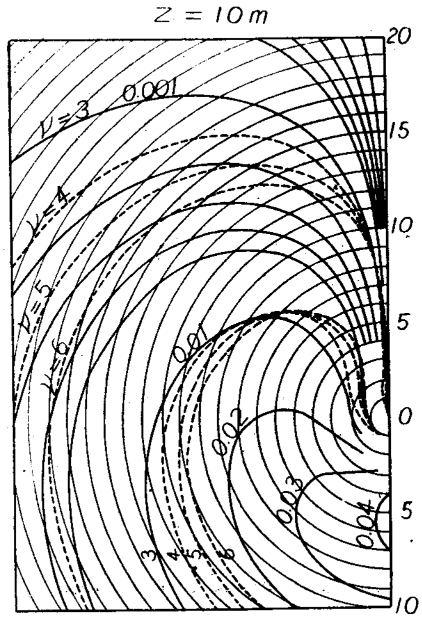

Fig. 1

Table 1

r	ν = 3		ν = 4		ν = 5		ν = 6	
	ΔK	α	ΔK	α	ΔK	α	ΔK	α
1	0.014925	24.12	0.019850	18.14	0.024751	14.54	0.029627	12.15
2	0.042153	8.54	0.055521	6.48	0.068558	5.25	0.081272	4.43
3	0.064255	5.60	0.083042	4.34	0.100619	3.58	0.117045	3.07
4	0.078196	4.60	0.098350	3.66	0.115976	3.10	0.131302	2.74
5	0.084995	4.23	0.103315	3.48	0.117750	3.06	0.128848	2.79
6	0.084951	4.23	0.099247	3.63	0.108717	3.31	0.114344	3.15
7	0.080769	4.45	0.090315	3.99	0.094694	3.80	0.095330	3.78
8	0.073559	4.89	0.078498	4.59	0.078548	4.58	0.075469	4.77
9	0.065528	5.49	0.066612	5.40	0.063495	5.67	0.058114	6.19
10	0.057126	6.30	0.055260	6.51	0.050124	7.18	0.043656	8.25
11	0.049129	7.33	0.045211	7.96	0.039013	9.23	0.032325	11.14
12	0.042024	8.57	0.036797	9.78	0.030213	11.92	0.023820	15.11
13	0.035707	10.08	0.029773	12.09	0.023276	15.47	0.017471	20.61
14	0.030357	11.86	0.024104	14.94	0.017948	20.06	0.012833	28.05
15	0.025618	14.05	0.019413	18.54	0.013792	26.10	0.009408	38.16
16	0.021803	16.51	0.015772	22.83	0.010698	33.64	0.006967	51.67
17	0.018531	19.38	0.012846	28.02	0.008328	43.25	0.005184	69.44
18	0.015792	22.80	0.010457	34.43	0.006492	55.45	0.003870	93.02
19	0.013533	26.60	0.008582	41.95	0.005104	70.53	0.002914	123.54
20	0.011544	31.19	0.007031	51.20	0.004014	89.69	0.002201	163.56
21	0.010003	35.99	0.005848	6.155	0.003206	112.29	0.001687	213.40
22	0.008580	41.96	0.004829	74.55	0.002548	141.29	0.001291	278.85
23	0.007478	48.14	0.004052	83.86	0.002059	174.84	0.001004	278.57
24	0.006490	55.47	0.003390	106.19	0.001660	216.87	0.000781	460.95
25	0.005560	63.60	0.002853	126.18	0.001349	266.86	0.000612	588.24

shows the case of $\nu=4$, 5, 6 for 0.001, 0.015.

The characteristics of this methsd can be considered as σ_z is obtainable not only in the case of Burimister's elasticity ($\nu=3$) but in any case by changing the value of ν and no difficult construction is required at each point as is the case with Krynine's method.

(2) Method 2

The following equation is derived from eq. (5).

$$\frac{\sigma_z}{p} = 1 - \frac{1}{\left[1 + \left(\frac{r_0}{z}\right)^2\right]^{\nu/2}} \qquad (5')$$

The relation between $\frac{\sigma_z}{p}$ and $\frac{r}{z}$ is as shown in Table 2.

Table 2

r/z	ν=3	ν=4	ν=5	ν=6
0.1	0.270	0.234	0.207	0.189
0.2	0.401	0.343	0.305	0.227
0.3	0.518	0.440	0.390	0.356
0.4	0.637	0.538	0.476	0.431
0.5	0.766	0.643	0.565	0.503
0.6	0.918	0.763	0.665	0.559
0.7	1.110	0.909	0.785	0.704
0.8	1.387	1.112	0.965	0.844
0.9	1.908	1.470	1.229	1.075

Concentric circles are drawn, the radii being the various values of corresponding to $\frac{\sigma_z}{p}$=0.1, 0.2, ……. Then when radial lines are drawn so as to divide the circumference into 100 equal parts, Fig. 2 is obtained. Each unit area surrounded by these lines is an influence plane of 0.001. The plan of the load is drawn with a scale corresponding to this figure. Therefore the characteristic of this method is that the value of σ_z at any depth can be obtained by this one diagram.

With this improvement the stress when the phenomenon of the concentration of stresses in the founation is considered can be obtained from the diagram which Newmark obtained assuming the foundation as an elastic body without doing the

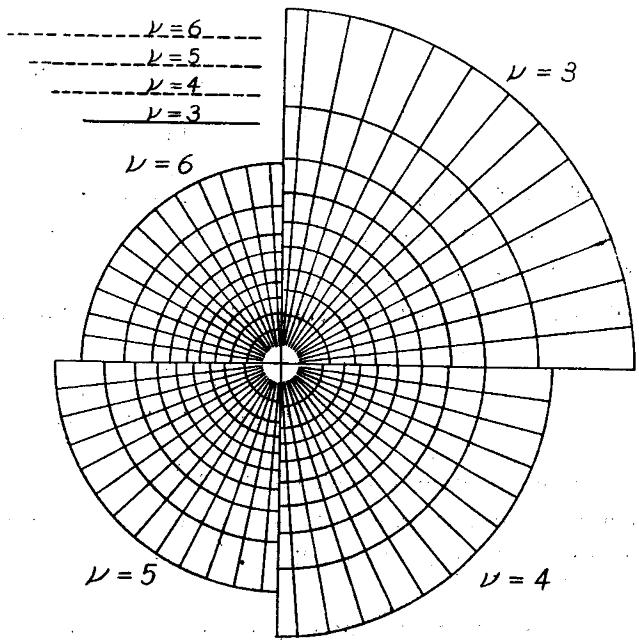

Fig. 2

difficult graphical construction required in Krynine's Method.

In Fig 2, the cases for $\nu=3$, $\nu=4$, $\nu=5$, $\nu=6$ are shown at the upper right, lower right, lower left and upper left corners respectively.

(3) Method 3

σ_z is obtained by employing the equation for elastic bodies, only the depth being computed according to the nature of the soil.

If the stress distribulion at a depth of Z in the ground is tne same as that at a depth of h in an elastic body, then

$$dh = \psi dZ \tag{8}$$

where ψ is a function of depth Z.

Converted depth $$h = \int_0^z \psi dZ \tag{9}$$

To make it simple, if ψ is taken as constant, then

$$h = \psi Z \tag{10}$$

If it is repeated until the stress distribution corresponding to the coefficient of concentration $\nu=4, 5, 6\cdots\cdots$ and that of a certain depth in the elastic body become approximately the same, then the next relation is obtained.

$$\psi = \sqrt{\frac{3}{\nu}}$$

ν	3	4	5	6	…
ψ	1.00	1.15	1.29	1.41	…

When this fact[8] is used it becomes possible to consider the coefficient of concentration in the graphical solution based upon the elastic theory. For instance, the ordinary line part in Fig. 1 corresponds to $\nu=3$, $Z=10$ m, but if the above relation is employed, it becomes the figure for the following.

ν	4	5	6	…
Z	11.5m	12.9m	14.1m	…

If this relation is applied to the figure at the upper right of Fig. 2 corresponding $\nu=3$, the unit length of the scale gradually becomes longer as shown in the upper left figure as the coefficient of concentration increases.

8) Newmark, N. M., "Discussion for distribution of stresses under a foundation by A. E. Cummings", Trans. A. S. C. E. Vol. 101, 1936.

Furthermore, the research of Burmister, Newmark and Krynine obscurely considers only the mechanically homogeneous layer, but although a problem for future study it seems useful also in the case when there is a mechanical change in the vertical direction, for instance in the solution of the case when clay, loam, sand etc. are found in layers.

4. Conclusion.

When the former methods and the author's methods are compared, the results are as shown in Table 3. It is noticed that the methods of Burmister and Newmark based upon the elastic theory are developed and improved and the laboursome construction required in Krynine's method is omitted. Also the clue for the solution of stresses for the case found in layers was obtained from method 3.

Table 3

<table>
<tr><th colspan="2">Fundamental Theory</th><th>Boussinesq</th><th>Fröhlich</th><th colspan="2">Remarks</th></tr>
<tr><th colspan="2">Coefficient of Concentration</th><th>3</th><th>3,4,5,6,···</th><th>Construction</th><th>No. of Deagrams</th></tr>
<tr><td rowspan="4">Nature of Foundation</td><td rowspan="3">Single Layer (homogeneous)</td><td rowspan="2">Burmister</td><td>Krynine</td><td>at Each Point</td><td></td></tr>
<tr><td>Matsuo I</td><td>Control of Scale</td><td>On Each Depth</td></tr>
<tr><td>Newmark</td><td>Matsuo II</td><td>Control of Scale</td><td>On Each Coefficient of Concentration</td></tr>
<tr><td>Several layer (Horizons)</td><td></td><td>Matsuo III</td><td>Control of Scale</td><td>1</td></tr>
</table>

Although the author recommends method 2, each method has its characteristics. Thus, if the most convenient method were employed resulting in contributing to this field where much is not yet clearly understood, the auther would consider it very fortunate.

The author thanks Prof. Yasuo Kondo for many precious instructions and would appreciate instructions and criticisms offered to him from the readers.