On the ⅓ Sub-harmonic Resonance of the System with Non-linear Restoring Force *

By

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Synopsis

In this report the sub-harmonic resonance of the dissipative system with the restoring force represented by cubic curve is treated. The authers tried to solve this problems by Fourier series and investigate the stability of the solutions, and then acquired the graphical solution which is useful actually. We made an experiment to see the validity of the calculation, that is, we could find how the magnitudes of damping force and external force affect occurrence of sub-harmonic resonance in the non-linear system.

Calculation

(1) Fundamental Equation.

In this case the equation of motion is expressed as

$$m\ddot{x}+c\dot{x}+\alpha x+\beta x^3 = P_0 \sin 3\omega t\,, \tag{1}$$

where m is the vibrating mass, c the viscous damping coefficient, $f(x)=\alpha x+\beta x^3$ the non-linear restoring force and $P_0 \sin 3\omega t$ the simple harmonic force acting on the vibrating mass, and x means displacement, t time, $\dot{x}$ and $\ddot{x}$ velocity and acceleration respectively. Now we can see the steady vibration which is represented as follows,

$$x = A_1 \sin(\omega t-\varphi_1)+A_3 \sin(3\omega t-\varphi_3)+A_5 \sin(5\omega t-\varphi_5)+\cdots\cdots, \tag{2}$$

where A_1, $A_3 \cdots\cdots$ and φ_1, $\varphi_3 \cdots\cdots$ are unknown constants, which will be determined by following calculations. By considering the following experimental results, we can neglect A_5, $A_7 \cdots\cdots$ as very small in comparison to A_1 and A_3. Then we can assume x approximately as

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$$x = A_1 \sin(\omega t - \varphi_1) + A_3 \sin(3\omega t - \varphi_3). \tag{2a}$$

If we apply eq. (2a) to eq. (1) and compare each coefficients of $\sin \omega t$, $\cos \omega t$, $\sin 3\omega t$ and $\cos 3\omega t$ in both sides of the equation, we obtain the following relations;

$$-mA_1\omega^2 + \alpha A_1 + \beta\left\{\frac{3}{4}A_1^3 - \frac{3}{4}A_1^2 A_3 \cos(3\varphi_1 - \varphi_3) + \frac{3}{2}A_1 A_3^2\right\} = 0, \tag{3}$$

$$cA_1\omega - \frac{3}{4}\beta A_1^2 A_3 \sin(3\varphi_1 - \varphi_3) = 0, \tag{4}$$

$$-9mA_3\omega^2 + \alpha A_3 + \beta\left\{-\frac{A_1^3}{4}\cos(3\varphi_1 - \varphi_3) + \frac{1}{4}A_3^3 + \frac{3}{2}A_1^2 A_3\right\} = P_0 \cos\varphi_3, \tag{5}$$

$$3cA_3\omega + \frac{\beta}{4}A_1^3 \sin(3\varphi_1 - \varphi_3) = P_0 \sin\varphi_3. \tag{6}$$

(2) Determination of A_1 and A_3.

We eliminate $\cos(3\varphi_1 - \varphi_3)$ and $\sin(3\varphi_1 - \varphi_3)$ from (3) and (4). Hence we obtain,

$$\frac{9}{16}\beta^2 A_1^4 + \frac{27}{16}\beta^2 A_1^2 A_3^2 + \frac{9}{4}\beta^2 A_3^4 - \frac{3\beta A_1^2}{2}(m\omega^2 - \alpha) - 3\beta A_3^2(m\omega^2 - \alpha) + (m\omega^2 - \alpha)^2 + c^2\omega^2 = 0. \tag{7}$$

We use the following notations;

$$\xi = A_1^2, \quad \eta = A_3^2, \quad L = \frac{3}{4}\beta, \quad M = m\omega^2 - \alpha, \quad N = c\omega.$$

Then eq. (7) becomes

$$L^2\cdot\xi^2 + 3L^2\cdot\xi\cdot\eta + 4L^2\cdot\eta^2 - 2L\cdot M\cdot\xi - 4L\cdot M\cdot\eta + M^2 + N^2 = 0. \tag{7'}$$

Eq. (7′) represents an elliptic curve in $\xi - \eta$ plane, and if we transform the coordinates ξ, η to the new coordinates ξ', η' by translation of ξ_0, η_0 and rotation of angle θ, eq. (7′) will be transformed to

$$\frac{\xi'^2}{a^2} + \frac{\eta'^2}{b^2} = 1 \tag{7''}$$

, where

$$\xi_0 = \frac{4}{7}\cdot\frac{M}{L}, \quad \eta_0 = \frac{2}{7}\cdot\frac{M}{L}, \quad \tan 2\theta = -1,$$

$$a = \sqrt{\frac{2}{7}\cdot\frac{M^2 - 7N^2}{5 + 3\sqrt{2}}\cdot\frac{1}{L^2}}, \quad b = \sqrt{\frac{2}{7}\cdot\frac{M^2 - 7N^2}{5 - 3\sqrt{2}}\cdot\frac{1}{L^2}}.$$

The envelope of the curves represented by eq. (7″) with parameter ω is a rectanglar hyperbola. If we apply eq. (3) and (4) to eq. (5) and (6) respectively, and eliminate $\sin\varphi_3$, $\cos\varphi_3$ from the two relations, we get the following formula, where we assume $A_1 \gg A_3$ and neglect the terms of A_3^6, A_3^8 as very small;

$$\left.\begin{aligned}&\left\{\frac{63}{8}\beta^2 A_1^4+\frac{3}{2}\beta(m\omega^2-\alpha)A_1^2-18\beta(9m\omega^2-\alpha)A_1^2+9(9m\omega^2-\alpha)^2+81c^2\omega^2\right\}A_3^4\\&+\left\{-\frac{9}{2}\beta^2 A_1^6+6\beta(m\omega^2-\alpha)A_1^4+\frac{9}{2}\beta(9m\omega^2-\alpha)A_1^4-6(m\omega^2-\alpha)(9m\omega^2-\alpha)A_1^2\right.\\&\left.+18c^2\omega^2A_1^2-9P_0^2\right\}A_3^2\\&+\left\{(m\omega^2-\alpha)^2A_1^4+c^2\omega^2A_1^4-\frac{3}{2}\beta(m\omega^2-\alpha)A_1^6+\frac{9}{16}\beta^2\cdot A_1^8\right\}=0.\end{aligned}\right\}\quad(8)$$

We can get the the values of A_1 and A_3 by eq. (7) and (8). Namely, in Fig. 1 elliptic curves represent eq. (7) and the curvs which is nearly parallel to the A_1^2 axis represents eq. (8), and the intersecting points of both curves give the values of A_1 and A_3. Thus obtained values of A_1 and A_3 are plotted in Fig. 2 against frequencies of external force. In the same figure, value of A_1 at contact point between abscissa and ellipse representing eq. (7) of the case when $c=0$, that is,

$$A_1=\frac{m\omega^2-\alpha}{\frac{3}{4}\beta}\qquad(9)$$

is plotted. This relation shows the one between amplitude and frequency in natural oscillation acquired by first approximation of Fourier series method. It is noticeable that the curve representing eq. (9) and the one representing A_1

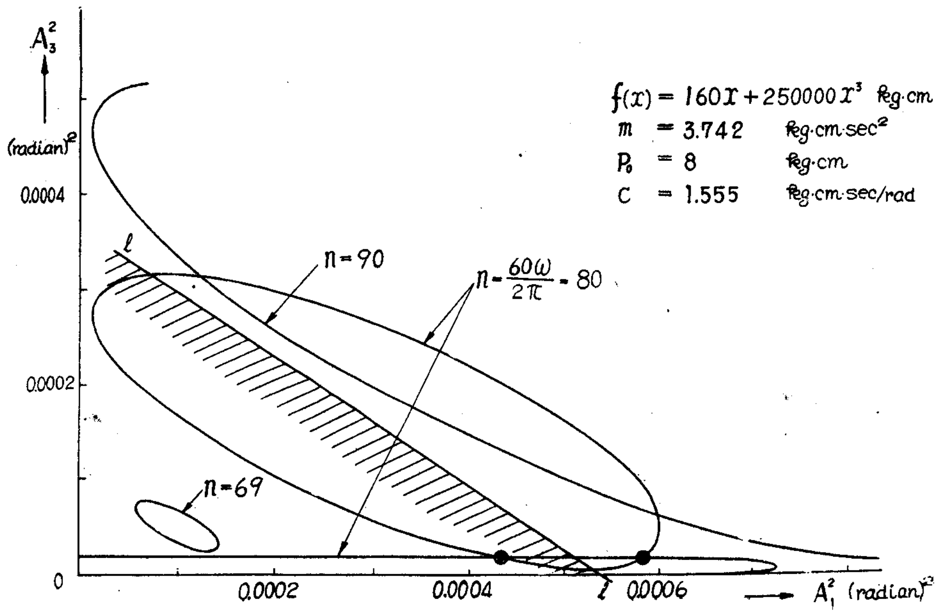

Fig. 1

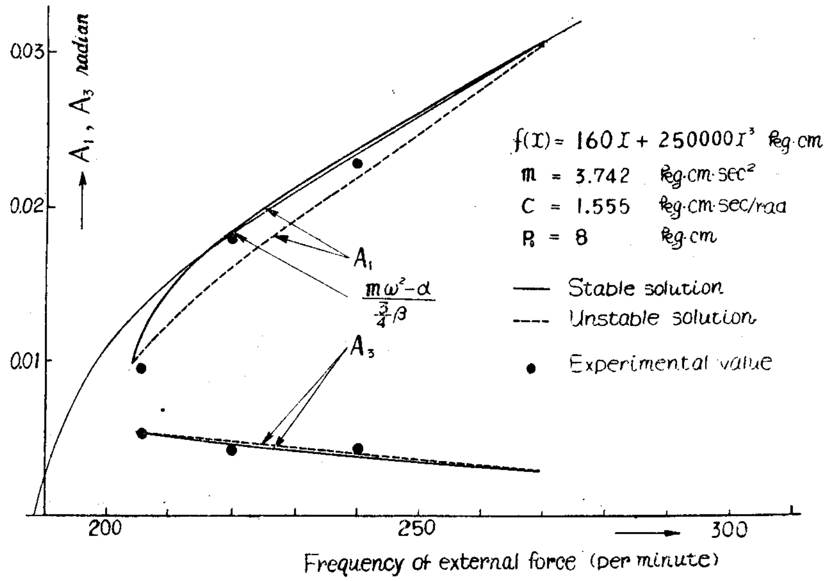

Fig. 2

obtained by graphical solution of Fig. 1 are very close to each other. From Fig. 2 we understand that the real values of A_1 and A_3 exist in some range of frequencies of external force.

(3) Stability of Solutions.

Generally two points exist as intersection of two curves represented by eq. (7) and (8), but these points are not necessarily stable. Therefore we must discuss on the stability of these solutions. First we assume

$$x = a_1 \sin \omega t + b_1 \cos \omega t + a_3 \sin 3\omega t + b_3 \cos 3\omega t \,, \tag{10}$$

where a_1, b_1 are not constants but functions of time t. After apllying (10) to (1), we get

$$\left.\begin{aligned} &-2m\frac{db_1}{dt}\omega + c\frac{da_1}{dt} + a_1\Big[(\alpha - m\omega^2) + \beta\Big\{\frac{3}{4}(a_1^2+b_1^2) + \frac{3}{2}(a_3^2+b_3^2)\Big\}\Big] \\ &-\beta\Big\{\frac{3}{4}a_3(a_1^2-b_1^2) + \frac{3}{2}a_1b_1b_3\Big\} = 0\,, \\ &2m\frac{da_1}{dt}\omega + c\frac{db_1}{dt} + b_1\Big[(\alpha - m\omega^2) + \beta\Big\{\frac{3}{4}(a_1^2+b_1^2) + \frac{3}{2}(a_3^2+b_3^2)\Big\}\Big] \\ &-\beta\Big\{\frac{3}{4}b_3(a_1^2-b_1^2) - \frac{3}{2}a_1a_3b_1\Big\} = 0\,, \end{aligned}\right\} \tag{11}$$

where we neglect $\dfrac{d^2a_1}{dt^2}$ and $\dfrac{d^2b_1}{dt^2}$ as very small in comparison to a_1, b_1, $\dfrac{da_1}{dt}$ and $\dfrac{db_1}{dt}$. Now, we consider an infinitesimal change in a_1, b_1. Then a_1, b_1 will become $a_1+\Delta a_1$ and $b_1+\Delta b_1$ respectively. Eq. (11) becomes

$$\left.\begin{aligned} -2m\omega\frac{d\Delta b_1}{dt}+c\frac{d\Delta a_1}{dt}+E\cdot\Delta a_1+F\cdot\Delta b_1 &= 0\,, \\ 2m\omega\frac{d\Delta a_1}{dt}+c\frac{d\Delta b_1}{dt}+G\cdot\Delta a_1+H\cdot\Delta b_1 &= 0\,, \end{aligned}\right\} \tag{12}$$

where

$$E = (\alpha-m\omega^2)+\frac{3}{4}\beta(3a_1^2+b_1^2)+\frac{3}{2}\beta(a_3^2+b_3^2)-\frac{3}{2}\beta(a_1a_3+b_1b_3)\,,$$

$$F = \frac{3}{2}\beta(a_1b_1+a_3b_1-a_1b_3)-c\omega\,,$$

$$G = \frac{3}{2}\beta(a_1b_1+a_3b_1-a_1b_3)+c\omega\,,$$

$$H = (\alpha-m\omega^2)+\frac{3}{4}\beta(a_1^2+3b_1^2)+\frac{3}{2}\beta(a_3^2+b_3^2)+\frac{3}{2}\beta(a_1a_3+b_1b_3)\,.$$

Eliminating Δb_1 from eq. (12), we get

$$\left.\begin{aligned} &2m\omega(c^2+4m^2\omega^2)\frac{d^2\Delta a_1}{dt^2}+\left\{2m\omega c(E+H)+4m^2\omega^2(G-F)\right\}\frac{d\Delta a_1}{dt} \\ &+2m\omega(EH-FG)\,\Delta a_1 = 0\,. \end{aligned}\right\} \tag{13}$$

If a_1 is stable, Δa_1 must tend to zero with time. These conditions are expressed as following formulae,

$$2m\omega c(E+H)+4m^2\omega^2(G-F) > 0\,,$$
$$2m\omega(EH-FG) > 0\,,$$

or,

$$(\alpha+m\omega^2)+\frac{3}{4}\beta(A_1^2+A_3^2) > 0, \tag{14}$$

$$\beta\left\{\frac{3}{2}\beta A_1^2+\frac{9}{4}\beta A_3^2-2(m\omega^2-\alpha)\right\} > 0\,. \tag{15}$$

The condition (14) is fulfilled always in this experiment. In Fig. 1 the straight line ll passing through the centre of ellipse shows the boundary of condition (15). Hence the shaded area represents the unstable region. Thus the values of A_1 and A_3 represented by the dotted line in Fig. 2 are unstable ones.

(4) Range of Sub-harmonic Resonance.

In Fig. 1 it is noticeable that the curve representing eq. (8) is nearly equal to parallel line to A_1^2 axis. We use $c=0$ and $A_1=0$ in eq. (8), then we get,

$$A_3 = \frac{P_0}{9m\omega^2-\alpha}\,. \tag{16}$$

Now we use eq. (16) as approximation of eq. (8), then the contact point between the ellipse and the line representing eq. (16) can be obtained easily. We calculate $\frac{d\eta}{d\xi}$ from eq. (7) and put it to zero.

Hence,

$$2L^2\xi+3L^2\eta-2LM = 0\,. \tag{17}$$

After eliminating ξ from eq. (17) and (7′), and applying (16), we get

$$\frac{63}{16}\cdot\frac{\beta^2}{(9m\omega^2-\alpha)^4}\cdot P_0^4-3\frac{\beta(m\omega^2-\alpha)}{(9m\omega^2-\alpha)^2}\cdot P_0^2+4c^2\omega^2 = 0\,. \tag{18}$$

From this equation we can obtain P_0 for given c and ω. Eq. (18) represents the boundary curve determining the range where sub-harmonic resonance is possible to occur. Putting $c=0$ in eq. (18), we get

$$P_0^2 = \frac{16}{21}\cdot\frac{(m\omega^2-\alpha)(9m\omega^2-\alpha)^2}{\beta}\,. \tag{19}$$

From eq. (19) we find that if β is positive, ω must be greater than $\sqrt{\frac{\alpha}{m}}$ and if β is negative, ω smaller than $\sqrt{\frac{\alpha}{m}}$. Fig. 3 shows the relation between P_0 and ω, which is shown in eq. (18) or (19). From this figure we can find that the range of sub-harmonic resonance is narrower with the increase of damping coefficient.

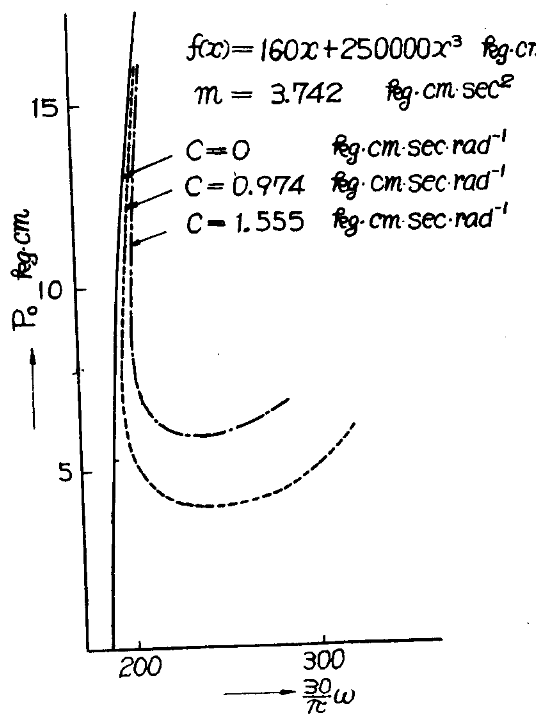

Fig. 3

(5) Determination of Phase Difference φ_1 and φ_3.

If we find the phase difference φ_1 and φ_3, we would be able to draw the wave form of the displacement by using already known A_1 and A_3. After eliminating $\cos(3\varphi_1-\varphi_3)$ from eq. (3) and (5), and $\sin(3\varphi_1-\varphi_3)$ from eq. (4) and (6), we get

$$\cos\varphi_3 = \frac{1}{3P_0\cdot A_3}\left\{-(\alpha-m\omega^2)A_1^2+3(\alpha-9m\omega^2)A_3^2-\frac{3}{4}\beta(A_1^4-A_3^4)+3\beta A_1^2A_3^2\right\}, \tag{20}$$

$$\sin\varphi_3 = \frac{1}{3P_0\cdot A_3}\cdot c\cdot\omega(A_1^2+9A_3^2)\,. \tag{21}$$

Next from eq. (3) and (4), we get

$$\sin(3\varphi_1-\varphi_3) = \frac{4c\omega}{3\beta A_1A_3}, \tag{22}$$

$$\cos(3\varphi_1-\varphi_3) = \frac{4}{3\beta A_1A_3}\left\{(\alpha-m\omega^2)+\frac{3}{4}\beta(2A_3^2+A_1^2)\right\}. \tag{23}$$

From eq. (20) to (23) we can find the values of φ_1 and φ_3. In Fig. 4 the calculated values of φ_1 and φ_3 are represented by full lines and dotted lines. It can be noticed that φ_3 is nearly equal to 180 degrees at any frequency of external force.

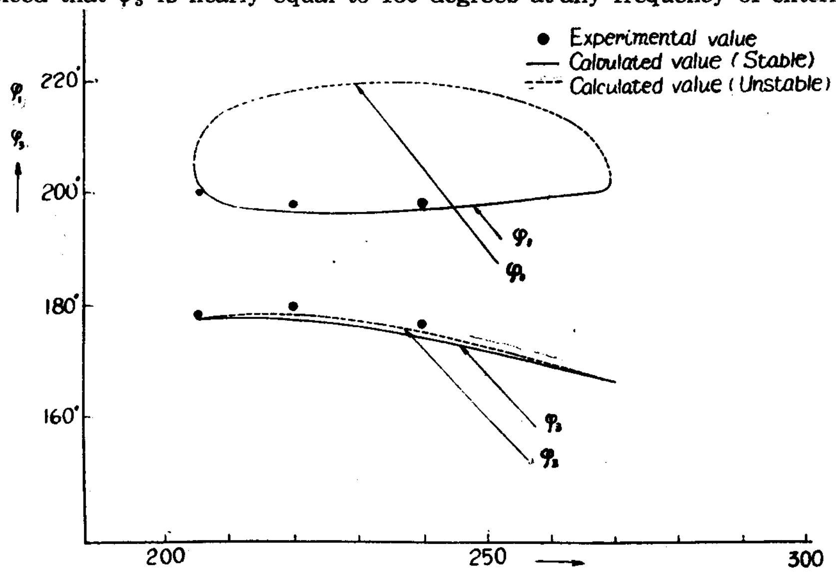

Fig. 4

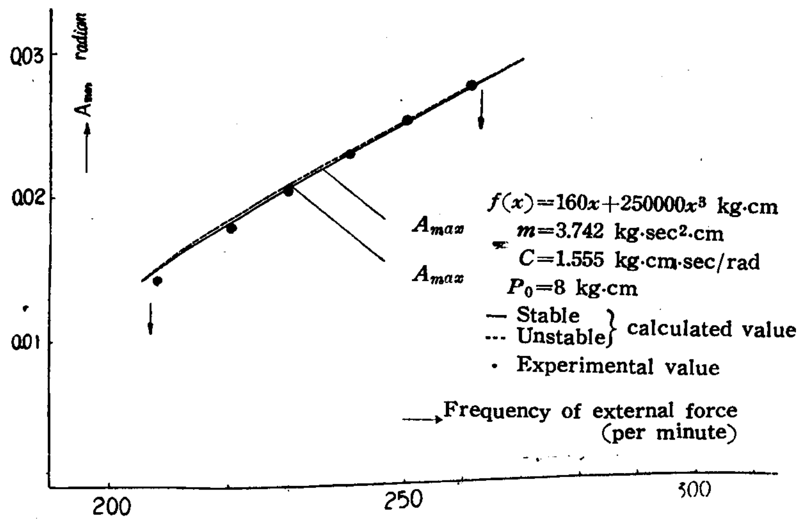

Fig. 5

We can draw the wave form of displacement by using the values of A_1, A_3, φ_1, and φ_3, in oder to obtain the maximum amplitude A_{max}. The values of A_{max} are plotted in Fig. 5 and the full line represents stable value and dotted line unstable one.

Experiment

(1) Apparatus Used in the Experiment.

Experiments were made to see the validity of the above calculation. In Fig. 6 diagramatical sketch of the apparatus is shown. In Fig. 6 vibrating bar hangs down vertically supported in pivot, and spring S_3 is connected to its lower end. By this spring we can give the non-linearity to the restoring force of this system. In order to exert the external force to this system, the same springs S_1 and S_2 were attached to the bar, and the other end of S_1 was fixed

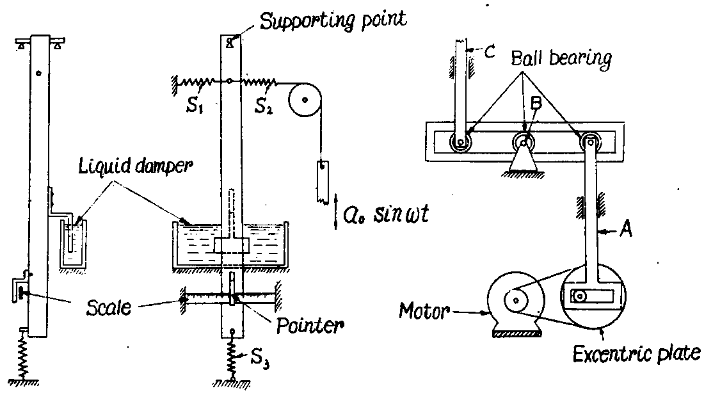

Fig. 6

to foundation, and that of S_2 was connected to the moving part, motion of which is sinusoidal. As for the damping force we used the fluid damper which is constructed to give the viscous friction to this system. That is, the thin plate attached to vibrating bar is movable in the fluid in a vessel. The moment of inertia of the vibrating bar about supported point was found by the natural oscillation without S_1, S_2, S_3 and dampers. The relation between restoring force and angular deflection was obtained experimentally, by means of which we get the value of α and β by least mean square method. The magnitude of external force is obtained by the product of spring coefficient of S_2, a_0 and the distance between supporting point and S_2. In order to alter the magnitude of the external force in running state, we employed the equipment shown in Fig. 6. That is, an excentric plate is driven by motor, giving A an up and down motion, which is transformed to C by lever action. In this equipment point B is able to move horizontally, hence amplitude of C may be changed to various values in running state. Then amplitude of the displacement was observed by the pointer attached to vibrating bar and the scale located on its background. The wave form of the displacement was observed by catching the shade of pointer optically in oscillograph paper. To determine the value of damping coefficient, the damped natural oscillation was observed in oscillograph paper. In this experiment water and machine oil were used as damping fluids. Thus we obtained following data in the experiment;

α = 160 kg·cm·rad^{-1}, β = 250000 kg·cm·rad^{-3}, m = 3.742 kg·cm·sec^2,
c = 1.555 kg·cm·sec·rad^{-1} (in the case of machine oil),
c = 0.974 kg·cm·sec·rad^{-1} (in the case of water),
P_0 = about 3∼11 kg·cm.

(2) Results of Experiment.

The experimental values of A_{max} are shown Fig. 7. From this figure we can see that the maximum amplitude of sub-harmonic resonance is almost independent from the magnitude of external force, but the range where sub-harmonic resonance may occur are dependent on it. In order to compare the experimental

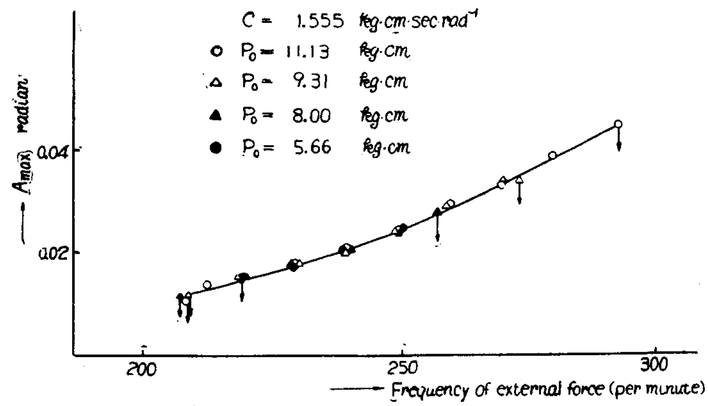

Fig. 7

results with the calculation, the case when P_0=8 was shown in Fig. 5. The possible range of sub-harmonic resonance in the case when P_0=8 with oil damper and water damper and shown as shaded area in Fig. 8 and Fig. 9 respectively. In these figures, the full lines and dotted lines represent the boundary determining the domain where sub-harmonic resonance may occur, and the former was drawn by eq. (7) and (8), the latter by eq. (18). From these figures we can see that the calculated values agree very closely with experimental results. We took the photograph of wave form of the displacement in the cases of $P_0=8$ and $c=1.555$ with frequency of 206, 220 and 240 cycles per minute and analizing the waves to Fourier series we got A_1, A_3, φ_1 and φ_3. These values are plotted in Fig. 2 and Fig. 4. We found that the higher harmonics than A_5 are negligible in comparison to A_1 and A_3.

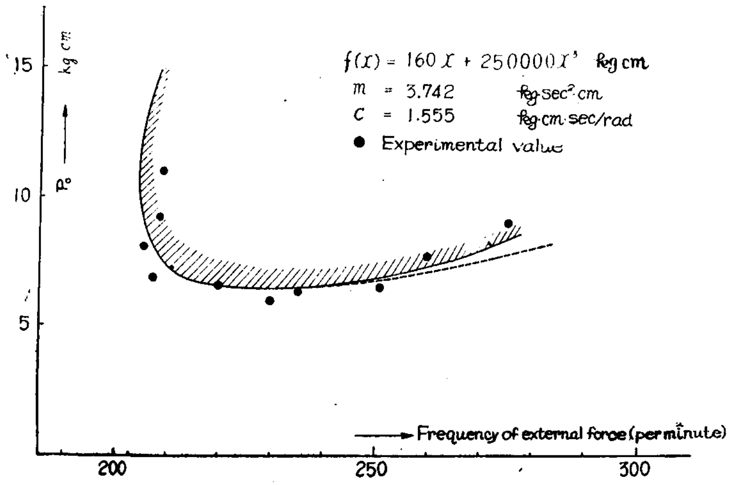

Fig. 8

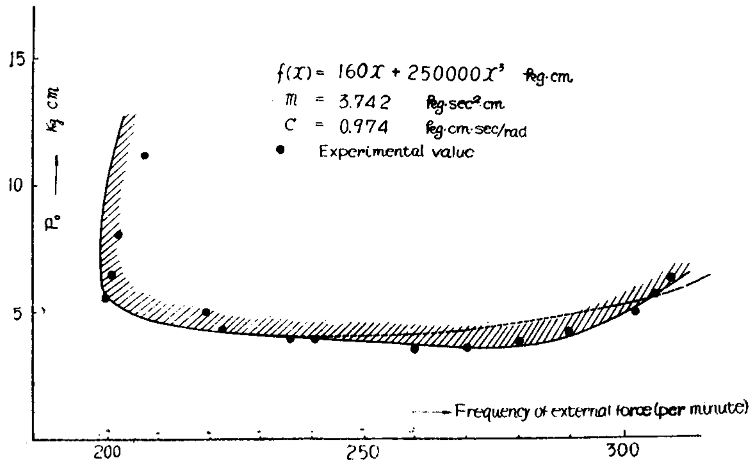

Fig. 9

Conclusions

It was proved that the calculations and the experiments have about the same result, therefore the above mentioned analysis was right. We can conclude the following points on the sub-harmonic resonance.

1. Sub-harmonic resomance may occur in some range of the magnitude and frequency of external and damping force, and if the magnitude of external force is given, the ferquency range would become narrower with increasing damping force.

2. The maximum amplitude of displacement in sub-harmonic resonance is nearly equal to the one in natural oscillation which is represented by eq. (9) and then almost independent of the magnitude of external force.

3. The value of A_3 is approximately determined by eq. (16).

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