On the Yielding of Steel Under Bending Moment

By

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Synopsis

It is an important problem to know how high rises the so-called yield point under uneven distribution of stress. Much efforts made in this field is found in the papers hitherto reported. Among which, F. Nakanishi's [1] theory concerning the yield point of mild steel under bending moment is noted. This is, however, not adequate for explaining the elasto-plastic bending of ordinary carbon steel except of mild steel. Therefore it encountered serious opposition of recent investigators such as German researchers and authors as well, who performed experiments by means of X-ray stress measurement.

This report is made on the recent experimental results of the measured stress existing on a yielding bar under bending moment. Basing on the experimental results the authors' opinion is proposed concerning the mechanism in yielding phenomena of bending a steel bar. The theoretical treatment of yielding mechanism is then explained with correlation between the theoretical and the experimental relation of moment and deflection.

I. Introduction

As for the yielding of steel bar under bending moment a number of experimental results are found [2]. It is well known that, in case of pure bending of mild steel beam with rectangular cross-section, the moment-deflection curve is nearly straight until the applied moment rises as high as 1.5 times of M_s, where M_s denotes the moment in which the stress of the outer-most layer reaches the lower yield stress σ_s as determined by tension test. In a simple view of the moment-deflection relation, the yield stress of mild steel beam loaded in pure bending might be taken to rise as high as 1.5 times of σ_s. For the purpose of confirming the problem, the stress existing on the surface of mild steel beam was measured by means of X-ray. The result has made clear that the stress exceeding σ_s is scarcely found on the surface of a beam even though the applied moment exceeds the magnitude of M_s. In Germany, as will be illustrated later on, H. Möller and J. Barbers[3], F. Bollenrath and E. Schiedt[4] reported independently the similar results of experiments. E. Siebel[5] pointed out the discrepancy of the above experimental results of the investigations and his view, derived from the observation of moment-deflection relation, of the yielding phenomenon of a steel beam under bending moment.

In this report, the authors point out a new idea on the yielding mechanism which preferably fulfils both views. It stands on the same fundamental conception as that of the surface effect which has been taken up by the authors concerning the yielding resistance of materials under tension. A theoretical treatment is then carried out. According to the theory the yielding of beams of any kind of steel, not to mention of mild steel, can be treated. The theoretical moment-deflection relations of beams of various kinds of steel were obtained by calculation with regard to a few kinds of cross-sectional forms and compared with the experimental ones.

II. Experiments

The stresses existing on the surface of mild steel beam of rectangular cross-section were measured by means of X-ray. The X-ray patterns photographed by

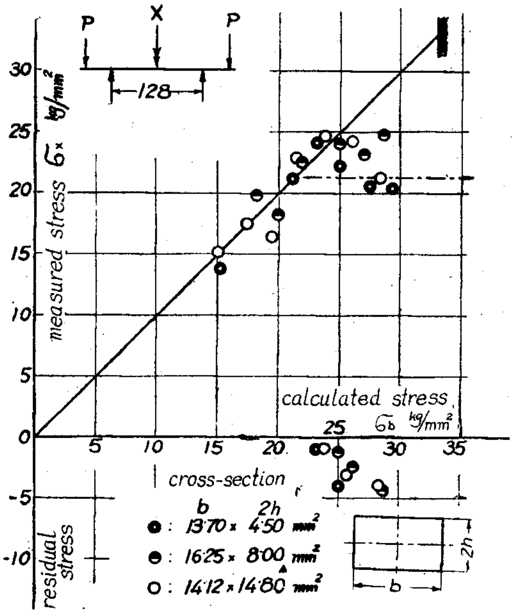


Fig. 1

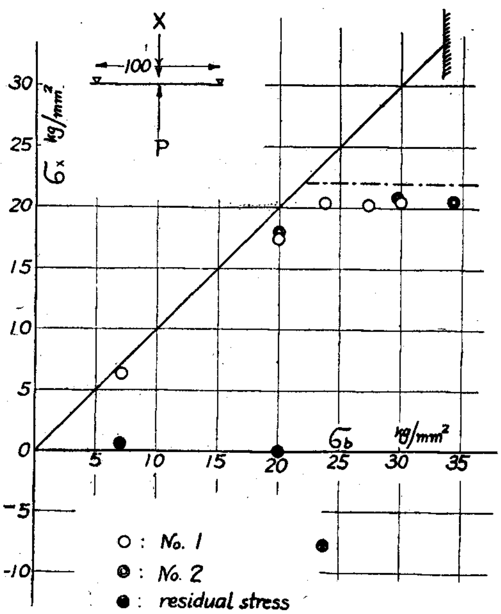


Fig. 2

both perpendicular and oblique incident ray were utilized for analysing the stress according to Glocker's method [7]. The specimens of the composition with 0.09% C, 0.38% Mn, 0.10% Si, 0.038% S and 0.045% P underwent machining with subsequent annealing at 900°C for one hour in a vacuum furnace. The principal arrangement of the experiment for constant bending moment and the results are illustrated in Fig. 1. As is seen in the figure, the measured stresses rise closely following the elastic theory so far as the applied moment does not exceeds M_s. For the higher value of the applied moment, they show a slight increase and subsequent decrease, whereas the observed moment-deflection curve is nearly straight until the moment of the highest value in the test is reached as is shown by small circles in Fig. 14 a. Fig. 2 shows the result of stress measurement carried out on annealed mild steel bar of 20 mm in breadth and 3 mm in thickness, when concentrated load is applied. Fig. 3 is an example of the results of the similar experiments by H. Möller and J. Barbers, where the stresses were measured on both sides of tension and compression. Fig. 4 shows that of F. Bollenrath and E. Schiedt for a bar with triangular cross-section. The tendency of measured stress are the same in all cases. Hence we can introduce the conclusion : on the yielding of bending bar, the stress at the outer-most fiber never rises beyond the yield stress σ_s.

The stress distribution in the section of yielding beams was observed by means of X-ray as well. Fig. 5 shows an example of experiment by F. Bollenrath

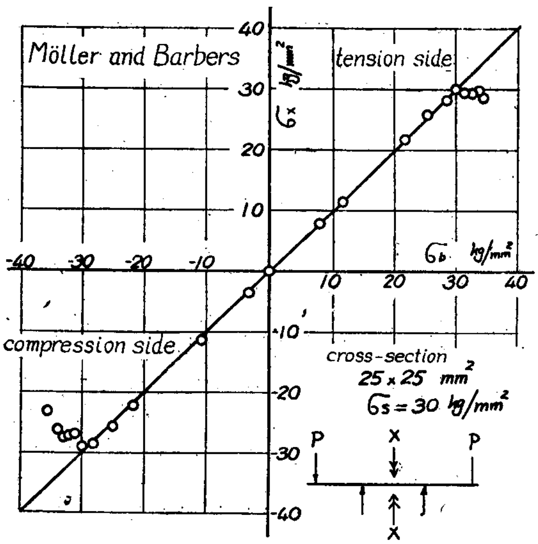


Fig. 3

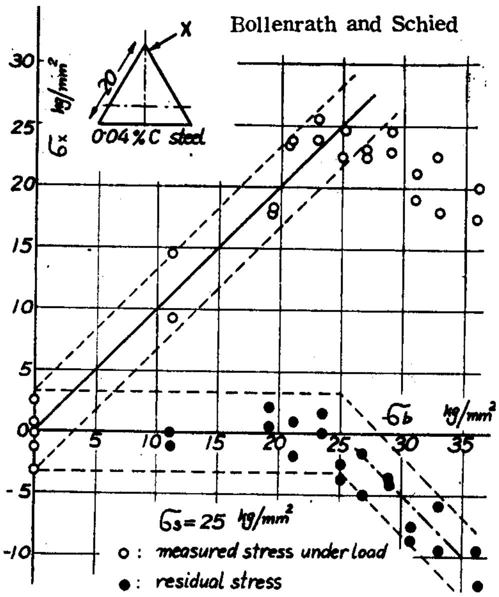


Fig. 4

and E. Schiedt for beams of triangular cross-section. The stress near the surface is rather lower than σ_s, increasing in the inner portion. For the purpose to corroborate the above, the stress distribution near the surface was investigated utilizing residual stress measurement by corrosion method [6]. The residual stress, existing in a beam subjected to a moment beyond M_s accompanied by a subsequent removal of the load, was measured[6]. Assuming the metal behaves like a perfectly elastic material during the process of unloading after plastic deformation, the stress distribution under loaded condition is obtainable, adding the residual stress to the linearly distributed one derived from the elastic theory. A few of the results obtained are shown in Figs. 6a and 6b. The size of rectangular cross-section and the applied moment are shown in the figure. The obtained distribution near the surface is seen to be the same as that of Fig. 5. Surveying the above results, it may be said that, in a bar subjected to a moment exceeding M_s, two portions coexist, the one is of linear stress distribution and the other of disturbed one caused by the occurence of local plastic deformation near the surface.

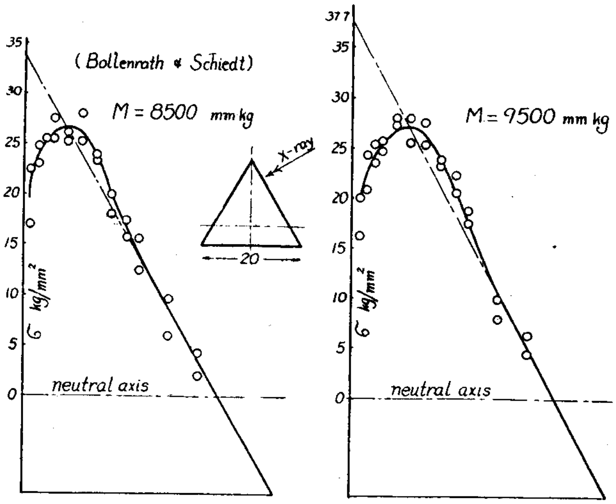


Fig. 5

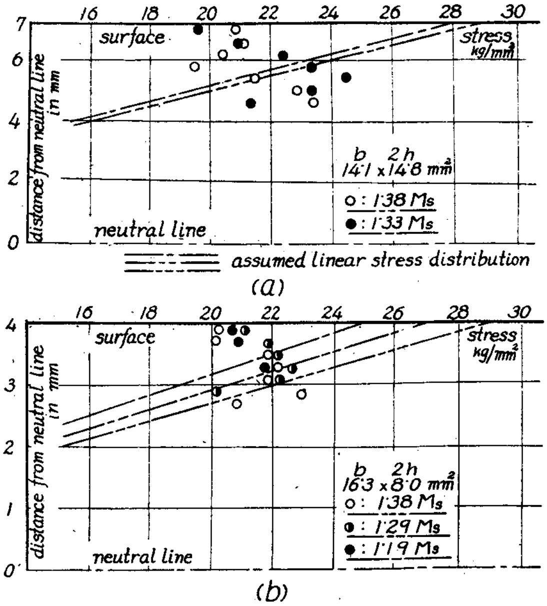


Fig. 6

III. Yielding mechanism

According to the authors' opinion, the yielding phenomena of steel bar under bending moment ought to be explained by the same mechanism as of the yielding in tension or compression. Loading a steel bar in tension or compression, the stress-strain diagram as is illustrated in Fig. 7 is usually obtained. It is well known that the strain of locally slipped portion takes instantaniously the value ε'' as soon as the yield point is reached, where ε'' means the value of strain at the end point of horizontal line in stress-strain curve of steel bar. If we denote by ε_e the elastic strain corresponding to the yield point σ_s, the local strain of yielding steel bar must be either ε'' or ε_e and no intermediate magnitude of strain occurs. In the case of yielding of bent beam, the same account must be taken in. When the applied moment just exceeds the value M_s, the local stress reaches σ_s here and there near the surface and initial slip begins.

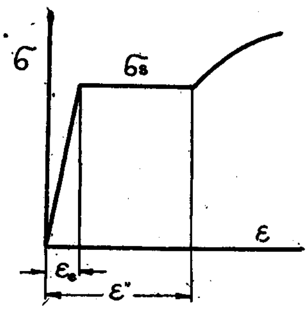


Fig. 7

Because of the surface effect, the grains near the surface, their forced-strain being the maximum in the section, is rather easier to deform plastically than those of inner portion. Therefore the above consideration is valid at least near the surface. The slipped portion spreads not only in width on the surface but also inwards along the plane inclined about 45 degrees to the axis with increase of the applied moment. Fig. 8 shows the slipped state schematically. In a beam subjected to a moment beyond M_s, two portions coexist as mentioned above, i. e. the one is plastic and the other elastic. Each of both portions is composed of elastic as well as plastic region as is shown in the figure. In the plastic region, the surface strain of plastic portion becomes as large as ε''. Therefore the Bernoulli–Navier's law is not valid in the disturbed outer region, while in the inner region it is valid. Considering unit length of beam, however, the law is

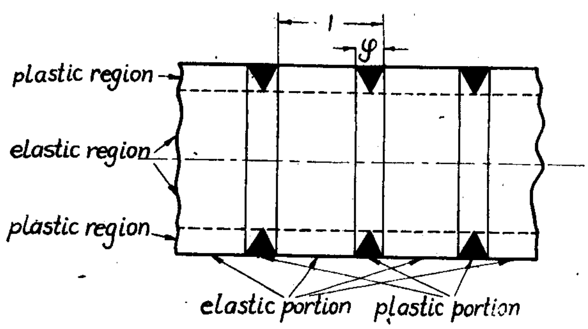


Fig. 8

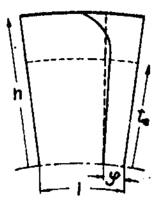


Fig. 9

necessarily valid through whole section of both end planes. The both portions may deform in such a manner as is shown in Fig. 9. The border of both portions, having been a plane in virgin state, is distorted into curved surface. The stress distribution in the section could be illustrated as is shown by C in Fig. 10, where A and are Bassumed stress distributions of elastic and perfectly plastic state respectively. That is to say, near the surface of disturbed plastic region the constrain of elastic portion is mitigated by the excessive strain of plastic portion. As the result, the stress of outer region in average yields to near the value σ_s as is seen in Fig. 1 and others. The above conception will be theorized as follows:

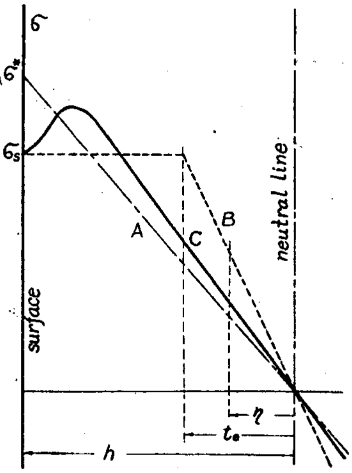


Fig. 10

Consider for unit length of beam, subjected to a moment exceeding M_s, to be composed of φ of plastic portion and $(1-\varphi)$ of elastic one. The strain ε' of the outer-most layer may be represented by

$$\varepsilon' = \varepsilon''\varphi + \varepsilon_e(1-\varphi) \tag{1}$$

In the undisturbed inner region, on the other hand, the resultant strain ε_η' of the fiber distant η from neutral axis (Fig. 10) may be written as

$$\varepsilon_\eta' = \varepsilon_\eta\varphi + \varepsilon_\eta^*(1-\varphi), \tag{2}$$

where ε_η^* and ε_η mean the strain obtained by assuming the stress distribution to be elastic and perfectly plastic respectively, sustaining the same moment. Because the Bernoulli–Navier's law must be valid at both ends of unit length, expanded value of ε_η' at the surface must be the same as ε'. Putting $(\varepsilon_\eta^*)_s = \varepsilon^*$ and $(\varepsilon_\eta)_s = \varepsilon$, where $(\varepsilon_\eta)_s$ means the expanded value at the surface, (2) becomes

$$\varepsilon' = \varepsilon\varphi + \varepsilon^*(1-\varphi). \tag{3}$$

From (1) and (3) we get

$$(\varepsilon^* - \varepsilon_e)(1-\varphi) = (\varepsilon'' - \varepsilon)\varphi .$$

Putting

$$k = \frac{\varphi}{1-\varphi} \quad \text{or} \quad \varphi = \frac{k}{1+k}, \tag{4}$$

it yields to

$$k = \frac{\varepsilon^* - \varepsilon_e}{\varepsilon'' - \varepsilon}. \qquad (5)$$

k means the ratio of summed length of local plastic portions to that of elastic ones. In the formula (5), ε_e as well as ε'', being inherent to materials, can be determined experimentally. ε^* and ε are given theoretically as a function of applied moment with respect to the kind of cross-sectional form. In the state $\varphi=0$ the beam deforms elastically and under the condition of $\varphi=1$ the plastic deformation developes along the whole length.

IV. Theoretical moment-deflection relation

As far as the applied moment does not exceed M_s, the moment-deflection relation is determined by elastic theory. Beyond this limit, the relation can be obtained in the same manner as above. Consider a beam subjected to a uniform bending moment. At any portion along the axis, the same ratio of plastic portion may be contained. Because the resultant strain of outer-most fiber is considered to be ε' and that, taking unit length, the Bernoulli-Navier's law is valid in average, the radius ρ of curvature of neutral axis may be written as

$$\frac{1}{\rho} = \frac{\varepsilon'}{h}.$$

The deflection δ at the middle point of the both supports is given by

$$\delta = \rho\left(1-\cos\frac{l}{2\rho}\right), \qquad (6)$$

where l is the distance between both supports. Let us consider a beam of any kind of steel and of any form of cross-section, ε' is given in term of applied moment from (1), because ε_e and ε'' are determined by tension or compression test and φ is given as a function of moment by substituting (5) for k of (4). Thus (6) leads to the theoretical moment-deflection relation.

V. Examples

According to the authors' theory, the yielding phenomena of bent bars can be treated with steels in general, to say nothing of mild steel. It leads us to a interesting discussion on the growth of plastic portion with increase of applied moment. Besides the moment-deflection relation is theoretically introduced. The propriety of the authors' idea could be corroborated by the comparison of theoretical view to the experimental results. The specimens made of the materials as shown in Table 1 were put into tension test. These have beforehand undergone such heat treatment as mentioned in the last column of the table. The experi-

Table 1

	C %	Mn %	Si %	S %	P %	heat treatment
A	0.10	0.62	0.16	0.040	0.044	annealed at 900°C for one hour
B	0.44	0.60	0.20	0.027	0.020	annealed at 780°C for one hour
C	0.65	0.62	0.20	0.031	0.020	annealed at 750°C for one hour
D	0.44	0.60	0.20	0.027	0.020	770°C oil quench, 150°C temper

mental values of ε_e and ε'' are tabulated in Table 2. The φ vs. moment and deflection vs. moment relations were obtained by calculation for the beams of rectangular, circular and cross-form cross-section and the latter relations are compared with the experimental results. Both tensile and bending specimens made of the same material were heat-treated at the same time in vacuum furnace. For tension as well as bending test the Matsumura's 5 ton universal testing machine was used.

Table 2

	ε_e %	ε'' %
A	0.114	3.14
B	0.140	0.60
C	0.258	0.70

1) Rectangular cross-section.

Consider a beam of rectangular cross-section, b in breadth and $2h$ in thickness. The moment M_s in which the stress of the outer-most layer reaches σ_s is given as

$$M_s \doteq W\sigma_s, \qquad \text{where} \qquad W = \frac{2}{3}bh^2. \tag{7}$$

Let the moment exceeding M_s be M, ε^* is represented, according to its definition, as

$$\varepsilon^* = \frac{M}{M_s}\frac{\sigma_s}{E} = \frac{M}{M_s}\varepsilon_e. \tag{8}$$

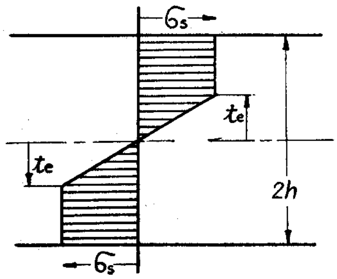


Fig. 11

For the purpose to obtain ε, the state of perfectly plastic stress distribution sustaining M, as is indicated in Fig. 11, is necessarily assumed. Hence

$$\varepsilon = \frac{h}{t_e}\varepsilon_e \tag{9}$$

is given. On the other hand, applying the plasticity,

$$M = \frac{3}{2} M_s \left(1 - \frac{t_e^2}{3h^2}\right) \tag{10}$$

is obtained. Eliminating h/t_e from (9) and (10), we have

$$\varepsilon = \frac{\varepsilon_e}{\sqrt{3\left(1 - \frac{2}{3}\frac{M}{M_s}\right)}} \tag{11}$$

Substituting ε^* and ε in (5) for those of (8) and (11), k is represented in term of M/M_s as

$$k = \frac{\frac{M}{M_s} - 1}{\frac{\varepsilon''}{\varepsilon_e} - \frac{1}{\sqrt{3\left(1 - \frac{2}{3}\frac{M}{M_s}\right)}}}. \tag{12}$$

For the specimens made of the materials indicated in Table 1, $\varphi - \frac{M}{M_s}$ relations are given by calculation utilizing (4) and (12) and shown in Fig. 12. The case of material A, the same mild steel as that used in the case of Fig. 1, attracts our attention, because it suggests the cause of so-called rise of yield point of mild steel bent beam. As soon as the applied moment exceeds M_s, local plastic deformation occurs and with increase of applied moment a slight increase of plastic portion is seen. At the vicinity of 1.47 M_s, it increases remarkably and at about 1.499 M_s the plastic region spreads all over the outer-most layer. The harder the material is, the more is the incressing ratio of plastic portion with increase of applied moment. The value of M/M_s when φ yields to unity is determined as follows. Denoting the radius of curvature of neutral axis in this case by ρ_y and the thickness of elastic region as t_{ey}, we get

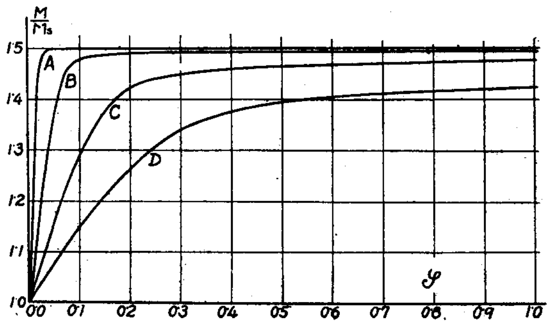


Fig. 12

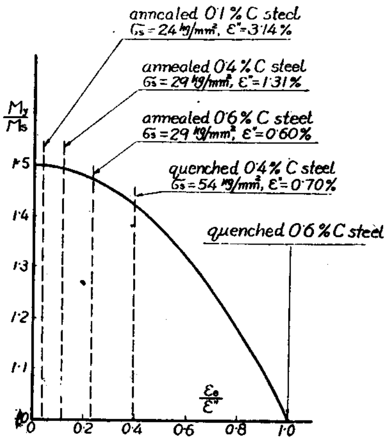
$$\frac{1}{\rho_y} = \frac{\varepsilon''}{h} \quad \text{and} \quad \frac{1}{\rho_y} = \frac{\varepsilon_e}{t_{ey}},$$

$$t_{ey} = \frac{\varepsilon_e}{\varepsilon''} h.$$

Substituting t_{ey} for t_e in (10) and denoting M in (3) as M_y, the relation

$$\frac{M_y}{M_s} = \frac{3}{2}\left[1 - \frac{1}{3}\left(\frac{\varepsilon_e}{\varepsilon''}\right)^2\right] \tag{13}$$

is introduced. For such materials as $\varepsilon_e \ll \varepsilon''$, e.g. annealed mild steel, we have $M_y \cong 1.5\ M_s$ and for the one as $\varepsilon_e = \varepsilon''$, e.g. quenched hard steel subjected to subsequent tempering, $M_y = M_s$, that is, the yield point rise in sight is not seen. Fig. 13 illustrates the theoretical view of the relation between M_y/M_s and $\varepsilon_e/\varepsilon''$. Utilizing the above results, the theoretical $\delta - \dfrac{M}{M_s}$ relations are easily given. The full lines in Figs. 14 a, b, and c show those of beams, 30 mm in breadth and 10 mm in thickness, made of materials A, B and C respectively. The experimental results are shown by small circles in the same figure. The calculated values by the plasticity is, for reference, indicated by the dotted lines. In case of mild steel it is far from result of the experiment while for high carbon steel it is rather favorable. For such materials as have no apparent yield point, in extreme case, the both curves given by the authors' theory and the



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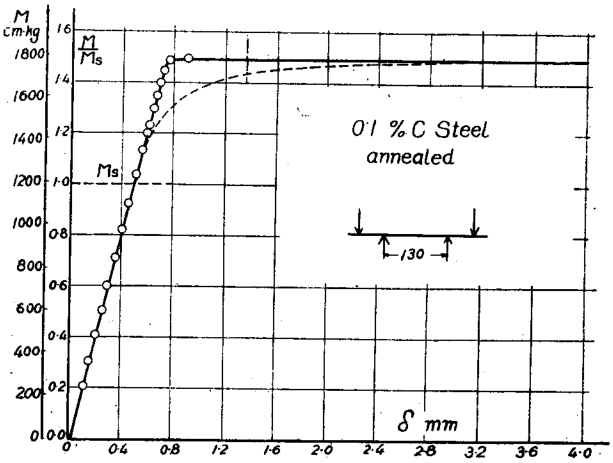


Fig. 14 a

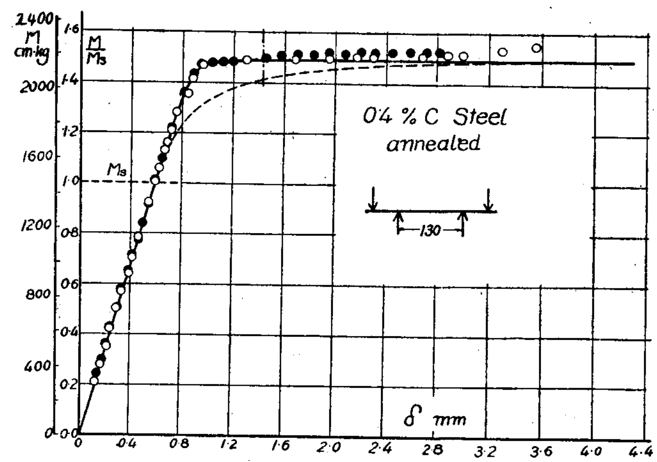


Fig. 14 b

plasticity tend, a matter of course, to coincide.

(2) Circular cross-section.

Let us consider a beam of circular cross-section of d in diameter. In the assumed stress distribution under perfectly plastic state, denoting the thickness of elastic region to be $2t_e$, we have

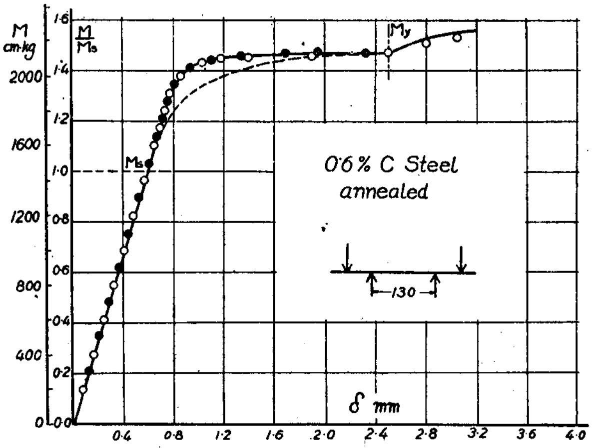


Fig. 14 c

$$\varepsilon = \frac{\varepsilon_e}{2D}, \quad \text{where} \quad D = \frac{t_e}{d}. \tag{15}$$

On the other hand, moment is given by

$$M = 2\sigma_s \left[\int_0^{t_e} \frac{2\eta^2}{t_e} \sqrt{\left(\frac{d}{2}\right)^2 - \eta^2}\, d\eta + \int_{t_e}^{\frac{d}{2}} 2\eta \sqrt{\left(\frac{d}{2}\right)^2 - \eta^2}\, d\eta \right].$$

Hence,

$$\frac{M}{M_s} = \frac{4}{3\pi}(1-4D^2)^{\frac{3}{2}} + \frac{2}{\pi}\left[(1-4D^2)^{\frac{1}{2}} + \frac{1}{2D}\sin^{-1} 2D\right] \tag{16}$$

(15) and (16) being combined through D as a parameter, ε is given as a function of moment ratio M/M_s. Substituting ε^* and ε in (5) and combining it with (4), k and accordingly φ are expressed in term of M/M_s with respect to any kind of steel. The theoretical $\varphi - \frac{M}{M_s}$ relations are shown in Fig. 15 in regard to the specimens made of the materials shown in Table 1. Figs. 16 a, b, c and d illustrate the moment-deflection relations, of which full lines and dotted lines are calculated curves by the authors' theory and by the

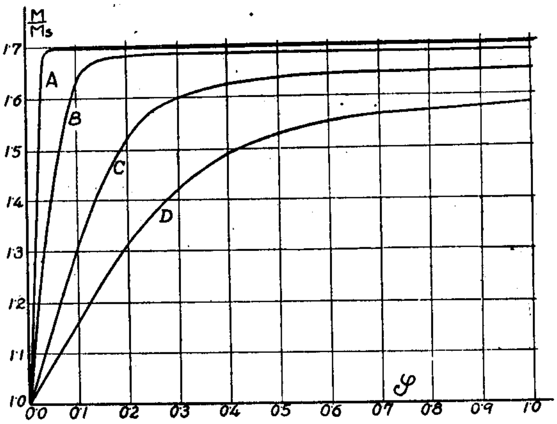


Fig. 15

Fig. 16 a

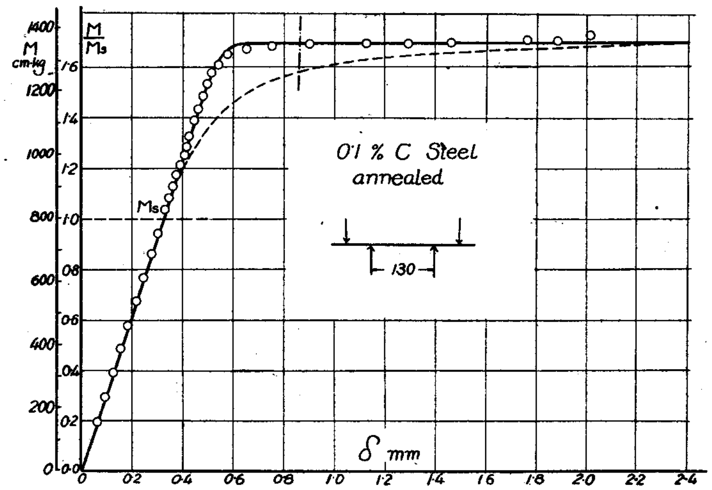


Fig. 16 c

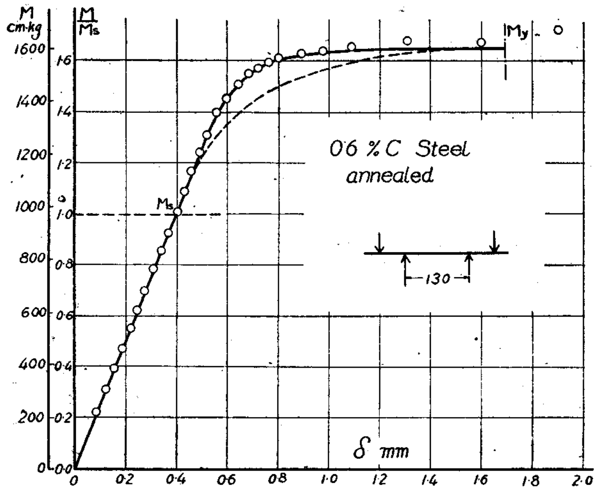


Fig. 16 b

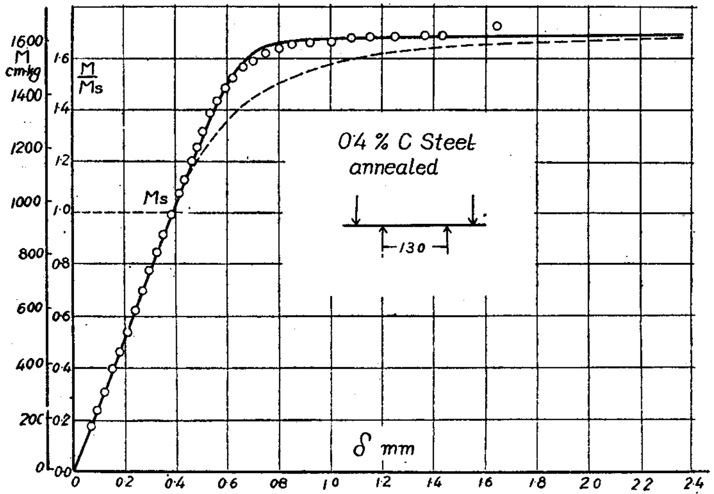
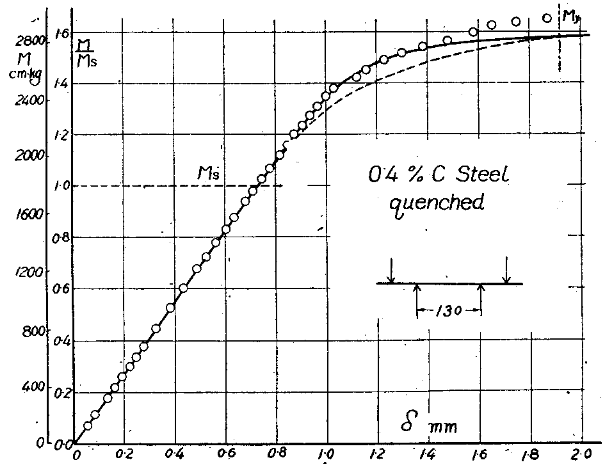


Fig. 16 d



plasticity respectively and the small circles represent the experimental results. The figures show the full lines agree with the experimental results rather than the dotted lines do. In case of mild steel, the break of the curve at the so-called yield point in bending occurs at so high moment as 1.7 M_s. Moreover, it is remarked that the bend of the diagram near the yielding moment is not so sharp in high carbon steel as compared with the case of mild steel.

(3) Cross-form cross-section.

As the third case of application, let us consider beams of the cross-section as is shown in Fig. 17, which are noteworthy on account of the fact that in the bending of these beams two yield points are observed.

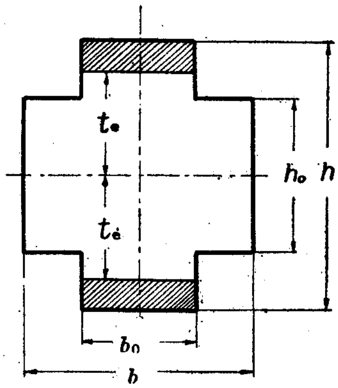


Fig. 17

$$M_{s1} = \frac{2}{3}\, b \left(\frac{h}{2}\right)^2 \sigma_s \left[B + (1-B)H^3\right]. \tag{17}$$

where $B = \dfrac{b_0}{b}$, and $H = \dfrac{h_0}{h}$.

$$\varepsilon^* = \varepsilon_e \frac{M}{M_s}.$$

From the assumed state of perfectly plastic stress distribution,

$$\varepsilon = \frac{\varepsilon_e}{C}, \quad \text{where} \quad C = \frac{2t_e}{h} \tag{18}$$

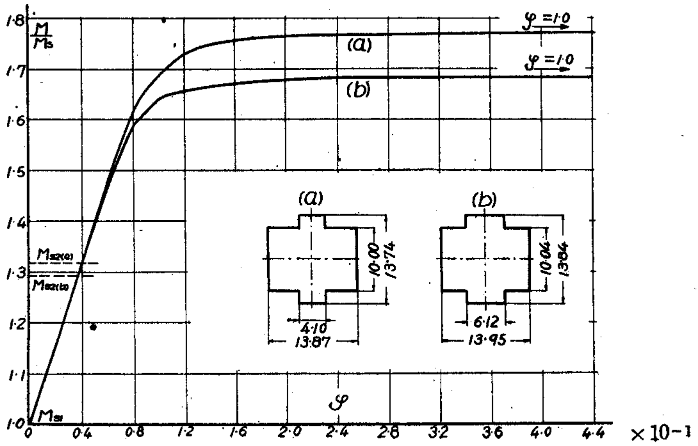


Fig. 18

is introduced. Moment is given as follows:

$$\left.\begin{array}{lll} \text{for} & \frac{h_0}{2} \leqq t_e \leqq \frac{h}{2}, & M = b\left(\frac{h}{2}\right)^2 \sigma_s \left[B - \frac{BC^2}{3} + \frac{2}{3}(1-B)\frac{H^3}{C}\right] \\ \text{for} & t_e \leqq \frac{h_0}{t_e}, & M = b\left(\frac{h}{2}\right)^2 \sigma_s \left[B(1-H^2) + H^3 - \frac{1}{3}C^2\right] \end{array}\right\} \quad (19)$$

Combining (18) and (19) through C as parameter, ε is given as a function of applied moment. In the same fashion as the preceding two cases, φ is indicated in term of moment. The calculated $\varphi - M$ relations of material B is exhibited in Fig. 18, where beams of two kinds of cross-sectional dimension are chosen. The break of the curve at A_1 and A_2 suggests the appearance of two yield points. The theoretical and experimental moment-deflection curves are shown in Figs. 19 a and b. The agreement is seen to be tolerably good. It is not the fact that the beam yields at first at the initial yielding point A_1 but when it just exceeds the moment M_{s1} the local plastic deformation occurs.

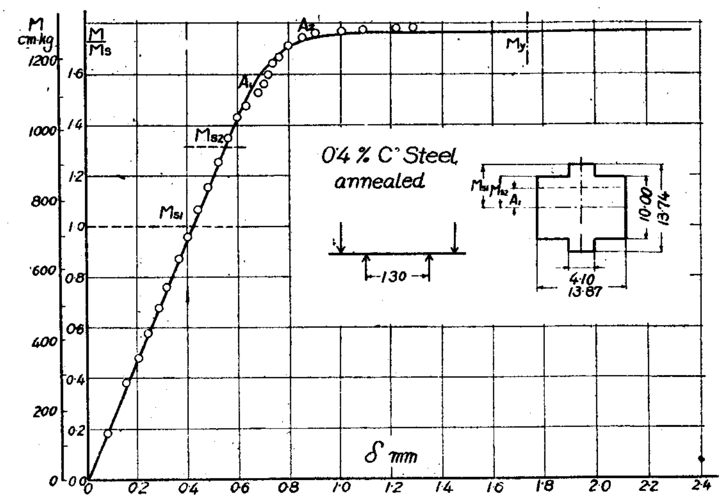


Fig. 19 a

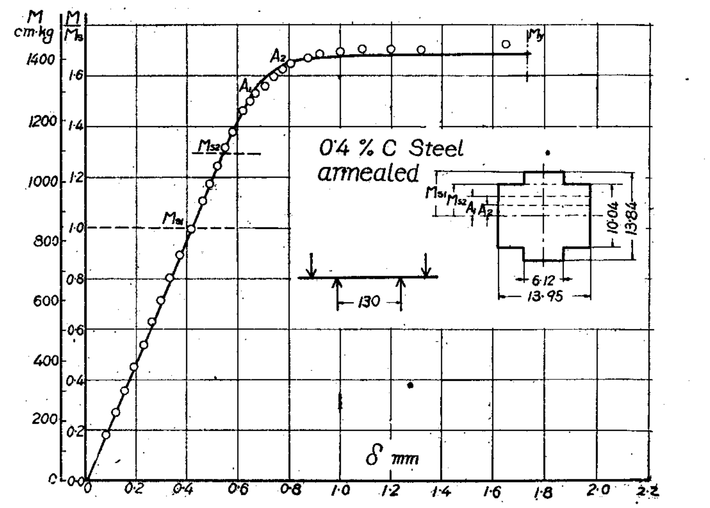


Fig. 19 b

VI. Concluding remarks

A new theory is proposed for the yielding of bending applicable for bars of all kinds of steel, without hesitation, even though it contains many problems in detail. It stands on the apposite conception of not only old observation but also recent experimental results. The agreement of the theoretical moment-deflection relation to the experimental one may be considered to prove the propriety of the authors' idea.

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