Theory of Flow on Road Surface

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(Received May 1951)

Synopsis

By solving the momentum equation of a thin sheet flow on road surface numerically with the condition of continuity obtained under the condition that rain falls on road uniformly, water depth and mean velocity of thin sheet flow and also frictional velocity related with soil-erosion of road surface are computed, and then the effects of camber shape and longitudinal slope of road surface on its drainage and stabilization are discussed.

1. Case when there is no longitudinal slope

(1) Fundamental equation.

As the flow on road surface may be considered as two dimensional when there is no longitudinal slope, x-axis is taken along the road surface perpendicular to the crown line, z-axis, vertically upwards and the original point at the crown, as shown in Fig. 1. $q=q_1-q_2$ is assumed, where q_1 represents rainfall intensity and q_2, infiltration capacity. Furthermore the following notations are used, u: velocity, h: water depth, J_x: slope, ρ: density of water, τ_x: frictional stress on bottom surface. Considering two cross sections a small distance δx apart along x-axis, the continuity and the mementum equations become as follows, neglecting the change in momentum during the rainfall on the road suface and the infiltration of the rainwater into the earth.

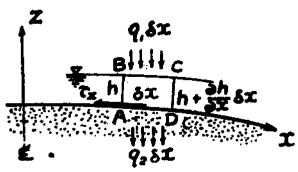

Fig. 1

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\int_0^h udz=q, \qquad (1)$$

$$\int_0^h \rho\frac{\partial u}{\partial t}dz+\frac{\partial}{\partial x}\int_0^h \rho u^2 dz=-\tau_x+\rho g h J_x-\rho g h\frac{\partial h}{\partial x}. \qquad (2)$$

As the steady state is discussed in this paper, Eq. (2) becomes as follows, if $\sqrt{\tau_x/\rho}=u^*$ is substituted.

$$u^{*2}=ghJ_x-gh\frac{dh}{dx}-\frac{d}{dx}\int_0^h u^2dz. \qquad (3)$$

In the continuity Eq. (1), the mean velocity is u_m and the discharge is zero for $x=0$, or at the crown, and if q is generally a certain function of x, then

$$u_m h=\int_0^x q dx. \qquad (4)$$

(2) Water surface curve, mean velocity and frictional velocity.

As the flow of rainwater on road surface is generally a laminar flow, it is assumed that, as for the velocity distribution, the following formula confirmed by the experiment of the thin sheet flow for smooth surface can also be applied universally in this case.

$$u/u^*=(u^*z/\nu)\left\{1-(z/2h)\right\}, \quad \nu: \text{kinematic viscosity.} \qquad (5)$$

Thus mean velocity u_m becomes as follows from Eq. (5).

$$u_m/u^*=(1/3)u^*h/\nu. \qquad (6)$$

Eliminating u_m and u^* from Eq. (4), (5), (6) and (3), the next equation is derived.

$$\left\{gh^3-\frac{6}{5}\left(\int_0^x qdx\right)^2\right\}\frac{dh}{dx}-gh^3J_x+\frac{12}{5}hq\int_0^x qdx+3\nu\int_0^x qdx=0.$$

Generally this equation cannot be solved, but if it is expressed by the following equation and numerical integration performed giving the boundary condition, the water surface curve can be obtained.

$$\frac{dh}{dx}=F_1(h,x)/F_2(h,x), \qquad (7)$$

where
$$F_1(h,x)=\left(\frac{h\sqrt{ghJ_x}}{\nu}\right)^2-\frac{12}{5}\cdot\frac{qh}{\nu}\cdot\frac{\int_0^x qdx}{\nu}-\frac{3\int_0^x qdx}{\nu},$$

$$F_2(h,x)=\left(\frac{h\sqrt{gh}}{\nu}\right)^2-\frac{6}{5}\left(\frac{\int_0^x qdx}{\nu}\right)^2.$$

If the relation between the water depth h and x is obtained, the mean velocity u_m is computed from Eq. (4). Furthermore, from Eq. (4) and (6)

$$u^* = \sqrt{3\nu \int_0^x q dx \Big/ h}\,, \tag{8}$$

from which the frictional velocity u^* can be computed.

2. Case when there is a longitudinal slope

(1) Fundamental equation.

As shown in Fig. 2, the crown line of road is taken as y-axis and the direction to which the rain water flows assumed as positive. The symbols are the same as those in the preceding case and v, J_y and τ_y respectively represent the velocity along y-direction, the slope and the component of the frictional stress on the road surface. If a hexahedron ABCDEFGH parallel to x, y and z-axis having a bottom surface whose two sides are small distances δx and δy is considered, as shown in Fig. 2, and the continuity and momentum equations in the x and y-directions are formed, then the case for the steady state is as follows.

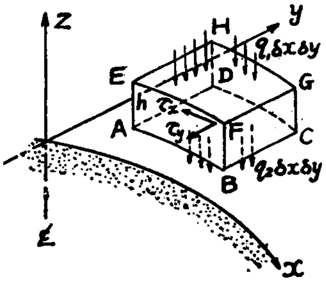

Fig. 2

$$\frac{\partial}{\partial x}\int_0^h \rho u^2 dz + \frac{\partial}{\partial y}\int_0^h \rho uv dz = -\tau_x + \rho g h J_x - \rho g h \frac{\partial h}{\partial x}\,, \tag{9}$$

$$\frac{\partial}{\partial y}\int_0^h \rho v^2 dz + \frac{\partial}{\partial x}\int_0^h \rho vu dz = -\tau_y + \rho g h J_y - \rho g h \frac{\partial h}{\partial y}\,, \tag{10}$$

$$\frac{\partial}{\partial x}\int_0^h u dz + \frac{\partial}{\partial y}\int_0^h v dz = q. \tag{11}$$

Except for the part near the boundary of the upper and lower ends of the longitudinal direction, viz. y-direction, generally there is no change in u, v and h in the y-direction, so terms of $\frac{\partial}{\partial y}$ in Eq. (9), (10) and (11) are eliminated and by putting $v^* = \sqrt{\tau_y/\rho}$, they become as simple as follows.

$$u^{*2} = g h J_x - g h \frac{dh}{dx} - \frac{d}{dx}\int_0^h u^2 dz\,, \tag{12}$$

$$v^{*2} = g h J_y - \frac{d}{dx}\int_0^h vu dz\,, \tag{13}$$

$$u_m h = \int_0^x q dx\,. \tag{14}$$

Eq. (12) and (14) are exactly the same as Eq. (3) and (4), respectively.

(2) Water suface curve, mean velocity and frictional velocity.

If the following same equations as in **case 1** are used for the velocity distribution,

$$u/u^{*}=(u^{*}z/\nu)\left\{1-(z/2h)\right\}, \tag{15}$$

$$v/v^{*}=(v^{*}z/\nu)\left\{1-(z/2h)\right\}, \tag{16}$$

then an equation exactly the same as Eq. (7) is obtained from Eq. (12)~(15),

$$\frac{dh}{dx}=F_1(h,x)/F_2(h,x). \tag{17}$$

From this, the important fact that the water surface curve does not change ever if there is a longitudinal slope is revealed.

Furthermore, the next relation is obtained from Eq. (14)~(16).

$$v^{*2}=3\nu v_m/h, \qquad \int_0^h vudz=(6/5)v_m\int_0^x qdx.$$

If these equations are put into Eq. (3), Eq. (3) becomes

$$\frac{6}{5}\int_0^x qdx\frac{dv_m}{dx}+\frac{6}{5}qv_m+\frac{3\nu v_m}{h}-ghJ_v=0\,, \tag{18}$$

and a differential equation of v_m and x is obtained. This being generally insolvable, a numerical integration must be preformed and transforming Eq. (18) by putting q=const. gives

$$\frac{d}{dx}(v_m x)=\frac{5gJ_v}{6q}h-\frac{5\nu}{2q}\cdot\frac{(v_m x)}{hx}\,. \tag{19}$$

Numerical integration is done using this equation.

As a special case when h=const., Eq. (19) is solved simply giving the following result.

$$v_m=\frac{gJ_vh^2}{3\nu}\Big/\left(1+\frac{2qh}{5\nu}\right). \tag{20}$$

If v_m is obtained, the resultant mean velocity is known as $V_m=\sqrt{u_m^2+v_m^2}$ using u_m obtained from Eq. (14), and the frictional velocity V^* becomes as follows.

$$V^*=\sqrt{3\nu V_m/h}\,. \tag{21}$$

3. Example of numerical calculation

(1) Water suface curve.

Cross sectional shape of road is expressed as follows by taking the original point at the crown of the road, η-axis vertically downwards and ξ-axis perpendicular to η-axis as shown in Fig. 3.

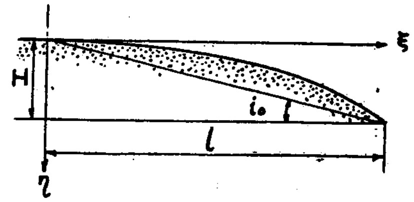

Fig. 3. Cross-sectional shape of road.

$$\eta = H(\xi/l)^n, \tag{22}$$

where l : half width of road, H : height of crown of road.

$$\frac{d\eta}{d\xi} = ni_0(\xi/l)^{n-1} \text{ from Eq. (22), where } i_0 = H/l,$$

so if the relation between J_x and x is approximated to $J_x = ni_0(x/l)^{n-1}$ and q=const., functions of F_1 and F_2 of Eq. (7) become

$$F_1(h, x) = \frac{gh^3 ni_0}{\nu^2}\left(\frac{x}{l}\right)^{n-1} - \frac{12}{5}\cdot\frac{q^2hx}{\nu^2} - \frac{3qx}{\nu},$$

$$F_2(h, x) = \frac{gh^3}{\nu^2} - \frac{6q^2x^2}{5\nu^2}.$$

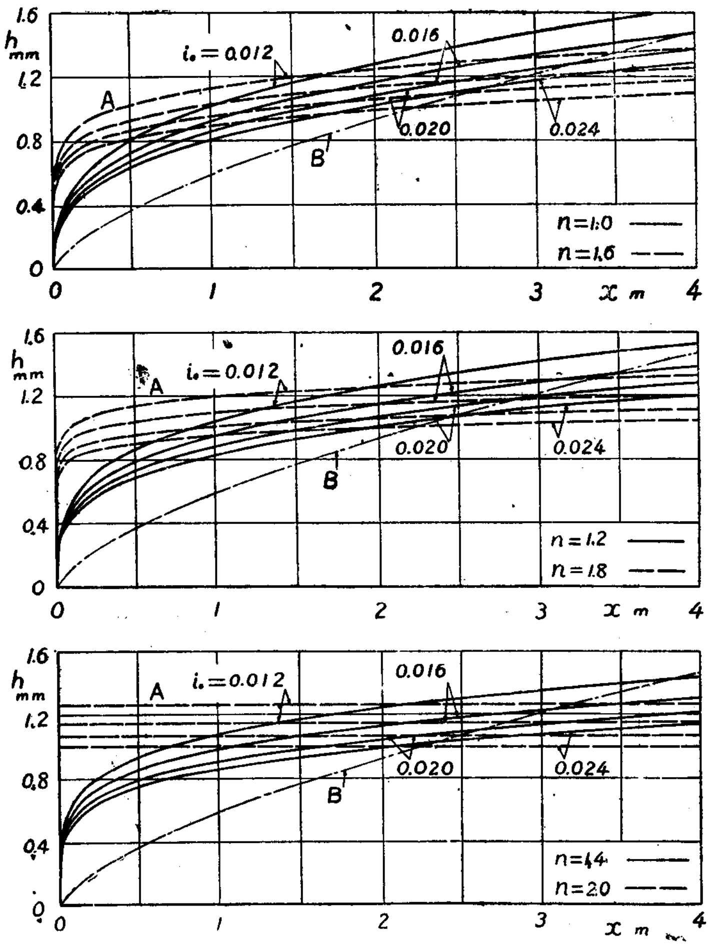

Fig. 4. Graphical representation of $F_1(h, x)=0$ and $F_2(h, x)=0$ curves.

As an example of numerical calculation, the results obtained by plotting curves of $F_1=0$ and $F_2=0$ for various combinations of $i_0=0.012$, 0.016, 0.020, 0.024 and $n=1.0$, 1.2, 1.4, 1.6, 1.8, 2.0, when $l=375$ cm, $q=0.004$ cm/s, $g=980$ cm/s^2 and $\nu=0.01$cm^2/s, are A- and B-curves in Fig. 4. B-curve represents the boundary at which ordinary flow changes to jet flow and $\frac{dh}{dx}=\infty$ at all points on this curve excluding the intersection point P with curve A and the original point, and also $\frac{dh}{dx}=0$ at all points on curve A excluding point P and the original point. The curve of the water suface can be obtained by doing numerical integration under the given boundary conditions, but when point P appears on the road surface in the case of a natural flow down, the curve of the water surface must be obtained step by step, beginning at the intersection point P and working towards the ordinary flow side of the upper stream and the jet flow side of the down stream. The value of $\frac{dh}{dx}$ at the intersection point P can be obtained by computing

$$\left(\frac{dh}{dx}\right)_p = \lim_p \frac{dF_1}{dx} \Big/ \frac{dF_2}{dx}.$$

The method of numerical integration employed here is the isoclinic method which P. Wilh. Werner* used in the calculation of the water surface

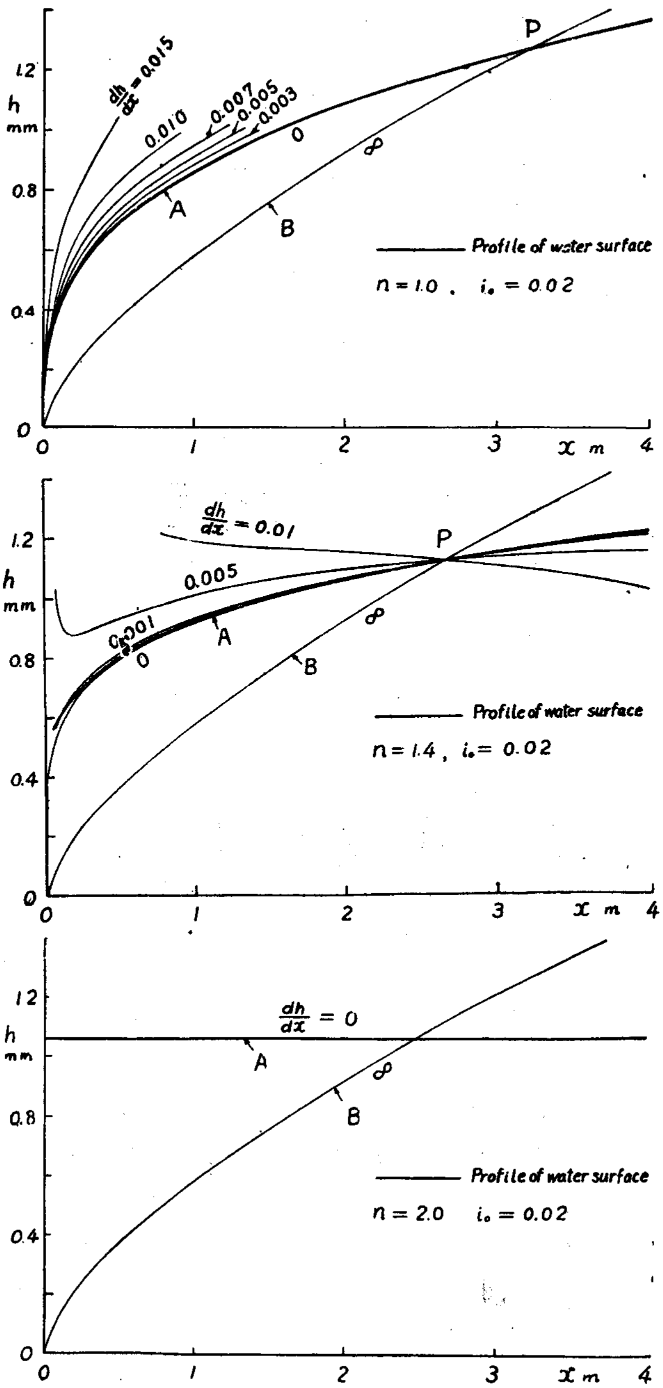

Fig. 5. Profile of water surface.

* P. Wilh. Werner: "Wasserspiegelberechnung von Kanälen bei gleichmässiger Bewegung und veränderlicher Wassermenge", Bautech., Heft 23, 1941.

profile of a channal with varying discharge. The curve of the water surface for the case of $i_0=0.02$ and $n=1.0$, 1.4, 2.0 obtained by this method is that shown in Fig. 5. What can readly be understood from this Fig. 5 are that, except for the part where x is small, the curve of the water surface is approximate to the curve of $\frac{dh}{dx}=0$, viz. A-curve, and that the water depth is constant when $n=2$, viz. the cross sectional shape of road is a quadratic parabola. Therefore, it may be said that A-curve in Fig. 4 is approximately the same as the curve of the water surface.

(2) Mean velocity and frictional velocity.

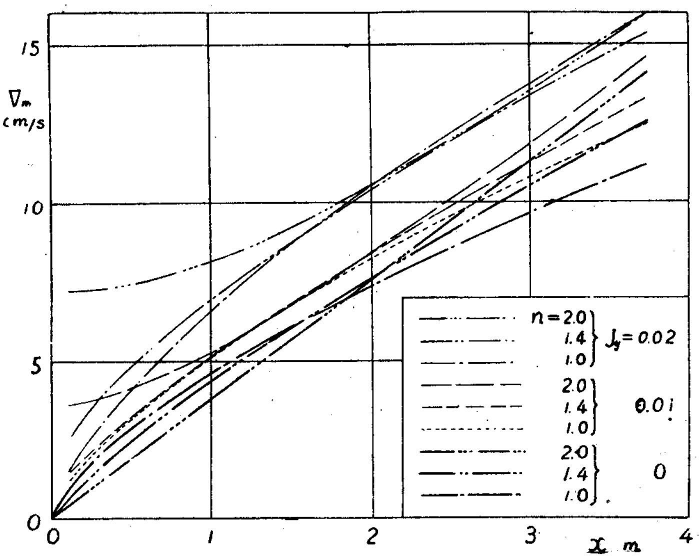

Fig. 6. Relation between mean velocity and n, J_y, x.

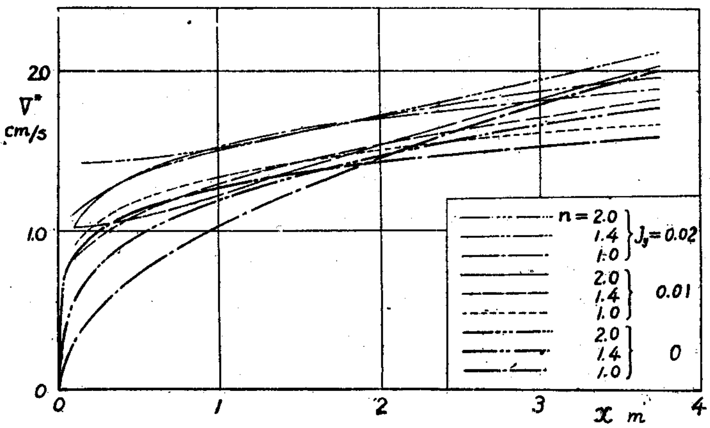

Fig. 7. Relation between frictional velocity and n, J_y, x.

If the relation between the water depth h and x is given by the above mentioned method, mean velocity u_m and frictional velocity u^* for the case when there is no longitudinl slope can be obtained from Eq. (4) and (8), and these results are the curves of $J_y=0$ shown in Fig. 6 and 7.

When there is a longitudinal slope, v_m must be obtained by a numerical integration of Eq. (19), and the isoclinic method was adopted as was done in obtaining the curve of the water surface. Broken lines in Fig. 8 represent the curves for $\frac{d(v_m x)}{dx}=0$ which express $v_m=gJ_yh^2/3\nu$, viz. the equation for uniform steady flow, Full lines are curves showing the required relation between $v_m x$ and x obtained from the condition of passing through the original point. The difference between the two is only about 4% at maximum, so if an error to this extent is admitted, then they can be obtained simply from the equation for uniform steady flow without resorting to a numerical integration by the isoclinic method. If v_m is obtained, the resultant mean velocity $V_m = \sqrt{u_m^2 + v_m^2}$ and the frictional velocity V^* is obtained from Eq. (21) using this.

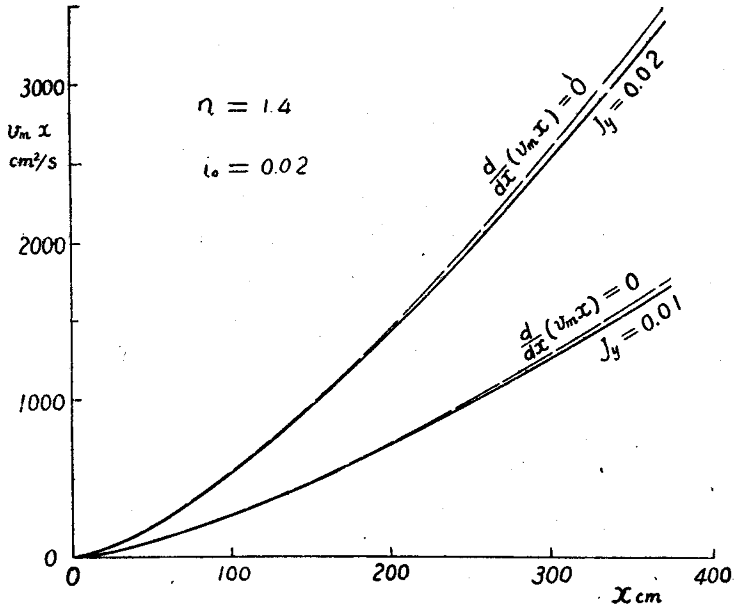

Fig. 8

Fig. 6 and 7 show the results of the calculation for the cases when the longitudinal slope J_y is 0, 0.01 and 0.02. Calculation was also done for the case of $J_y=0.002$, but except for the small values of x the curve can not be illustrated as it almost coincides with the curve of $J_y=0$.

5. Conclusion

Conclusions obtained from the above theory and the results calculated are as follows.

(1) Profile of the water surface is invariable regardless of the longitudinal slope.

(2) In the case of natural flow down, the profile of the water surface may be approximated by the curve $\frac{dh}{dx}=0$ except for the part near the crown of the road.

(3) The water depth is constant when the cross sectional shape of the road is a quadratic parabola, viz. $n=2$, and when it approaches a straight line, viz. $n=1$, the water depth is smaller near the crown and becomes greater towards the sides.

(4) When there is a longitudinal slope, both the mean velocity and the

frictional velocity become greater as can be seen from Fig. 6 and 7. The effect of the longitudinal slope on the mean velocity and the frictional velocity is the greater near the crown of the road the larger n is and the greater near the sides the smaller n is.

(5) When $i_0=1/50$, a longitudinal slope smaller than 1/500 has hardly no effect on the mean velocity and frictional velocity.

(6) As the critical tractive force, namely the value of the frictional velocity when small sand particles begin to be moved is about 1.5 cm/s for sand particles 0.2 mm in diameter and $V^*=2.0$ cm/s for those 0.6 mm in diameter, it can be seen from Fig. 7 that for the case when there is no longitudinal slope and the difference of the rainfall intensity and the infiltration capacity is $q=0.004$ cm/s and $i_0=1/50$, particles about 0.2 mm in diameter are moved on the road surface between $x=2.7$ m and sides for $n=1$ and also between near $x=2.1$ m and sides for both $n=1.4$ and $n=2.0$. It is also noted that when there is a longitudinal slope of 1/50, particles 0.2 mm in diameter are moved on the road surface between near $x=1.0$ m and sides for $n=1.0$, 1.4 and 2.0 and even particles 0.6 mm in diameter are also moved on the road surface near the sides for $n=2$.

In summarizing the above facts, in the case of paved roads where there is practically no necessity of considering the erosion due to rain water, $n=2$, namely a quadratic parabola, seems suitable because it is best to keep the water depth on road surface constant from the point of view of traffic. As longitudinal slope has no effect on the profile of the water surface, it must be determined by other conditions if water depth is in question. In the case of unpaved roads, a cross sectional shape close to a straight line, namely $n=1$, is desirable in order to make V^* as near constant as possible so as to make the grade of erosion uniform. If only the erosion of the road surface is considered, no longitudinal slope is better in view of making V^* small, but when there are inevitable conditions as configuration and etc. it is better to make the longitudinal slope as small as possible. In the example of numerical calculation explained in this paper, V^* is about 6% larger when there is a longitudinal slope of 1/100 than when there is no longitudinal slope, for the road surface with cross sectional shape of straight line, viz. $n=1$.

The writer is greatly indebted to Prof. Dr. Tojiro Ishihara for his constant instruction in completing this study.