Heat Transfer by Natural Convection in Laminar Boundary Layer on Vertical Flat Wall

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In this paper, we have studied theoretically the heat transfer by laminar natural convection from a hot vertical flat wall to the surrounding fluid. This problem has already been theoretically studied by Pohlhausen,[1] but his method is very troublesome and since he uses Schmidt and Beckmann's[2] experimental results for air, Pohlhausen's solution is not applicable to other kinds of fluids except air. We have analysed this problem for gas and liquid by means of three different approximate methods, and compared each result with the other.

1. Fundamental ordinary differential equations

For the steady flow past a vertical flat wall, which is at temperature T_1 (absolute) and contacts with the fluid at temperature T_0, the equation of motion, the equation for thermal equilibrium, and the equation of continuity are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_0)+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=k\frac{\partial^2 T}{\partial y^2}; \tag{2}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3}$$

respectively, provided that the temperature difference between the wall and the fluid is small as compared with the absolute temperature. In the above equation, T, ν, k and β are absolute temperature, kinematic viscosity, thermometric conductivity and coefficient of thermal expansion of fluid, respectively, and g is the acceleration of gravity, and velocity u and v, and coordinates x and y are shown in Fig. 1. When

1) E. Pohlhausen: Forschung, Vol. 1, 1930 p. 391.
2) E. Schmidt and W. Beckmann: Forschung, Vol. 1, 1930 p. 341.

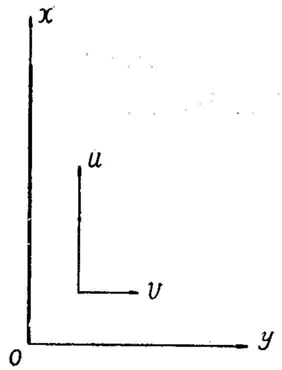

Fig. 1.

the temperature of heated surface of the vertical wall is not constant but varies in the direction of the height x, such as $T_1 = T_0 + Bx^{\varepsilon}$, where T_0, B and ε are constant, Eqs. (1) and (2) become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T_1 - T_0)\,\Theta + \nu\frac{\partial^2 u}{\partial y^2}, \quad (4)$$

$$u\left(\frac{\partial \Theta}{\partial x} + \frac{\varepsilon}{x}\Theta\right) + v\frac{\partial \Theta}{\partial y} = k\frac{\partial^2 \Theta}{\partial y^2}, \quad (5)$$

where

$$\Theta = \frac{T - T_0}{T_1 - T_0}. \quad (6)$$

The boundary conditions are $u = v = 0$, $\Theta = 1$ at $y = 0$, and $u = 0$, $\Theta = 0$ at $y = \infty$.

The partial differential equations can now be transformed into ordinary differential equations by the substitutions

$$\xi = Ax^{-\frac{1}{4}}\,y, \quad (7)$$

$$\psi = 4\nu Ax^{\frac{3}{4}}\,\varsigma(\xi), \quad (8)$$

$$\Theta = \vartheta(\xi), \quad (9)$$

where A is a function of x, and ψ is the stream-function defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

so that

$$u = 4\nu A^2 x^{\frac{1}{2}}\frac{d\varsigma}{d\xi} \quad (10)$$

$$v = -\left[4\nu A' x^{\frac{3}{4}}\,\varsigma + 4\nu AA' x^{\frac{1}{2}}\,y\frac{d\varsigma}{d\xi} + 3\nu Ax^{-\frac{1}{4}}\varsigma - \nu A^2 x^{-\frac{1}{2}}\,y\frac{d\varsigma}{d\xi}\right], \quad (11)$$

where the dash indicates the first differential coefficient of x.
The equations for ς and ϑ are

$$\frac{d^3\varsigma}{d\xi^3} + \left(3 + 4\frac{A'}{A}x\right)\varsigma\frac{d^2\varsigma}{d\xi^2} - 2\left(1 + 4\frac{A'}{A}x\right)\left(\frac{d\varsigma}{d\xi}\right)^2 + \frac{g\beta(T_1 - T_0)}{4\nu^2 A^4}\vartheta = 0, \quad (12)$$

$$\frac{d^2\vartheta}{d\xi^2} + \frac{\nu}{k}\left(3 + 4\frac{A'}{A}x\right)\varsigma\frac{d\vartheta}{d\xi} - 4\frac{\nu}{k}\frac{d\varsigma}{d\xi}\vartheta = 0. \quad (13)$$

When we put

$$A = \left[\frac{g\beta(T_1 - T_0)}{4\nu^2}\right]^{\frac{1}{4}} = \left[\frac{g\beta B}{4\nu^2}\right]^{\frac{1}{4}} x^{\varepsilon/4} \quad (14)$$

in Eqs. (12) and (13), they become

$$\frac{d^3\varsigma}{d\xi^3} + (3+\varepsilon)\varsigma\frac{d^2\varsigma}{d\xi^2} - 2(1+\varepsilon)\left(\frac{d\varsigma}{d\xi}\right)^2 + \vartheta = 0, \quad (15)$$

$$\frac{d^2\vartheta}{d\xi^2}+P_r(3+\varepsilon)\varsigma\frac{d\vartheta}{d\xi}-4P_r\varepsilon\frac{d\varsigma}{d\xi}\vartheta=0, \tag{16}$$

where P_r is Prandtl number (equal to ν/k), and the boundary conditions are

$$\left.\begin{array}{llll} \xi=0: & \varsigma=0 & \dfrac{d\varsigma}{d\xi}=0 & \vartheta=1 \\ \xi=\infty: & \dfrac{d\varsigma}{d\xi}=0 & & \vartheta=0 \end{array}\right\}. \tag{17}$$

Eqs. (15) and (16) are the fundamental ordinary differential equations, and they will be solved with the boundary conditions (17). Now, suppose ε is equal to zero, that is, if the surface temperature is constant, two equations become

$$\frac{d^3\varsigma}{d\xi^3}+3\varsigma\frac{d^2\varsigma}{d\xi^2}-2\left(\frac{d\varsigma}{d\xi}\right)^2+\vartheta=0, \tag{18}$$

$$\frac{d^2\vartheta}{d\xi^2}+3P_r\varsigma\frac{d\vartheta}{d\xi}=0, \tag{19}$$

which have been given by Pohlhausen. Pohlhausen has solved these equations for air in the form of power series, adopting Schmidt and Beckmann's experimental results as the value of $\left.\frac{d^2\varsigma}{d\xi^2}\right|_{\xi=0}$ and $\left.\frac{d\vartheta}{d\xi}\right|_{\xi=0}$. Therefore its solution can not be applied to other fluids. But we think that Eqs. (15) and (16) or Eqs. (18) and (19) may be solved for any kinds of fluids by means of Runge-Kutta's[3] method, in stead of Pohlhausen's, but since this method is very troublesome to calculate, we have analysed by the following three approximate methods.

2. The first method

When the fluid flows along a wall, the boundary layer exists near the surface of the wall. Now let δ be the thickness of the velocity layer and δ' the thickness of the temperature layer, then the momentum equation for δ becomes

$$\frac{d}{dx}\int_0^\delta \rho u^2 dy=\int_0^\delta \rho g\beta(T-T_0)\,dy-\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{20}$$

and the energy equation for δ' becomes

$$JC_p\left[\frac{d}{dx}\int_0^{\delta'}\rho uT dy-T_0\frac{d}{dx}\int_0^{\delta'}\rho u dy\right]=-J\lambda\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{21}$$

Since the pressure is constant (the variation of pressure with height is neglected), the density ρ for fluid is expressed by the equation

$$\rho=\rho_0\Big/\Big[1+\beta(T-T_0)\Big], \tag{22}$$

3) C. Runge: Mathematische Annalen, Vol. 46, 1895 p. 167.
W. Kutta: Zeit. f. Mathe. u. Physik, Vol. 46, 1901 p. 435.

and in the case of gas

$$\rho T = \rho_0 T_0. \tag{23}$$

For gas, Eqs. (20) and (21) are transformed by using Eq. (23):

$$\frac{d}{dx}\int_0^\delta \frac{T_0}{T}u^2 dy = g\int_0^\delta \left(1-\frac{T_0}{T}\right)dy-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{20'}$$

$$\frac{d}{dx}\int_0^{\delta'} u\left(1-\frac{T_0}{T}\right)dy = -k\left[\frac{\partial}{\partial y}\left(\frac{T}{T_0}\right)\right]_{y=0}. \tag{21'}$$

For liquid, β is less than the thermal expansion of gas, and is so small that $\beta(T-T_0)$ is less than $(T-T_0)/T_0$: therefore, we can neglect the variation of density ρ with temperature when the temperature difference (T_1-T_0) is small as compared with T_0; hence Eqs. (20) and (21) become

$$\frac{d}{dx}\int_0^\delta u^2 dy = g\int_0^\delta \beta(T-T_0)\,dy-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{20''}$$

$$\frac{d}{dx}\int_0^{\delta'} u(T-T_0)\,dy = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{21''}$$

Here the suitable approximate expressions for u and T which satisfy the boundary conditions are demanded. For this purpose, we approximate u and T to the following expressions containing unknown factors:

$$u = A_0(x)+A_1(x)\,y+A_2(x)\,y^2+A_3(x)\,y^3, \tag{24}$$

$$T = B_0(x)+B_1(x)\,y+B_2(x)\,y^2+B_3(x)\,y^3, \tag{25}$$

and decide A_0, B_0, A_1, B_1, etc. from the following boundary conditions:

For u—

(i) $u=0$ at $y=0$

(ii) $u=v=0$ at $y=0$, hence from Eq. (1)

$$g\beta(T_1-T_0)+\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}=0$$

(iii) $u=0$ at $y=\delta$

(iv) $\frac{\partial u}{\partial y}=0$ at $y=\delta$.

For T—

(i) $T=T_1$ at $y=0$

(ii) $u=v=0$ at $y=0$, hence from Eq. (2)

$$\left|\frac{\partial^2 T}{\partial y^2}\right|_{y=0}=0$$

(iii) $T=T_0$ at $y=\delta'$

(iv) $\frac{\partial T}{\partial y}=0$ at $y=\delta'$.

Then we obtain u and T as functions of y/δ and y/δ':

$$u = \frac{g\beta(T_1 - T_0)}{4\nu}\delta^2\left[\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3\right] \tag{26}$$

$$\theta = \frac{T - T_0}{T_1 - T_0} = 1 - \frac{3}{2}\frac{y}{\delta'} + \frac{1}{2}\left(\frac{y}{\delta'}\right)^3. \tag{27}$$

Because the natural convection occurs owing to the difference of gravitational forces due to the temperature difference in the fluid, we can consider the velocity field is closely related with the temperature field. Now, we put δ as approximately equal to δ', and we use only the energy equation for the temperature layer, and assume $(T_1 - T_0)/T_0 < 1$ for gas, so that from Eq. (27) we can obtain

$$\frac{T_0}{T} = 1 - \left(\frac{T_1 - T_0}{T_0}\right)\left[1 - \frac{3}{2}\frac{y}{\delta'} + \frac{1}{2}\left(\frac{y}{\delta'}\right)^3\right] \tag{27'}$$

to a sufficient approximation. With these expressions for u and T, the energy equation (21′) becomes

$$\frac{1}{105}\frac{g(T_1 - T_0)}{\nu T_0}\frac{d}{dx}(\delta'^3) = \frac{3}{2}k\frac{1}{\delta'}, \tag{28}$$

when the temperature of surface is constant.[4] (The following discussion is only for the case of the constant temperature of surface.) This equation can be solved easily, and we obtain the following result from the boundary condition that $\delta' = 0$ at $x = 0$:

$$\delta' = [210]^{1/4} P_r^{-1/4}\left[\frac{g(T_1 - T_0)}{\nu^2 T_0}\right]^{-1/4} x^{1/4}. \tag{29}$$

Similarly we obtain for liquid by using Eq. (21″),

$$\delta' = [210]^{1/4} P_r^{-1/4}\left[\frac{g\beta(T_1 - T_0)}{\nu^2}\right]^{-1/4} x^{1/4}. \tag{29'}$$

The local coefficient of surface heat transmission at x is given as follows:

$$\alpha_x = -\frac{\lambda}{T_1 - T_0}\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{30}$$

And the mean coefficient of surface heat transmission from zero to x is

$$\alpha_m = \frac{1}{x}\int_0^x \alpha_x dx. \tag{31}$$

When we use Eqs. (27), (30), (31) and (29) or Eq. (29′), we obtain the following relation among non dimensional numbers Nu_m, P_r and G_r for both gas and liquid:

$$Nu_m = 0{\cdot}525\, P_r^{1/4}\, G_r^{1/4}, \tag{32}$$

4) For the case where the temperature of surface is not constant, we have prepared another paper that was read at the lecture meeting of J. S. M. E. at Kobe, Dec. 3, 1949.

where

$$Nu_m = \frac{\alpha_m x}{\lambda} \quad \text{Nusselt number}$$

$$P_r = \frac{\nu}{k} \quad \text{Prandtl number}$$

$$G_r = \frac{g\beta(T_1 - T_0)\,x^3}{\nu^2} \quad \text{Grashoff number,}$$

and in the case of gas,

$$G_r = \frac{g(T_1 - T_0)\,x^3}{\nu^2 T_0}.$$

This result agrees well with Schmidt and Beckmann's experimental results for air which are shown in Fig. 3, but the velocity gradient on the surface in the direction of y is about 8% higher than their experimental results.

3. The second method

In the first method, we have assumed that u is independent of T and that the thickness of velocity layer is equal to that of temperature layer. But for the natural convection, the velocity distribution is closely related with the temperature distribution, so that it becomes necessary to consider the relation between temperature field and velocity field. For this purpose, we assume that velocity u is a function of temperature T, that is,

$$u = f(T).$$

Then Eq. (1) becomes

$$\frac{du}{dT}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = g\beta(T - T_0) + \nu\frac{\partial^2 u}{\partial y^2},$$

and by putting Eq. (2) into this equation,

$$k\frac{\partial^2 T}{\partial y^2}\frac{du}{dT} = g\beta(T - T_0) + \nu\left[\left(\frac{\partial T}{\partial y}\right)^2\frac{d^2 u}{dT^2} + \frac{\partial^2 T}{\partial y^2}\frac{du}{dT}\right],$$

and by using Prandtl number, the above equation becomes

$$\frac{d^2 u}{dT^2} + \frac{P_r - 1}{P_r}\frac{\partial^2 T}{\partial y^2}\left(\frac{\partial T}{\partial y}\right)^{-2}\frac{du}{dT} + \frac{g\beta(T - T_0)}{\nu}\left(\frac{\partial T}{\partial y}\right)^{-2} = 0. \tag{33}$$

By substituting Θ for T, Eq. (33) is transformed as follows:

$$\frac{d^2 u}{d\Theta^2} + \frac{P_r - 1}{P_r}\frac{\partial^2 \Theta}{\partial y^2}\left(\frac{\partial \Theta}{\partial y}\right)^{-2}\frac{du}{d\Theta} + \frac{g\beta(T_1 - T_0)}{\nu}\,\Theta\left(\frac{\partial \Theta}{\partial y}\right)^{-2} = 0. \tag{33'}$$

This equation expresses that, when $\partial\Theta/\partial y$ and $\partial^2\Theta/\partial y^2$ are given as functions of Θ, the relation between u and T is obtainable, and that this relation varies with Prandtl number which is constant for gas. [For liquid, Prandtl number varies with temperature, so that Eq. (33') can not be solved so easily. But if we assume

$(T_1 - T_0)/T_0 < 1$, we may adopt its mean value as Prandtl number of liquid.] Now, from the above assumption $u = f(T)$,

$$\frac{\partial u}{\partial y} = \frac{du}{d\theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{d^2 u}{d\theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{du}{d\theta}\frac{\partial^2 \theta}{\partial y^2}.$$

When we transform Eq. (33′) with these relations, the following equation is obtained:

$$\frac{\partial^2 u}{\partial y^2} - \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2}\left(\frac{\partial \theta}{\partial y}\right)^{-1}\frac{\partial u}{\partial y} + \frac{g\beta(T_1 - T_0)}{\nu}\theta = 0. \tag{34}$$

Here we assume that δ is equal to δ' as in the 1st method, and both u and T are functions of y/δ only, then Eq. (34) becomes

$$\frac{d^2 u}{d\eta^2} - \frac{1}{P_r}\frac{d^2 \theta}{d\eta^2}\left(\frac{d\theta}{d\eta}\right)^{-1}\frac{du}{d\eta} + \frac{g\beta(T_1 - T_0)}{\nu}\theta\delta^2 = 0, \tag{35}$$

where

$$\eta = y/\delta.$$

We approximate θ to the following expressions which satisfy the boundary conditions as in the 1st method:

$$\theta_1 = 1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3, \tag{36}$$

or

$$\theta_2 = 1 - 2\eta + 2\eta^3 - \eta^4. \tag{37}$$

Putting Eq. (36) or (37) into Eq. (35), we can solve the differential equation from the boundary conditions that $u=0$ at $y=0$ and $u=0$ at $y=\delta$.

(I) Using Eq. (36), we obtain:

1. for $P_r=2$,

$$u = g\beta(T_1 - T_0)\nu^{-1}\delta^2[1{\cdot}429\{1/2\sin^{-1}\eta + 1/2\,\eta(1-\eta^2)^{1/2}\} - 1/4(\sin^{-1}\eta)^2 - 1/2\,\eta(1-\eta^2)^{1/2} - 7/6\,\eta + 1/4\,\eta^2 + 4/9\,\eta^3 - 1/30\,\eta^5] \tag{38}$$

2. for $P_r=1$,

$$u = g\beta(T_1 - T_0)\nu^{-1}\delta^2[\{1/3(\eta+1)^3 - (\eta+1)^2\}\log(\eta+1) - 1/20\,\eta^5 + 1/12\,\eta^3 - 1/9(\eta+1)^3 + 1/2(\eta+1)^2 + 182{\cdot}132/720(-1/3\,\eta^3 + \eta) - 7/18] \tag{39}$$

3. for $P_r=1/2$,

$$u = 1/2\,g\beta(T_1 - T_0)\nu^{-1}\delta^2[1/25(\eta+1)^5 - 1/4(\eta+1)^4 + 4/9(\eta+1)^3 + 1/4\,\eta^4 - 1/3\,\eta^3 - 1/2\,\eta^2 + \eta - \{1/5(\eta+1)^5 - (\eta+1)^4 + 4/3(\eta+1)^3\}\log(\eta+1) - 250{\cdot}579/480(1/5\,\eta^5 - 2/3\,\eta^3 + \eta) - 211/900] \tag{40}$$

4. for $P_r=1/3$,

$$u = 1/2\, g\beta(T_1-T_0)\nu^{-1}\delta^2 [3/8\,\{1/7\,(1-\eta)^7-(1-\eta)^6+12/5\,(1-\eta)^5 -2(1-\eta)^4\}\log(1-\eta)-3/8\,\{1/49\,(1-\eta)^7-1/6\,(1-\eta)^6+12/25\,(1-\eta)^5 -1/2\,(1-\eta)^4\}+3/8\{1/7\,(1+\eta)^7-(1+\eta)^6+12/5\,(1+\eta)^5-2(1+\eta)^4\}\log(1+\eta) +0{\cdot}070\,(1+\eta)^7-0{\cdot}605\,(1+\eta)^6+1{\cdot}972\,(1+\eta)^5-2{\cdot}773\,(1+\eta)^4+(1+\eta)^3 +(1+\eta)^2-0{\cdot}727]. \tag{41}$$

(II) Using Eq. (37), we obtain

1. for $P_r=2$,

$$u = 1/4096\, g\beta(T_1-T_0)\nu^{-1}\delta^2 [1/21\,(4\eta+2)^6-8/7\,(4\eta+2)^5+8\,(4\eta+2)^4 +32\,(4\eta+2)^3-432\,(4\eta+2)^2-3937{\cdot}595/16\,\{2/5\,(4\eta+2)^{5/2}-4\,(4\eta+2)^{3/2}\} -849{\cdot}916] \tag{42}$$

2. for $P_r=1$,

$$u = 1/64\, g\beta(T_1-T_0)\nu^{-1}\delta^2 [4\,\{4/3\,\eta^6-4\eta^5+3/2\,\eta^4+13/3\,\eta^3-2\eta^2-3\eta\} +3/16\,\{1/16\,(4\eta+2)^4-4/3\,(4\eta+2)^3+9\,(4\eta+2)^2\}-3/16\,\{1/4\,(4\eta+2)^4 -4\,(4\eta+2)^3+18\,(4\eta+2)^2\}\log(4\eta+2)+20{\cdot}898\,(\eta^4-2\eta^3+2\eta)+0{\cdot}781] \tag{43}$$

3. for $P_r=1/2$,

$$u = 1/4\, g\beta(T_1-T_0)\nu^{-1}\delta^2 [0{\cdot}178\,(16/7\,\eta^7-8\eta^6+36/5\,\eta^5+4\eta^4-8\eta^3+4\eta) +32/441\,(1-\eta)^7-2/27\,(1-\eta)^6-2/25\,(1-\eta)^5-2/9\,\{16/7\,(1-\eta)^7-8(1-\eta)^6 +36/5\,(1-\eta)^5\}\log(1-\eta)-1/4608\,\{1/7\,(4\eta+2)^7-4\,(4\eta+2)^6+216/5\,(4\eta+2)^5 -216\,(4\eta+2)^4+432\,(4\eta+2)^3\}\log(4\eta+2)+1/4608\,\{1/49\,(4\eta+2)^7 -2/3\,(4\eta+2)^6+216/25\,(4\eta+2)^5-54\,(4\eta+2)^4+144\,(4\eta+2)^3\}+0{\cdot}140]. \tag{44}$$

Next, we calculate the thickness of boundary layer by means of the energy equation, and obtain the following result:

$$\delta = P(P_r)\left[\frac{g\beta(T_1-T_0)}{\nu^2}\right]^{-1/4} x^{1/4}, \tag{45}$$

where $P(P_r)$ is a function of Prandtl number only and its value is given in Table 1.

Table 1.

P_r	2	1	½	⅓
P_1	3·125	3·817	4·739	5·412
P_2	3·912	4·833	6·083	

The local coefficient of surface heat transmission α_x and its average value α_m are obtained from Eqs. (30) and (31). Then the relation among non dimensional numbers Nu_m, P_r and G_r becomes as follows:

$$Nu_m = K(P_r)\, G_r^{1/4}. \tag{46}$$

$K(P_r)$ is a function of Prandtl number only and its value is given in Table 2. Now we plot the value of K_1 and K_2 in the logarithmic graph as Fig. 2; then the calculated points are situated on a straight line which has the same gradient in both cases (K_1 and K_2) for Prandtl number, and the gradient is about 1/3. Hence, Eq. (46) becomes

Table 2.

P_r	2	1	½	⅓
K_1	0·640	0·524	0·422	0·369
K_2	0·681	0·552	0·438	

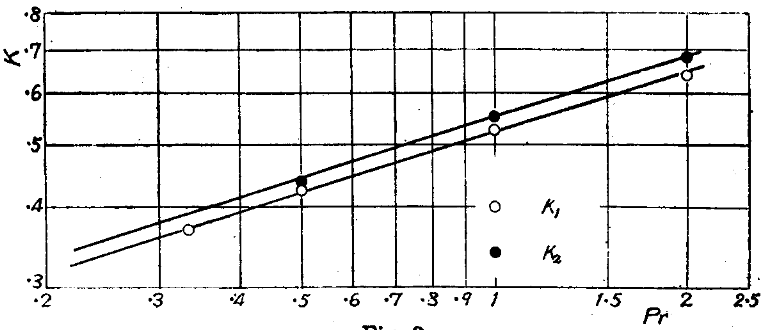

Fig. 2.

(I) $$Nu_m = 0{\cdot}525\, P_r^{1/3}\, G_r^{1/4}, \tag{47}$$

(II) $$Nu_m = 0{\cdot}550\, P_r^{1/3}\, G_r^{1/4}. \tag{48}$$

These relations show that Nusselt number is proportional to $G_r^{1/4}$ and $P_r^{1/3}$. But the relation obtained by the 1st method indicates in Eq. (32) that Nusselt number is proportional to $G_r^{1/4} P_r^{1/4}$, and this relation well agrees with the expression obtained from the law of similarity, if it is assumed that the natural convection is a "slack flow" and in the equation of motion the terms of inertia force are neglected when compared with the gravitaitional force and the frictional force. When we compare Eq. (47) or (48) with Eq. (32), we find the difference between them that while one is proportional to $P_r^{1/3}$, the other is proportional to $P_r^{1/4}$. This difference is due to the fact that, in the 2nd method, u is dependent of T. That is to say, we solve the equation of motion and temperature distribution according to the relation $u=f(T)$; while in the 1st method, we use only the energy equation and assume u is independent of T, except a relation that

$$\left.\begin{array}{c}\text{at } \ y=0, \ \ u=v=0\,; \ \ \text{hence from Eq. (1)} \\ g\beta(T_1-T_0)+\nu\left|\dfrac{\partial^2 u}{\partial y^2}\right|_{y=0}=0.\end{array}\right\} \tag{49}$$

But since the relation in Eq. (49) is applicable likewise in the case of "slack flow", we can say that the relation deduced by the 1st method is owing to the assumption that the natural convection is a "slack flow". On the other hand, we believe that the relation obtained from the 2nd method satisfy more completely the fundamental differential equations, and it is close to the experimental results. However, we can not compare very satisfactorily our theoretical results with the experimental results, since those kinds of fluids with which many investigators have conducted experiments are very few, namely, air, water[5] and one kind of oil.[6] Now, when we compare Eq. (48) with the experimental results relative to air (P_r=0.733), we

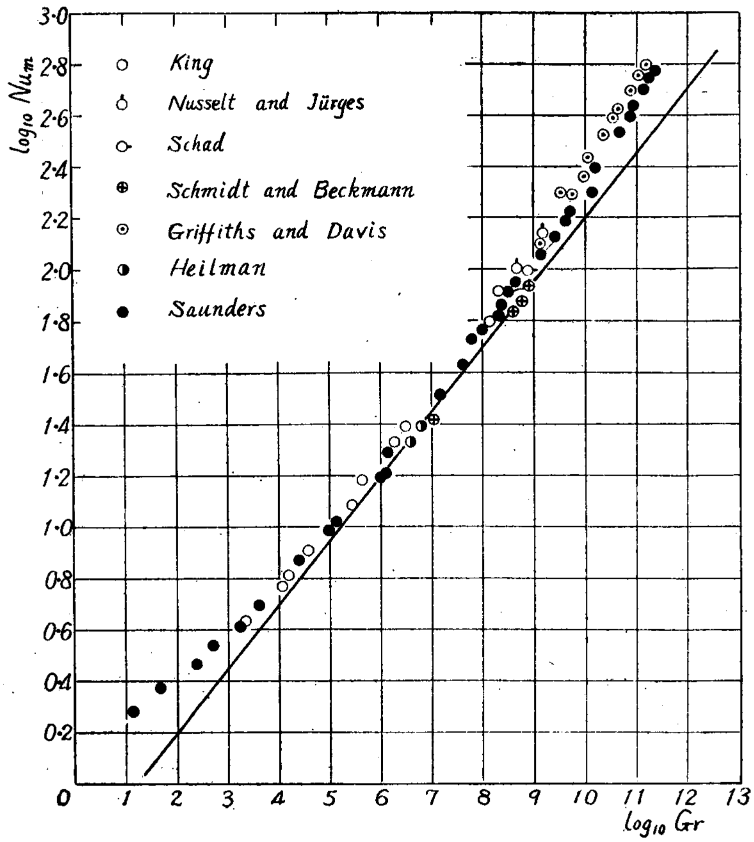

Fig. 3.

5) C. W. Rice: Trans. Am. Inst. Elec. Engrs, 42, 1923 p. 653.
6) H. H. Lorenz: Zeit. f. tech. Physik, Jahrg. 15, 1934 p. 362.

obtain Fig. 3. In this figure, the experimental results are shown by points, and the straight line shows Eq. (48). From this figure, we see that Eq. (48) agrees sufficiently well with the experimental results (to be precise, Eq. (48) is a few % less than experimental results) over the range of $\log_{10} G_r$ from 4 to 9.[7]

4. The third method

In the two methods mentioned above, we assume that the thickness δ of velocity layer is equal to the thickness δ' of temperature layer. But δ is not generally equal to δ'. Hence we must consider what difference appears on δ'/δ corresponding to the Prandtl number. For this purpose, we use the momentum equation (20″) and the energy equation (21″) for fluid as simultaneous equations, and approximate u and T to the following expressions as in the 1st method:

(I)
$$u_1 = \frac{g\beta(T_1 - T_0)}{4\nu}\delta^2\left[\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3\right], \tag{50}$$

$$\Theta_1 = 1 - \frac{3}{2}\frac{y}{\delta'} + \frac{1}{2}\left(\frac{y}{\delta'}\right)^3, \tag{51}$$

(II)
$$u_2 = \frac{g\beta(T_1 - T_0)}{4\nu}\delta^2\left[\frac{y}{\delta} - 3\left(\frac{y}{\delta}\right)^2 + 3\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right], \tag{50'}$$

$$\Theta_2 = 1 - 2\frac{y}{\delta'} + 2\left(\frac{y}{\delta'}\right)^3 - \left(\frac{y}{\delta'}\right)^4. \tag{51'}$$

Putting Eqs. (50) and (51) or Eqs. (50′) and (51′) into Eqs. (20″) and (21″), we calculate as follows:

(I) When Prandtl number is larger than unity, namely, $\delta'/\delta<1$,

$$\frac{1}{210}\frac{g\beta(T_1 - T_0)}{\nu^2}\frac{d(\delta^5)}{dx} = (3\chi - 2)\,\delta \tag{52}$$

$$\frac{1}{4}\frac{g\beta(T_1 - T_0)}{\nu}\frac{d}{dx}\left[\chi\left(\frac{1}{10}\chi - \frac{1}{12}\chi^2 + \frac{3}{140}\chi^3\right)\delta^3\right] = \frac{3}{2}k\frac{1}{\chi}\frac{1}{\delta}, \tag{53}$$

and when Prandtl number is smaller than unity, namely, $\delta'/\delta>1$,

$$\frac{1}{210}\frac{g(T_1 - T_0)}{\nu^2 T_0}\frac{d(\delta^5)}{dx} = \left(6 - 6\frac{1}{\chi} + \frac{1}{\chi^3}\right)\delta \tag{54}$$

$$\frac{1}{4}\frac{g(T_1 - T_0)}{\nu T_0}\frac{d}{dx}\left[\left(\frac{1}{12} - \frac{1}{20}\frac{1}{\chi} + \frac{1}{210}\frac{1}{\chi^3}\right)\delta^3\right] = \frac{3}{2}k\frac{1}{\chi}\frac{1}{\delta}, \tag{55}$$

where $\chi=\delta'/\delta$, i.e., χ is the ratio between the thickness of the temperature and the velocity layer. If we assume that χ is independent of x, we can solve δ and χ from Eqs. (52) and (53) for $P_r>1$, and Eqs. (54) and (55) for $P_r<1$. Hence we obtain the following:

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7) In the range larger than 9, the boundary layer changes to the turbulent boundary layer, and as for the range smaller than 4, a paper has been read by us at the lecture meeting of J. S. M. E. at Osaka, March 26, 1950.

$P_r>1$, $\chi<1$

$$21\,(3\chi-2)\left(\frac{1}{10}\chi^3-\frac{1}{12}\chi^4+\frac{3}{140}\chi^5\right)=\frac{1}{P_r} \tag{56}$$

$$\delta=[168]^{1/4}\,(3\chi-2)^{1/4}\,[g\beta(T_1-T_0)\nu^{-2}]^{-1/4}\,x^{1/4}\,, \tag{57}$$

$P_r<1$, $\chi>1$

$$21\left(6-6\frac{1}{\chi}+\frac{1}{\chi^3}\right)\left(\frac{1}{12}-\frac{1}{20}\frac{1}{\chi}+\frac{1}{210}\frac{1}{\chi^3}\right)\chi=\frac{1}{P_r} \tag{58}$$

$$\delta=[168]^{1/4}\left(6-6\frac{1}{\chi}+\frac{1}{\chi^3}\right)^{1/4}\left[\frac{g(T_1-T_0)}{\nu^2 T_0}\right]^{-1/4}x^{1/4}\,. \tag{59}$$

(II) $P_r>1$, $\chi<1$

$$\frac{2268}{5}\left(\frac{3}{10}\chi-\frac{1}{6}\right)\left(\frac{1}{15}\chi-\frac{1}{14}\chi^2+\frac{9}{280}\chi^3-\frac{1}{180}\chi^4\right)\chi^2=\frac{1}{P_r} \tag{56'}$$

$$\delta=\left[\frac{36288}{5}\right]^{1/4}\left(\frac{3}{10}\chi-\frac{1}{6}\right)^{1/4}\left[g\beta(T_1-T_0)\nu^{-2}\right]^{-1/4}x^{1/4}\,, \tag{57'}$$

$P_r<1$, $\chi>1$

$$\frac{2268}{5}\left(\frac{5}{6}-\frac{1}{\chi}+\frac{1}{2}\frac{1}{\chi^3}-\frac{1}{5}\frac{1}{\chi^4}\right)\left(\frac{1}{20}-\frac{1}{30}\frac{1}{\chi}+\frac{1}{140}\frac{1}{\chi^3}-\frac{1}{504}\frac{1}{\chi^4}\right)\chi=\frac{1}{P_r} \tag{58'}$$

$$\delta=\left[\frac{36288}{5}\right]^{1/4}\left(\frac{5}{6}-\frac{1}{\chi}+\frac{1}{2}\frac{1}{\chi^3}-\frac{1}{5}\frac{1}{\chi^4}\right)^{1/4}\left[\frac{g(T_1-T_0)}{\nu^2 T_0}\right]^{-1/4}x^{1/4}\,. \tag{59'}$$

Fig. 4 shows the relation between χ and Prandtl number. When we calculate the surface heat transmission according to Eqs. (30) and (31), we obtain the following relation among non dimensional numbers Nu_m, P_r and G_r:

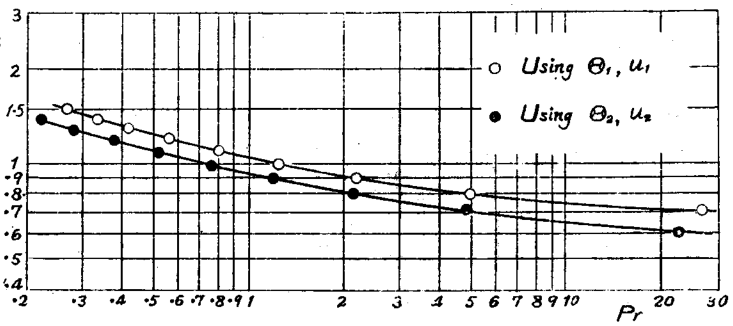

Fig. 4.

$$Nu_m=M(P_r)\,G_r^{1/4}\,. \tag{60}$$

$M(P_r)$ is a function of Prandtl number only, and its values are plotted in the logarithmic graph as Fig. 5, where a dotted line shows $K_2(P_r)$ in Eq. (46). From this figure, we find that the relation between Nusselt number and Prandtl number

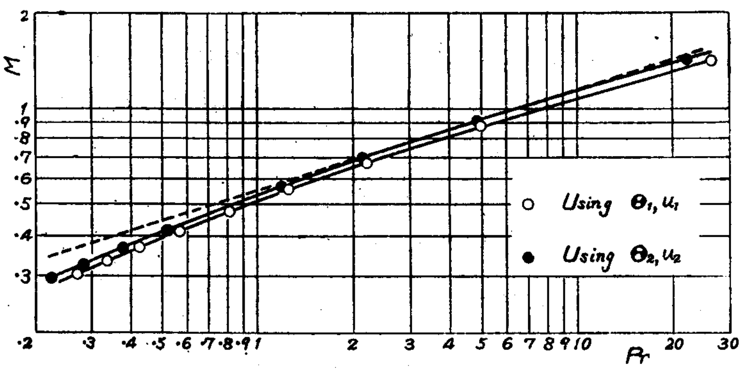

Fig. 5.

obtained by the 3rd method agrees well with the relation obtained from the 2nd method except its absolute value. And we also find some interesting facts. They are: when Prandtl number is smaller than 0.6 or larger than 20, the relation differs from Eq. (48), and Nusselt number is not proportional to $Pr^{1/3}$. But we can not compare these facts with the experimental results, because experiments have been carried out only in relation to a few kinds of fluids such as air, water and one kind of oil as already mentioned.