On the Solution of a Circular Plate of Non-uniform Thickness

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Synopsis

A method giving an approximate solution is explained by the application of a formula analogous to that of the slope deflection method used in the solution of rigid frames to the case when a symmetrical load is applied to a circular plate whose thickness is only a function of the radial distance.

1. Introduction

Circular plates of non-uniform thickness are sometimes encountered in the design of machine parts, such as diaphragms of steam turbines and pistons of reciprocating engines. The thickness of such plates is usually a function of the radial distance, and the acting load is symmetrical with respect to the center of the plate.

The investigation of the bending of the circular plates of non-uniform thickness was made about such cases first by H. Holzer[1] and latter in detail by O. Pichler[2], R. Gran Olsson[3] and H. D. Conway[4].

Criticizing some of these investigations, Pichler's method by the form of power series and Olsson's by the hypergeometric series are both very complicated as they faithfully solve according to the equation of thickness, viz. flexural rigidity.

For the solution of rectangular plates of non-uniform thickness, the author has formerly derived a formula analogous to the slope deflection method used in the solution of rigid frames and by applying this formula has shown that a satisfactorily approximate solution can be easily obtained without troublesome calculation[5]. A solution was done following the same principle in the case of a circular plate of non-uniform thickness, and as satisfactory results were obtained, a general outline will be explained in the following.

2. Derivation of the fundamental equation.

As shown in Fig. 1, the case when a ring plate with a uniform thickness is subjected to a uniform load p symmetrical about the center will be considered.

As the differential equation of deflection surface is unrelated to θ in this case,

$$\frac{d}{dr}\left\{r\frac{d}{dr}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)\right\}=\frac{p}{N} \qquad (1)$$

where, w: deflection

$N=Eh^3/12(1-\nu^2)$: flexural rigidity

h: thickness of plate

ν: Poisson's ratio

Solving eq. (1) and putting $r/a=\rho$ gives,

$$w=\frac{pa^4}{64N}\rho^4+C_1+C_2\rho^2+C_3\rho^2\log\rho+C_4\log\rho \qquad (2)$$

From the above equation

$$\frac{dw}{dr}=\frac{1}{a}\left\{\frac{pa^4}{16N}\rho^3+2C_2\rho+C_3\rho(1+2\log\rho)+\frac{C_4}{\rho}\right\} \qquad (3)$$

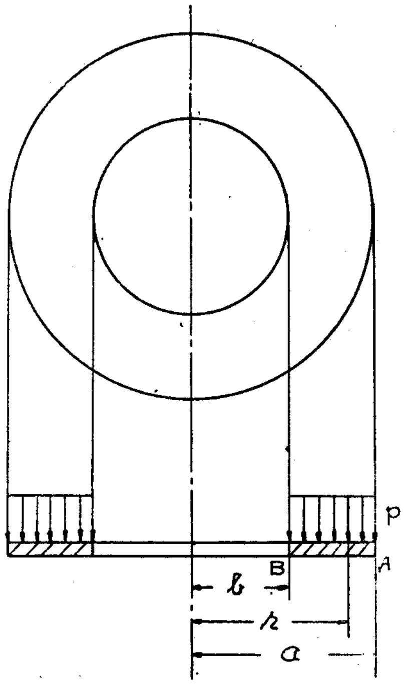

Fig. 1

$$M_r=-\frac{2(1+\nu)}{a^2}N\left[\frac{pa^4}{16N}\cdot\frac{3+\nu}{2(1+\nu)}\rho^2+C_2+C_3\left\{\frac{3+\nu}{2(1+\nu)}+\log\rho\right\}-\frac{C_4}{\rho^2}\cdot\frac{1-\nu}{2(1+\nu)}\right] \qquad (4)$$

$$p_r=-\frac{4N}{a^3}\cdot\frac{C_3}{\rho}-\frac{pa}{2}\rho \qquad (5)$$

Where, M_r = radial bending moment

p_r = radial shearing force

Now, let the deflection and the deflection angle at both circumferences $\rho=\alpha=1$ and $\rho=\beta$ $(0<\beta<1)$ be represented as follows,

$$\begin{aligned} A:&\quad \rho=\alpha=1 \quad w=\delta_A,\quad dw/dr=\theta_A \\ B:&\quad \rho=\beta \qquad\quad w=\delta_B,\quad dw/dr=\theta_B \end{aligned} \qquad (6)$$

For the sake of convenience the next notations will be used,

$(3+\nu)/2(1+\nu)=e$, $(1-\nu)/2(1+\nu)=f$, $\alpha^2-\beta^2=g$, $\alpha^2\log\alpha-\beta^2\log\beta=h$,
$\log\alpha-\log\beta=j$, $e+\log\alpha=k$, $f/\alpha^2=l$, $e+\log\beta=m$, $f/\beta^2=n$,
$\alpha(1+2\log\alpha)=q$, $\beta(1+2\log\beta)=s$

Using the above notations, the following three equations are derived from eq. (2), (3) and (6),

$$\begin{aligned} C_2\cdot g+C_3\cdot h+C_4\cdot j &= (\delta_A-\delta_B)-pa^4(\alpha^4-\beta^4)/64N \\ 2C_2\cdot\alpha+C_3\cdot q+C_4(1/\alpha) &= a\theta_A-pa^4\cdot\alpha^3/16N \\ 2C_2\cdot\beta+C_3\cdot s+C_4(1/\beta) &= a\theta_B-pa^4\cdot\beta^3/16N \end{aligned} \qquad (7)$$

The values of C_2, C_3 and C_4 are obtained as follows by solving eq. (7),

$$
\begin{aligned}
\Delta\cdot C_2 &= (pa^4/16N)D+(\delta_A-\delta_B)E+a\theta_A\cdot F+a\theta_B\cdot G\\
\Delta\cdot C_3 &= (pa^4/16N)H+(\delta_A-\delta_B)I+a\theta_A\cdot J+a\theta_B\cdot K\\
\Delta\cdot C_4 &= (pa^4/16N)L+(\delta_A-\alpha_B)M+a\theta_A\cdot P+a\theta_B\cdot Q
\end{aligned}
\qquad (8)
$$

where $\Delta=(1/\alpha)(2\beta h-gs)+(1/\beta)(gq-2\alpha h)+2j(\alpha s-\beta q)$,

$$D=\frac{1}{4}(\beta^4-\alpha^4)E-\alpha^3F-\beta^3G,\quad H=\frac{1}{4}(\beta^4-\alpha^4)I-\alpha^3J-\beta^3K,$$

$$L=\frac{1}{4}(\beta^4-\alpha^4)M-\alpha^3P-\beta^3Q,\quad E=\frac{q}{\beta}-\frac{s}{\alpha},\quad F=-\left(\frac{h}{\beta}-sj\right),\quad G=-\left(qj-\frac{h}{\alpha}\right),$$

$$I=\frac{2\beta}{\alpha}-\frac{2\alpha}{\beta},\quad J=-\left(2\beta j-\frac{g}{\beta}\right),\quad K=-\left(\frac{g}{\alpha}-2\alpha j\right),\quad M=2\alpha s-2\beta q,$$

$$P=-(sg-2\beta h),\quad Q=-(2\alpha h-gq)$$

Following the example of eq. (6), the radial bending moment and the radial shearing force at both circumferences will be represented as,

$$
\begin{aligned}
A:&\quad \rho=\alpha=1,\quad M_r=M_{r,A},\quad p_r=p_{r,A}\\
B:&\quad \rho=\beta\qquad\ ,\quad M_r=M_{r,B},\quad p_r=p_{r,B}
\end{aligned}
\qquad (9)
$$

Therefore, from eq. (4), (5), (8) and (9)[6]

$$
\begin{aligned}
M_{r,A} &= -\frac{2(1+\nu)N}{a^2}\Big\{\frac{pa^4}{16N}(e\alpha^2+D+kH-lL)+(\delta_A-\delta_B)(E+kI-lM)\\
&\qquad +a\theta_A(F+kJ-lP)+a\theta_B(G+kK-lQ)\Big\}\\
M_{r,B} &= -\frac{2(1+\nu)N}{a^2}\Big\{\frac{pa^4}{16N}(e\beta^2+D+mH-lL)+(\delta_B-\delta_B)(E+mI-nM)\\
&\qquad +a\theta_A(F+mJ-nP)+a\theta_B(G+mK-nQ)\Big\}\\
p_{r,A} &= -\frac{4N}{a^3}\Big\{(\delta_A-\delta_B)\frac{I}{\alpha}+a\theta_A\frac{J}{\alpha}+a\theta_B\frac{K}{\alpha}\Big\}-\Big(\frac{pa}{2}\alpha+\frac{pa}{4}\frac{H}{\alpha}\Big)\\
p_{r,B} &= -\frac{4N}{a^3}\Big\{(\delta_A-\delta_B)\frac{I}{\beta}+a\theta_A\frac{J}{\beta}+a\theta_B\frac{K}{\beta}\Big\}-\Big(\frac{pa}{2}\beta+\frac{pa}{4}\frac{H}{\beta}\Big)
\end{aligned}
$$

Simplifying the above equations gives,

$$
\left.
\begin{aligned}
M_{r,A} &= \frac{N}{a}\Big(c'\theta_A+d'\theta_B-\frac{\delta_B-\delta_A}{a}e'\Big)+s'(pa^2)\\
M_{r,B} &= \frac{N}{a}\Big(f'\theta_A+g'\theta_B-\frac{\delta_B-\delta_B}{a}h'\Big)+t'(pa^2)
\end{aligned}
\right\}
\qquad (10)
$$

$$
\left.
\begin{aligned}
p_{r,A} &= -\frac{4N}{a^2}\Big(i'\theta_A+j'\theta_B-\frac{\delta_B-\delta_A}{a}k'\Big)+u'(pa)\\
p_{r,B} &= -\frac{4N}{a^2}\Big(l'\theta_A+m'\theta_B-\frac{\delta_B-\delta_A}{a}h'\Big)+v'(pa)
\end{aligned}
\right\}
\qquad (11)
$$

If the rule of signs adopted in the slope deflection method used in the solution of rigid frames is employed here for the sake of convenience, then

1. In eq. (10), $M_{r,A}$ becomes $-M_{r,AB}$, $M_{r,B}$ becomes $M_{r,BA}$ and the sign stands as it is.
2. The signs of θ_A and θ_B remain unchanged.
3. As for the signs of s' and t', only s' changes in accordance with 1.
4. In eq. (11), as $p_\theta=0$, suffix r is omitted, and $p_{r,A}$ and $p_{r,B}$ are expressed as p_{AB} and p_{BA}.

By rewritting eq. (10) and (11) following the above explanation, the next two fundamental equations are finally derived[7].

$$\left.\begin{aligned} M_{r,AB} &= \frac{N}{a}\left(c\theta_A+d\theta_B-\frac{\delta_B-\delta_A}{a}e\right)+s(pa^2) \\ M_{r,BA} &= \frac{N}{a}\left(f\theta_A+g\theta_B-\frac{\delta_B-\delta_A}{a}h\right)+t(pa^2) \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} p_{AB} &= -\frac{4N}{a^2}\left(i\theta_A+j\theta_B-\frac{\delta_B-\delta_A}{a}k\right)+u(pa) \\ p_{BA} &= -\frac{4N}{a^2}\left(l\theta_A+m\theta_B-\frac{\delta_B-\delta_A}{a}n\right)+v(pa) \end{aligned}\right\} \tag{13}$$

If the coefficients of the above equations are calculated for the four cases of $\beta=4/5$, $3/4$, $2/3$ and $1/2$ when $\alpha=1$, the result is as shown in Table 1, where ν is assumed as 0.3.

Table 1. Values of several coefficients.

	$\alpha=1,\ \beta=4/5$	$\alpha=1,\ \beta=3/4$	$\alpha=1,\ \beta=2/3$	$\alpha=1,\ \beta=1/2$
c	19.285 538	15.281 077	11.273 461	7.256 414
d	8.948 278	6.932 972	4.905 348	2.838 228
e	−139.689 902	− 87.683 915	− 47.677 338	− 19.668 074
f	11.185 364	9.243 991	7.358 039	5.676 456
g	20.856 963	16.908 145	13.010 216	9.312 827
h	−162.060 727	−106.171 681	− 62.393 412	− 31.020 988
i	34.922 472	21.920 952	11.919 334	4.917 018
j	32.412 133	19.907 203	10.398 899	3.877 623
k	−338.067 754	−168.463 392	− 67.866 966	− 18.282 213
l	43.653 093	29.227 936	17.879 001	9.834 037
m	40.515 167	26.542 937	15.598 349	7.755 247
n	−422.584 693	−224.617 856	−101.800 449	− 36.564 427
s	0.003 220	0.004 946	0.008 624	0.018 657
t	− 0.003 498	− 0.005 551	− 0.010 149	− 0.024 689
u	− 0.095 333	− 0.117 705	− 0.157 387	− 0.220 793
v	0.105 834	0.134 727	0.186 119	0.303 414

Next, the case when a circular plate of a radius a is subjected to a uniform load p and a radial bending moment is uniformly distributed at the circumference will be considered (Fig. 2). This is the case when C_3 and $C_4=0$ in equations (1) and the following eq. are obtained.

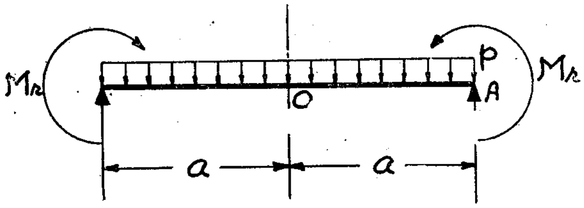

Fig. 2

$$w = \frac{pa^4}{64N}\rho^4 + C_1 + C_2\rho^2, \quad \frac{dw}{dr} = \frac{1}{a}\left(\frac{pa^4}{16N}\rho^3 + 2C_2\rho\right)$$

$$M_r = -\frac{2(1+\nu)N}{a^2}\left\{\frac{pa^4}{16N}\cdot\frac{3+\nu}{2(1+\nu)}\rho^2 + C_2\right\}$$

Putting $\rho=1$, $dw/dr=\theta_A$ in the above equation gives,

$$M_{r,A} = -\frac{(1+\nu)N}{a}\theta_A - \frac{pa^2}{8}$$

If expressed following the expression of the slope deflection method,

$$M_{r,AO} = \frac{N}{a}(1+\nu)\theta_A + \frac{pa^2}{8} \tag{14}$$

In this case, $p_{AO} = -\frac{pa}{2}$ is easily obtained.

3. Solution of a circular plate of non-uniform thickness by the slope deflection method[8].

Here, a circular plate whose flexural rigidity is a function of the radial distance is refered to as a circular plate of non-uniform thickness and the case when the load is symmetrical with respect to the center of the plate will be considered.

The following method is applied to obtain the bending, the deflection and etc. of a circular plate of non-uniform thickness by the slope deflection method. Fig. 3 shows the circular plate of which the bending, the deflection and etc. will be obtained. Let the rigidity at the center of the plate be represented as N_0 and the radius as a. The plate is divided into $n-1$ circular ring plates and a center circular plate. It is convenient to make the width of the rings constant.

The rigidity of each circular ring and the center circular plate is considered as approximately constant. Thus, the rigidity of the original circular plate changes step by step. In the calculation of a circular plate of non-uniform thickness by the slope deflection method, the circular plate is substituted by the above mentioned step shaped circular plate, and the calculation is done concerning the step shaped circular plate. Therefore, the result of the calculation is that for the step shaped circular plate and not that for the actual given plate. However, by increasing the number of circular ring plates into which the circular plate is divided, a value which is practically sufficient is obtained.

Next let us consider the load. As for the load, it is usual to consider a uniform load for such a problem, but when it is a distributed load which is a function of the radial distance then the load is divided in the same manner as the plate was, the load being considered as constant where the rigidity of the ring and the center circular plate are assumed constant. That is, the load is considered as step shaped just like the rigidity of the plate was.

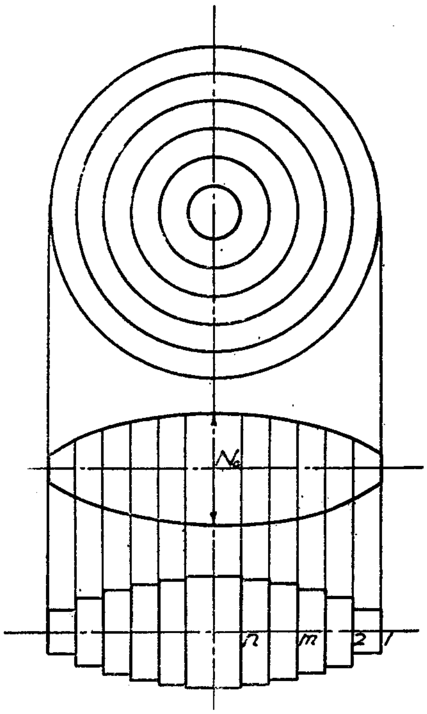

Fig. 3

As shown in Fig. 3, numbers $1, 2, 3, \ldots, m, \ldots, n$ will be attached to the stepped points of the step shaped circular plate which was substituted for the plate of non-uniform thickness whose bending, deflection and etc. are to be obtained.

At any stepped point m, equilibrium equations between the bending moments and between the shearing forces must exist. The equilibrium condition equation of the radial bending moment at point m is

$$M_{r,\, m-(m-1)} + M_{r,\, m-(m+1)} = 0 \tag{15}$$

The equlibrium condition equation of the shearing force is as follows except the case when the ring load exists along the division circle between the neighbouring two circular ring plates

$$-p_{m-(m-1)} + p_{m-(m+1)} = 0 \tag{16}$$

For the case of the ring load, the right hand of the above equation is substituted by the intensity of the ring load instead of 0.

There is no need of considering the condition of the deflection and the deflection angle at the stepped point, as they are already shown in formula (12) and (13).

As for the boundary conditions, $\delta=0$ and $M=0$ can be used if simply supported, and $\delta=0$ and $\theta=0$, if fixed.

The values of δ and θ are obtained by solving the equation of the first order concerning δ and θ, derived from eq. (12)~(16), and the value of M_r is obtained from eq. (12). The deflection and the bending moment at the center circular plate can also be obtained from the well known formulas[9]. The tangential bending moment can be obtained by the method explained in the next article.

4. Numerical Example

a) The case when a circular plate with $N=N_0 \cdot \exp(-1.5\rho^4)$, as shown in Fig. 4, is subjected to a uniform load p will be solved[10].

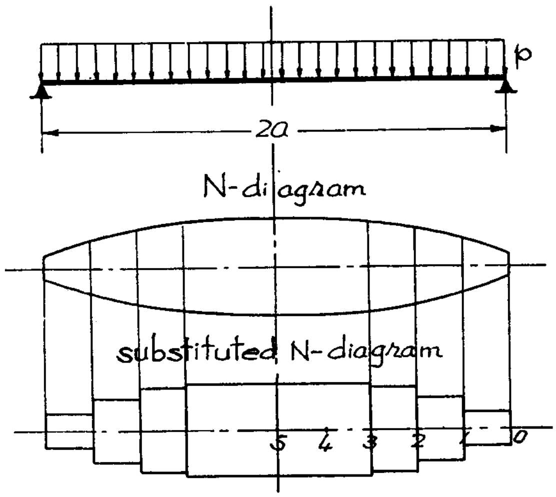

Fig. 4

The value of N for the various values of ρ from 0 to 1.0 are shown below.

$\rho=$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$N/N_0=$	1.00000	0.99985	0.99760	0.98792	0.96233	0.91051	0.82333	0.69757	0.54096	0.37374	0.22313

As is clear from the above values, the change in the value of N is very small while the value of ρ ranges from 0 to 0.4. Thus, the given circular plate will be divided into three ring plates and a center circular plate, the dividing radius being $\rho=0.8$, 0.6 and 0.4. As for the plate rigidity which is assumed constant, half of that of both ends may be taken, but in this case as the number of division is small, the rigidity of the middle point $\rho=0.9$, 0.7, 0.5 and 0.2 is taken for the sake of convenience.

As the ratios of inner and outer radii of the rings are 4/5, 3/4 and 2/3, Table 1 can be employed.

The equlibrium condition equation of the radial bending moment becomes as shown in Table 2 and that of the shearing force as shown in Table 3. Table 4 gives the values of θ and δ obtained by solving the above equations. Computing M_r from Table 4 gives Table 5. The computed values of $M_{m-(m-1)}$ take negative signs, but become positive when changed to the ordinary rule of sign of bending moment.

Table 2.

$$1.\quad \frac{0.37374\ N_0}{a}\left\{19.285538\,\theta_0+\ 8.948278\,\theta_1-(-139.689902)\ \frac{\delta_1}{a}\right\}+0.003220\,pa^2=0$$

$$2.\quad \frac{0.37374\ N_0}{a}\left\{11.185364\,\theta_0+20.856963\,\theta_1-(-162.060727)\ \frac{\delta_1}{a}\right\}-0.003498\,pa^2$$

$$+\frac{0.69757\ N_0}{0.8a}\left\{15.281077\,\theta_1+\ 6.932972\,\theta_2-(-\ 87.683915)\ \frac{\delta_2-\delta_1}{0.8a}\right\}+0.004946\,p\,(0.8a)^2=0$$

$$3.\quad \frac{0.69757\ N_0}{0.8a}\left\{\ 9.243991\,\theta_1+16.908145\,\theta_2-(-106.171681)\ \frac{\delta_2-\delta_1}{0.8a}\right\}-0.005551\,p\,(0.8a)^2$$

$$+\frac{0.91051\ N_0}{0.6a}\left\{11.273461\,\theta_2+\ 4.905348\,\theta_3-(-\ 47.677338)\ \frac{\delta_9-\delta_2}{0.6a}\right\}+0.008624\,p\,(0.6a)^2=0$$

$$4.\quad \frac{0.91051\ N_0}{0.6a}\left\{\ 7.358039\,\theta_2+13.010216\,\theta_3-(-\ 62.393412)\ \frac{\delta_3-\delta_2}{0.6a}\right\}-0.010149\,p\,(0.6a)^2$$

$$+\frac{0.99760\ N_0}{0.4a}\,1.3\,\theta_3 \qquad +0.125\,p\,(0.4a)^2=0$$

Table 3.

$$1.\quad 4\,\frac{0.37374\ N_0}{a^2}\left\{43.653093\,\theta_0+40.515167\,\theta_1-(-422.584693)\ \frac{\delta_1}{a}\right\}+0.105834\,pa$$

$$-4\,\frac{0.69757\ N_0}{(0.8a)^2}\left\{21.920952\,\theta_1+19.907203\,\theta_2-(-168.463392)\ \frac{\delta_2-\delta_1}{0.8a}\right\}-0.117705\,p\,(0.8a)=0$$

$$2.\quad 4\,\frac{0.69757\ N_0}{(0.8a)^2}\left\{29.227936\,\theta_1+26.542937\,\theta_1-(-224.617856)\ \frac{\delta_2-\delta_1}{0.8a}\right\}+0.134727\,p\,(0.8a)$$

$$-4\,\frac{0.91051\ N_0}{(0.6a)^2}\left\{11.919334\,\theta_2+10.398899\,\theta_3-(-\ 67.866966)\ \frac{\delta_3-\delta_2}{0.6a}\right\}-0.157387\,p\,(0.6a)=0$$

$$3.\quad 4\,\frac{0.91051\ N_0}{(0.6a)^2}\left\{17.879001\,\theta_2+15.598349\,\theta_3-(-101.800449)\ \frac{\delta_3-\delta_2}{0.6a}\right\}+0.186119\,p\,(0.6a)$$

$$-0.5\,p\,(0.4a)=0$$

Table 4.

point	0	1	2	3
θ	−0.134 981	−0.120 374	−0.095 463	−0.066 825
$\delta\ (pa^4/N_0)$	0.0	0.026 285	0.048 174	0.064 558

Table 5. (unit pa^2)

$M_{r,1}$	$M_{r,2}$	$M_{r,3}$
0.085 944	0.151 732	0.196 658

Next, the deflections and the bending moments of points $\rho=0.2$ and 0.0 will be obtained. As the center part is assumed as a circular plate with a rigidity $0.99760\,N_0$, the formula for the circular plate with a constant rigidity subjected to a uniform load p and uniformly distributed bending moment $M_{r,3}=0.196658\,pa^2$ at the circumference $\rho=0.4$ can be applied. Thus, the deflection and the bending moment at the center are, respectively[9]

$$\delta_5 = \delta_3 + \frac{p(0.4a)^4}{64N} \cdot \frac{5+\nu}{1+\nu} + \frac{M_{r,3}\cdot(0.4a)^2}{2N(1+\nu)} = 0.064558 + \frac{(0.4)^4}{64\cdot 0.99760} \cdot \frac{5.3}{1.3}$$

$$+ \frac{0.196658\cdot(0.4)^2}{2\cdot 0.99760\cdot 1.3} = 0.064558 + 0.001635 + 0.012141 = 0.078324\frac{pa^4}{N_0}$$

$$M_{r,5} = \frac{p(0.4a)^2}{16}(3+\nu) + M_{r,3} = \frac{(0.4)^2}{16}3.3 + 0.196658 = 0.229658\, pa^2$$

At point $\rho=0.2$

$$\delta_4 = \delta_3 + \frac{p(0.4a)^2}{64N(1+\nu)}\left\{2(3+\nu)\phi_1 - (1+\nu)\phi_0\right\} + \frac{M_{r,3}\cdot(0.4a)^2}{2N(1+\nu)}\phi_1$$

$$= 0.070807\frac{pa^4}{N_0}\left(\because\ \phi_0 = 1-(0.5)^4 = 0.9375\,,\quad \phi_1 = 1-(0.5)^2 = 0.75\,,\right)$$

$$M_{r,4} = \frac{p(0.4a)^2}{16}(3+\nu)\phi_1 + M_{r,3} = 0.221408\, pa^2$$

A comparison of the obtained values with the solution of R. Gran Olsson computed by using hypergeometric series is shown in Table 6 in which the calculation is done concerning the values of $\delta/aq\,(q=pa^3/2N_0)$ and $\sigma_r/\sigma_0=(6M_r/h^2)/(3pa^2/h_0)=(2M_r/pa^2)(h_0/h)^2$, where h_0 is the thickness at the center.

Table 6.

author	$\rho=$	0.0	0.2	0.4	0.6	0.8	1.0
M. Naruoka	δ/aq	0.1566	0.1496	0.1291	0.0963	0.0526	0.0000
	σ_r/σ_0	0.4593	0.4435	0.4035	0.3454	0.2589	0.0000
R. G. Olsson	δ/aq	0.1566	0.1495	0.1289	0.0960	0.0523	0.0000
	σ_r/σ_0	0.4613	0.4453	0.4039	0.3454	0.2585	0.0000

Next, the tangential bending moment M_t is obtained as follows.

$$M_r = -N\left(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right) = -N\left(\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right)$$

$$M_t = -N\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) = -N\left(\frac{\theta}{r} + \nu\frac{d\theta}{dr}\right)$$

Here, the values of M_r and θ are obtained from Tables 5 and 4, and the actual value of N must be used.

M_t can be obtained by substituting the value of $d\theta/dr$ obtained from the first equation into the second equation.

As for the value of $M_{t,4}$, it is calculated from the formula for a circular plate with a radius of $0.4a$ subjected to a uniform load p and a bending moment of $0.196658\,pa^2$ uniformly distributed at the circumference $\rho=0.4$. Moreover it is clear that $M_{t,5}=M_{r,5}$.

If the values of the tangential bending moment obtained above are shown in Table 7, they are as follows.

Table 7.

author	$\rho=$	0.0	0.2	0.4	0.6	0.8	1.0
M. Naruoka	σ_t/σ_0	0.4593	0.4505	0.4212	0.3750	0.3008	0.1491
R. G. Olsson		0.4613	0.4518	0.4237	0.3779	0.3032	0.1460

b) The case when a fixed circular plate of non-uniform thickness $h=h_0\cdot\exp\ (-\rho^2/6)$, shown in Fig. 5, is subjected to a uniform load will be solved[11].

The method of division is quite the same as before. Showing only the result obtained by solving the given plate as a step shaped plate of $0.66698N_0$, $0.78271N_0$, $0.88250N_0$, $0.98020N_0$, it becomes as given in Table 8. For comparison, the result obtained by Otto Pichler who solved by using the power series is shown together.

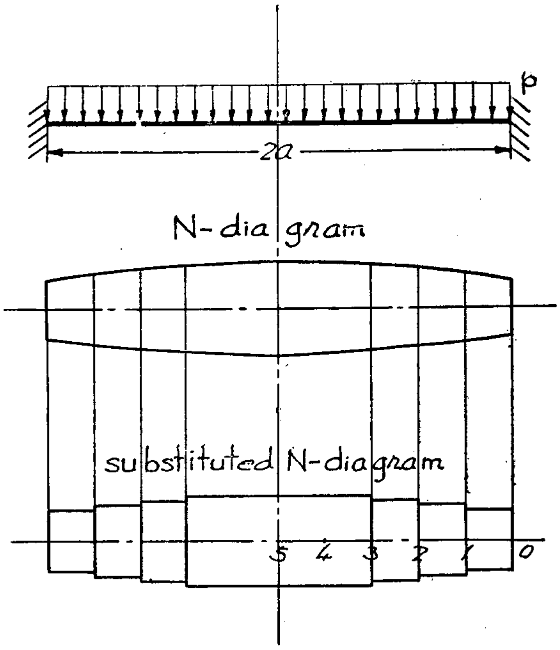

Fig. 5

Table 8.

ρ	0.0	0.2	0.4	0.6	0.8	1.0	author
δ/aq	0.0392	0.0363	0.0284	0.0171	0.0056	0.0000	M. Naruoka
	0.0398	0.0370	0.0289	0.0175	0.0058	0.0000	O. Pichler
σ_r/σ_0	0.1843	0.1700	0.1247	0.0368	−0.1060	−0.3295	M. Naruoka
	0.1869	0.1718	0.1250	0.0377	−0.1052	−0.3293	O. Pichler
σ_t/σ_0	0.1843	0.1781	0.1475	0.0974	0.0187	−0.0989	M. Naruoka
	0.1869	0.1779	0.1498	0.0991	0.0198	−0.0988	O. Pichler

5. Conclusion

Many solutions which have been proposed in the past are either so complicated that they are not suitable for engineers or are solutions for so special cases that they are not applicable to other cases when flexural rigidity is somewhat different from the treated special cases.

Author's solution by the slope deflection method applied to a circular plate of non-uniform thickness, that is, variable flexural rigidity is always possible, so far as the flexural rigidity and intensity of the distributed load are only a function of the radial distance and the plate is symmetrically loaded. Also the obtained result coincides satisfactorily with the strictly solved result as can be seen from the two examples.

The values of the coefficients necessary to the calculation can be given beforehand. If the table is given in various values of β for $\alpha=1$, a result very approximate to the true value can be obtained.

References

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4) H. D. Conway, Journal of Applied Mechanics, Trans. ASME, Vol. 30 (1948), March, p. 1.
5) M. NARUOKA, Trans. of Japan Society of Civil Engineers, Vol. 5 (1950), p. 55.
6) The notations $D, E, \ldots, Q$ used in the following four equations are conveniently used and actually mean the values divided by Δ.
7) The coefficients used in eq. (12) and (13) are different in meaning from the formerly used notations in p. 2.
8) Author's method is similar in principle to that of R. Grammel for rotating disk of non-uniform thickness (Dinglers Polytechnic Journal, Vol. 338 (1923), p. 217), but is different in the point of analytical solution.
9) K. Beyer, Die Statik im Eisenbetonbau II. 1934.
10) See reference 3). This is the case $\lambda=6$, $n=4$ in his solution.
11) See reference 2). This is the case $\beta=1$ in his solution.