# Deflections of Laterally Loaded Square Plates under Various Edge Conditions

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In this paper, the authors calculated the deflections of laterally loaded square plates under various edge conditions. The authors treated the problem assuming the plate is clamped or supported at the edges and deriving a fundamental solution suitable to any boundary condition, determined the coefficients included in the solution by the various boundary conditions.

## 1. Fundamental Equation and Its Solution

For the sake of simplicity, it is assumed that the distributed pressure $p$ is uniform and the boundaries of plate are $x=\pm 1$, $y=\pm 1$. It is easy to extend the solution to that of rectangular plate and it also can to lead into solutions of any other form of load distribution.

The deflection $w$ of the plate which is loaded with uniform pressure $p$ must satisfy the following fundamental differential equation

$$\Delta_1 \Delta_1 w = p/D, \tag{1}$$

where

$$1/D = 12(m^2-1)/m^2 E h^3,$$

$$\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$m$ is Poisson's number, $E$ the modulus of elasticity, and $h$ the thickness of the plate.

Putting it as

$$\Delta_1 w = W, \tag{2}$$

Eq. (1) becomes

$$\Delta_1 W = p/D. \tag{3}$$

This is Poisson's partial differential equation. Assigning $W_1$ to the particular solution and $W_2$ to the general solution of $\Delta_1 W=0$, then we may put

$$W = W_1 + W_2. \tag{4}$$

The particular solution $W_1$ is easily obtained by

$$W_1 = p(x^2+y^2-2)/4D. \tag{5}$$

It is convenient to choose the form in which the particular solution is zero at the four points $x=\pm1$, $y=\pm1$.

The boundary values of $W_2$ are supposed to be as follows;

$$\begin{aligned} W_2|_{x=1} &= G(y), \\ W_2|_{x=-1} &= G'(y), \\ W_2|_{y=1} &= H(x), \\ W_2|_{y=-1} &= H'(x). \end{aligned} \tag{6}$$

As $W_2$ satisfies the equation

$$\Delta_1 W_2 = 0, \tag{7}$$

the solution is obtained as

$$\begin{aligned} W_2 &= \sum_{n=1}^{\infty} \frac{A_n \sinh\frac{n\pi}{2}(1+x) + A_n' \sinh\frac{n\pi}{2}(1-x)}{\sinh n\pi} \sin\frac{n\pi}{2}(1+y) \\ &+ \sum_{n=1}^{\infty} \frac{B_n \sinh\frac{n\pi}{2}(1+y) + B_n' \sinh\frac{n\pi}{2}(1-y)}{\sinh n\pi} \sin\frac{n\pi}{2}(1+x), \end{aligned} \tag{8}$$

where

$$\begin{aligned} A_n &= \int_{-1}^{1} G(y') \sin\frac{n\pi}{2}(1+y')dy', \\ A_n' &= \int_{-1}^{1} G'(y') \sin\frac{n\pi}{2}(1+y')dy', \\ B_n &= \int_{-1}^{1} H(x') \sin\frac{n\pi}{2}(1+x')dx', \\ B_n' &= \int_{-1}^{1} H'(x') \sin\frac{n\pi}{2}(1+x')dx'. \end{aligned} \tag{9}$$

Using Eqs. (2), (4), (5) and (8),

$$\begin{aligned} \Delta_1 w &= p(x^2+y^2-2)/4D \\ &+ \sum_{n=1}^{\infty} \frac{A_n \sinh\frac{n\pi}{2}(1+x) + A_n' \sinh\frac{n\pi}{2}(1-x)}{\sinh n\pi} \sin\frac{n\pi}{2}(1+y) \\ &+ \sum_{n=1}^{\infty} \frac{B_n \sinh\frac{n\pi}{2}(1+y) + B_n' \sinh\frac{n\pi}{2}(1-y)}{\sinh n\pi} \sin\frac{n\pi}{2}(1+x). \end{aligned} \tag{10}$$

Denoting the particular solution of Eq. (10) by $w_1$ and the general solution of $\Delta_1 w=0$ by $w_2$, then we may put

$$w = w_1 + w_2. \tag{11}$$

We can choose the particular solution $w_1$ as

$$
\begin{aligned}
w_1 &= p[(x^4+y^4)-6(x^2+y^2)+10]/48D \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\left[(1+x)\frac{n\pi}{2}\cosh(1+x)\frac{n\pi}{2}-\sinh(1+x)\frac{n\pi}{2}\right]\sin(1+y)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{A_n'}{\sinh n\pi}\left[(1-x)\frac{n\pi}{2}\cosh(1-x)\frac{n\pi}{2}-\sinh(1-x)\frac{n\pi}{2}\right]\sin(1+y)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{B_n}{\sinh n\pi}\left[(1+y)\frac{n\pi}{2}\cosh(1+y)\frac{n\pi}{2}-\sinh(1+y)\frac{n\pi}{2}\right]\sin(1+x)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{B_n'}{\sinh n\pi}\left[(1-y)\frac{n\pi}{2}\cosh(1-y)\frac{n\pi}{2}-\sinh(1-y)\frac{n\pi}{2}\right]\sin(1+x)\frac{n\pi}{2}. \quad (12)
\end{aligned}
$$

Let the boundary values of $w_2$ be as follows;

$$
\begin{aligned}
w_2|_{x=1} &= g(y), \\
w_2|_{x=-1} &= g'(y), \\
w_2|_{y=1} &= h(x), \\
w_2|_{y=-1} &= h'(x).
\end{aligned} \quad (13)
$$

The solution $w_2$ has the same form as Eq. (8). However, the constants $A_n$, $A_n'$, $B_n$ and $B_n'$ are replaceable by $a_n$, $a_n'$, $b_n$ and $b_n'$ as:

$$
\begin{aligned}
a_n &= \int_{-1}^{1} g(y')\sin\frac{n\pi}{2}(1+y')dy', \\
a_n' &= \int_{-1}^{1} g'(y')\sin\frac{n\pi}{2}(1+y')dy', \\
b_n &= \int_{-1}^{1} h(x')\sin\frac{n\pi}{2}(1+x')dx', \\
b_n' &= \int_{-1}^{1} h'(x')\sin\frac{n\pi}{2}(1+x')dx'.
\end{aligned} \quad (14)
$$

Substituting $w_1$ and $w_2$ into Eq. (11), we obtain the solution of $w$ by

$$
\begin{aligned}
w &= p[(x^4+y^4)-6(x^2+y^2)+10]/48D \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\left[(1+x)\frac{n\pi}{2}\cosh(1+x)\frac{n\pi}{2}-\sinh(1+x)\frac{n\pi}{2}\right]\sin(1+y)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{A_n'}{\sinh n\pi}\left[(1-x)\frac{n\pi}{2}\cosh(1-x)\frac{n\pi}{2}-\sinh(1-x)\frac{n\pi}{2}\right]\sin(1+y)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{B_n}{\sinh n\pi}\left[(1+y)\frac{n\pi}{2}\cosh(1+y)\frac{n\pi}{2}-\sinh(1+y)\frac{n\pi}{2}\right]\sin(1+x)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\frac{B_n'}{\sinh n\pi}\left[(1-y)\frac{n\pi}{2}\cosh(1-y)\frac{n\pi}{2}-\mathrm{shin}(1-y)\frac{n\pi}{2}\right]\sin(1+x)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{1}{\sinh n\pi}\left[a_n\sinh(1+x)\frac{n\pi}{2}+a_n'\sinh(1-x)\frac{n\pi}{2}\right]\sin(1+y)\frac{n\pi}{2} \\
&+\sum_{n=1}^{\infty}\frac{1}{\sinh n\pi}\left[b_n\sinh(1+y)\frac{n\pi}{2}+b_n'\sinh(1-y)\frac{n\pi}{2}\right]\sin(1+x)\frac{n\pi}{2}. \quad (15)
\end{aligned}
$$

The constants $A_n$, $A_n'$, $B_n$, $B_n'$, $a_n$, $a_n'$, $b_n$ and $b_n'$ are determined by the boundary conditions whether or not the plate is supported or clamped at four edges.

## 2. Boundary Conditions

In the case of supported edges, the deflection and the moment along the boundary are both zero.

In the case of built-in, the deflection and the slope along the boundary are both zero.

Suppose that the boundary coincides with $y=y_0$, then the boundary conditions can be written as follows;

i) in case of simply supported edge,

$$\begin{aligned} &(w)_{y=y_0}=0 \\ &\left(\frac{\partial^2 w}{\partial y^2}+\frac{1}{m}\frac{\partial^2 w}{\partial x^2}\right)_{y=y_0}=0\,, \end{aligned} \tag{16}$$

ii) in case of built-in edge,

$$\begin{aligned} &(w)_{y=y_0}=0 \\ &\left(\frac{\partial w}{\partial y}\right)_{y=y_0}=0\,. \end{aligned} \tag{17}$$

Plate may assume six kinds of edge conditions;

1. all four edges supported,
2. all four edges built-in,
3. three edges supported and one edge built-in,
4. two opposite edges supported and the other two edges built-in,
5. two neighbouring edges supported and the other two neighbouring edges built-in,
6. one edge supported and the others built-in.

The cases of 1～4 are solved by S. Timoshenko[1] and as for the cases of 1～4 and 6 the solutions were obtained by calculation of finite differences equations[2] changed from differential equations.

## 3. Equations Representing Stresses

Placing $z$-axis in the direction of thickness of the plate and the origine at the middle plane of the plate, we may write for the condition of $\sigma_z=0$

$$\begin{aligned} \sigma_x &= \frac{-mE}{(m^2-1)}z\left(m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right), \\ \sigma_y &= \frac{-mE}{(m^2-1)}z\left(\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial y^2}\right), \\ \tau_z &= \frac{-mE}{m+1}z\frac{\partial^2 w}{\partial x\partial y}. \end{aligned} \tag{18}$$

On the surface of the plate $z=\pm h/2$, the following maximum stresses are found

$$\sigma_x = \frac{mEh}{2(m^2-1)}\left(m\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right),$$
$$\sigma_y = \frac{mEh}{2(m^2-1)}\left(\frac{\partial^2 w}{\partial x^2}+m\frac{\partial^2 w}{\partial y^2}\right). \tag{19}$$

## 4. Solutions for Various Edge Conditions

1. The case of all edges supported.

From the condition of symmetry, it will be

$$A_n = A_n' = B_n = B_n',$$
$$a_n = a_n' = b_n = b_n'. \tag{20}$$

By Eq. (15), it will consummate in

$$\begin{aligned} w &= p[(x^4+y^4)-6(x^2+y^2)+10]/48D \\ &+\sum_{\substack{n=1\\ n:\,\text{odd}}}^{\infty}\frac{2}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\Big[\Big\{(1+x)\frac{n\pi}{2}\cosh(1+x)\frac{n\pi}{2}-\sinh(1+x)\frac{n\pi}{2} \\ &\qquad +(1-x)\frac{n\pi}{2}\cosh(1-x)\frac{n\pi}{2}-\sinh(1-x)\frac{n\pi}{2}\Big\}\sin(1+y)\frac{n\pi}{2} \\ &\qquad +\Big\{(1+y)\frac{n\pi}{2}\cosh(1+y)\frac{n\pi}{2}-\sinh(1+y)\frac{n\pi}{2} \\ &\qquad +(1-y)\frac{n\pi}{2}\cosh(1-y)\frac{n\pi}{2}-\sinh(1-y)\frac{n\pi}{2}\Big\}\sin(1+x)\frac{n\pi}{2}\Big] \\ &+\sum_{\substack{n=1\\ n:\,\text{odd}}}^{\infty}\frac{a_n}{\sinh n\pi}\Big[\Big\{\sinh(1+x)\frac{n\pi}{2}+\sinh(1-x)\frac{n\pi}{2}\Big\}\sin(1+y)\frac{n\pi}{2} \\ &\qquad +\Big\{\sinh(1+y)\frac{n\pi}{2}+\sinh(1-y)\frac{n\pi}{2}\Big\}\sin(1+x)\frac{n\pi}{2}\Big]. \end{aligned} \tag{21}$$

When the edge is supported, condition (16) is replaced by

$$(w)_{y=y_0} = 0,$$
$$(W)_{y=y_0} = 0. \tag{22}$$

Hence the values of $G(y)$, $H(x)$ that $W_2$ must satisfy along the boundary are

$$G(y) = G'(y) = p(1-y^2)/4D,$$
$$H(x) = H'(x) = p(1-x^2)/4D. \tag{23}$$

Using Eq. (9), we get

$$\begin{aligned} A_n &= \int_{-1}^{1}\frac{p}{4D}(1-y'^2)\sin\frac{n\pi}{2}(1+y')dy' \\ &= \frac{p}{D}\frac{4}{(n\pi)^3}(1-\cos n\pi). \end{aligned}$$

Because $n$ is odd number, it becomes

$$A_n = \frac{p}{D}\frac{8}{(n\pi)^3}. \tag{24}$$

From these results, the boundary values of $w_2$, namely, $g(y)$ and $h(x)$ are determined to be

$$\begin{aligned} g(y) &= g'(y) \\ &= \frac{-p}{48D}(y^4-6y^2+5) - \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{2}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\Big[n\pi\cosh n\pi - \sinh n\pi\Big]\sin(1+y)\frac{n\pi}{2}, \\ h(x) &= h'(x) \\ &= \frac{-p}{48D}(x^4-6x^2+5) - \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{2}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\Big[n\pi\cosh n\pi - \sinh n\pi\Big]\sin(1+x)\frac{n\pi}{2}. \end{aligned} \tag{25}$$

Operating the integrals of Eq. (14), we obtain

$$\begin{aligned} a_n &= -\frac{p}{D}\frac{16}{(n\pi)^5}(1-\cos n\pi) - \frac{2}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\Big[n\pi\cosh n\pi - \sinh n\pi\Big] \\ &= -\frac{p}{D}\frac{16}{(n\pi)^5}(1+n\pi\coth n\pi). \end{aligned} \tag{26}$$

Table 1 gives the calculated values of $A_n$ and $a_n$.

Table 1.

| $n$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $A_n(\times p/D)$ | 0.258012 | 0.009556 | 0.002064 | 0.000752 | 0.000354 |
| $a_n(\times p/D)$ | -0.217154 | -0.002243 | -0.000280 | -0.000072 | -0.000026 |

Next, by denoting $w_0$ for the deflection at $x=0$, $y=0$, we get

$$\begin{aligned} w_0 &= \frac{4}{25}\frac{p}{D} + \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{4}{(n\pi)^2}\frac{A_n}{\sinh n\pi}\Big[n\pi\cosh\frac{n\pi}{2} - 2\sinh\frac{n\pi}{2}\Big]\sin\frac{n\pi}{2} \\ &\quad + \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{4a_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\sin\frac{n\pi}{2} \\ &= \frac{4}{25}\frac{p}{D} + \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{2(-1)^{\frac{n-1}{2}}}{\cosh\frac{n\pi}{2}}\Big[A_n\Big(\frac{1}{n\pi}\coth\frac{n\pi}{2} - \frac{2}{(n\pi)^2}\Big) + a_n\Big]. \end{aligned} \tag{27}$$

The following is obtained if the numerical values of $A_n$ and $a_n$ are substituted:

$$w_0 = 0.065946\, p/D. \tag{28}$$

Assuming that the side length of the square is $a$, we may express $w_0$ as

$$w_0 = 0.004060\, pa^4/D. \tag{29}$$

2. The case of all edges being built-in.

In this case the preceding Eqs. (20) and (21) can be applied in their exact form.

The deflection and slope along the boundary $y=\pm 1$ become

$$w|_{y=\pm 1} = \frac{p}{48D}(x^4-6x^2+5) + \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \left[\frac{2A_n}{n\pi}\left(\coth n\pi - \frac{1}{n\pi}\right) + a_n\right] \sin(1+x)\frac{n\pi}{2}, \tag{30}$$

$$\begin{aligned} \frac{\partial w}{\partial y}\bigg|_{y=\pm 1} &= \frac{p}{6D} - \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \left[A_n + \frac{n\pi}{2} a_n \left(\coth n\pi - \frac{1}{\sinh n\pi}\right)\right] \sin(1+x)\frac{n\pi}{2} \\ &+ \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \left[\frac{A_n}{\sinh n\pi}\left(\cosh\frac{n\pi}{2} - \frac{2}{n\pi}\sinh\frac{n\pi}{2}\right) + \frac{n\pi a_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\right] \cosh\frac{n\pi}{2}x \\ &+ \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \left[\frac{A_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\right] x \sinh\frac{n\pi}{2}x. \end{aligned} \tag{31}$$

Eqs. (30) and (31) must be zero regardless of $x$.

If we put

$$\frac{2A_n}{n\pi}\left(\coth n\pi - \frac{1}{n\pi}\right) + a_n = K_n, \tag{32}$$

Eq. (30) becomes

$$w|_{y=\pm 1} = \frac{p}{48D}(x^4-6x^2+5) + \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} K_n \sin(1+x)\frac{n\pi}{2}. \tag{33}$$

The values of $A_n$ and $a_n$ in Table 1 should satisfy Eq. (30), and consequently they are supposed to satisfy Eq. (33) exactly. The relation between $A_n$ and $a_n$ can be known by Eq. (32), where $K_n$ are determined by $A_n$ and $a_n$ shown in Table 1.

Expanding hyperbolic functions into trigonometric series, we get

$$\cosh\frac{n\pi}{2}x = \frac{1}{2}\lambda_0 + \sum_l \lambda_l \cos\frac{l\pi}{2}x, \tag{34}$$

where

$$\lambda_0 = \frac{2}{n\pi}\sinh n\pi,$$

$$\lambda_l = \frac{2n(-1)^l}{(n^2+l^2)\pi}\sinh n\pi$$

and

$$x\sinh\frac{n\pi}{2}x = \frac{1}{2}\lambda_0' + \sum_l \lambda_l' \cos\frac{l\pi}{2}x, \tag{35}$$

where

$$\lambda_0' = \frac{4}{n^2\pi}\left[n\cosh n\pi - \frac{1}{\pi}\sinh n\pi\right],$$

$$\lambda_l' = \frac{4}{(n^2+l^2)\pi}\left[n\cosh n\pi - \frac{n^2-l^2}{(n^2+l^2)\pi}\sinh n\pi\right](-1)^l.$$

To simplify the formulae, we put

$$\sinh\frac{n\pi}{2}A_n=\bar{A}_n\,,\qquad n\pi\sinh\frac{n\pi}{2}\,a_n=\bar{a}_n\,. \tag{36}$$

Eqs. (32) and (31) can be rewritten using $\bar{A}_n$ and $\bar{a}_n$ as follows;

$$\frac{2\bar{A}_n}{n\pi}\Big(\coth n\pi-\frac{1}{n\pi}\Big)\Big/\sinh\frac{n\pi}{2}+\bar{a}_n/n\pi\sinh\frac{n\pi}{2}=K_n\,, \tag{37}$$

$$\begin{aligned}\frac{\partial w}{\partial y}\Big|_{y=\pm1}&=\frac{p}{6D}+\sum_{\substack{n=1\\ n:\text{ odd}}}^{\infty}\Big[\frac{\bar{A}_n}{\sinh\frac{n\pi}{2}}+\frac{\bar{a}_n}{2\sinh\frac{n\pi}{2}}\Big(\coth n\pi-\frac{1}{\sinh n\pi}\Big)\Big](-1)^{\frac{n-1}{2}}\cos\frac{n\pi}{2}x\\ &+\sum_{\substack{n=1\\ n:\text{ odd}}}^{\infty}\Big[\Big\{\bar{A}_n\Big(\coth\frac{n\pi}{2}-\frac{2}{n\pi}\Big)+\bar{a}_n\Big\}\frac{2n}{\pi}\Big\{\frac{1}{2n^2}+\sum_l\frac{n(-1)^l}{(n^2+l^2)}\cos\frac{l\pi}{2}x\Big\}\Big]\\ &+\sum_{\substack{n=1\\ n:\text{ odd}}}^{\infty}\frac{4}{\pi}\bar{A}_n\Big[\frac{1}{2n^2}\Big(n\coth n\pi-\frac{1}{\pi}\Big)\\ &\qquad+\sum_l\frac{(-1)^l}{(n^2+l^2)}\Big(n\coth n\pi-\frac{n^2-l^2}{(n^2+l^2)\pi}\Big)\cos\frac{l\pi}{2}x\Big]\,.\end{aligned} \tag{38}$$

Taking terms up to $n=5$, we set the constant term and each coefficient of $\cos\frac{\pi}{2}x$ and $\cos\pi x$ terms at zero. The coefficients are determined after some numerical calculation and the results are shown in Table 2.

Table 2.

| $n$ | 1 | 3 | 5 |
|---|---|---|---|
| $K_n(\times p/D)$ | -0.104568 | -0.000430 | -0.000034 |
| $A_n(\times p/D)$ | 0.45112 | -0.02908 | 0.00195 |
| $a_n(\times p/D)$ | -0.30142 | 0.00509 | -0.00027 |

The deflection of plate at the midpoint is

$$w_0=0.01992\,p/D \tag{39}$$

and

$$w_0'=0.00124\,pa^4/D\,. \tag{40}$$

3. The case of three edges supported and one edge built-in.

Supposing the plate is built-in along the edge $y=-1$, we get

$$A_n=A_n'=B_n\,,\qquad a_n=a_n'=b_n\,. \tag{41}$$

If we use the values of $A_n$ and $a_n$ shown in Table 1, the conditions of $w=0$ and of $W=0$ along $x=1$, $x=-1$ and $y=1$ are satisfied. Further, it is necessary to satisfy the zero condition of the deflection and the slope along $y=-1$. We have

$$w|_{y=-1} = \frac{p}{48D}(x^4-6x^2+5) + \sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty}\left[\frac{2B_n'}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+b_n'\right]\sin(1+x)\frac{n\pi}{2}, \tag{42}$$

$$\begin{aligned}\left.\frac{\partial w}{\partial y}\right|_{y=-1} &= \frac{p}{6D}+\sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty}\left[B_n'+\frac{n\pi}{2}\left(b_n'\coth n\pi-\frac{a_n}{\sinh n\pi}\right)\right]\sin(1+x)\frac{n\pi}{2}\\ &+\sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty}\left[\frac{A_n}{2\cosh\frac{n\pi}{2}}\left(\coth\frac{n\pi}{2}-\frac{2}{n\pi}\right)+\frac{n\pi a_n}{2\cosh\frac{n\pi}{2}}\right]\cosh\frac{n\pi}{2}x\\ &+\sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty}\left[\frac{A_n}{2\cosh\frac{n\pi}{2}}\right]x\sinh\frac{n\pi}{2}x,\end{aligned} \tag{43}$$

and putting

$$\begin{aligned}\alpha &= \frac{p}{6D}+\sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty}\left[\frac{A_n}{2\cosh\frac{n\pi}{2}}\left(\coth\frac{n\pi}{2}-\frac{2}{n\pi}\right)+\frac{n\pi a_n}{2\cosh\frac{n\pi}{2}}\right]\cosh\frac{n\pi}{2}x\\ &+\sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty}\left[\frac{A_n}{2\cosh\frac{n\pi}{2}}\right]x\sinh\frac{n\pi}{2}x,\end{aligned} \tag{44}$$

$$\beta_n = B_n'+\frac{n\pi}{2}\left(b_n'\coth n\pi-\frac{a_n'}{\sinh n\pi}\right), \tag{45}$$

we obtain

$$\left.\frac{\partial w}{\partial y}\right|_{y=-1} = \alpha+\sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty}\beta_n(-1)^{\frac{n-1}{2}}\cos\frac{n\pi}{2}. \tag{46}$$

To make the integral

$$\int_0^1\left[\left(\frac{\partial w}{\partial y}\right)_{y=-1}\right]^2 dx = I \tag{47}$$

minimum, $I$ is differentiated by $\beta_n$ and put equal to zero, the following is obtained

$$\beta_n = 2(-1)^{\frac{n-1}{2}}\int\alpha\cos\frac{n\pi}{2}x\,dx. \tag{48}$$

Thus one relation between $B_n'$ and $b_n'$ is obtained by Eq. (45).

On the other hand, another relation between $B_n'$ and $b_n'$ is obtained from Eq. (42) as

$$\frac{2B_n'}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+b_n' = K_n, \tag{49}$$

here the values of $K_n$ are shown in Table 2. As far as $B_n'$ and $b_n'$ are bounded by Eq. (49), the condition of $w=0$ along $y=-1$ is satisfied.

The values obtained from Eqs. (48) and (49) are shown in Table 3,

Table 3.

| $n$ | 1 | 3 |
|---|---|---|
| $\beta_n(\times p/D)$ | 0.055248 | 0.000636 |
| $A_n(\times p/D)$ | 0,258012 | 0.009556 |
| $a_n(\times p/D)$ | -0.217154 | -0.002243 |
| $B_n(\times p/D)$ | 0.611555 | 0.039472 |
| $b_n(\times P/D)$ | -0.371110 | -0.008116 |

The deflection at the centre becomes

$$w_0 = 0.044521\, p/D\,, \tag{50}$$

and

$$w_0 = 0.002783\, pa^4/D\,. \tag{51}$$

The deflection $w_0$ does not agree strictly with the maximum deflection $w_{max}$. The maximum deflection is found to occur near the point $x=0$, $y=0.184$, and it is calculated as

$$w_{max} = 0.04478\, p/D\,,$$
$$w_{max} = 0.00280\, pa^4/D\,.$$

But $w_{max}$ does not differ much from $w_0$. $w_{max}$ may be approximated with $w_0$. This fact is applicable in another unsymmetric case as far as they are square plate.

4. The case of two opposite edges supported and the other two edges clamped.

Supposing the plate is supported along the edges $x=\pm1$ and clamped along the edges $y=\pm1$, we have

$$\begin{aligned} A_n &= A_n'\,, \quad B_n = B_n'\,, \\ a_n &= a_n'\,, \quad b_n = b_n'\,. \end{aligned} \tag{52}$$

The values $A_n$ and $a_n$ in Table 1 are again utilized for this problem to make the supporting condition along $x=\pm1$ satisfied.

Performing the calculation, we get the deflection and the slope along $y=\pm1$ as

$$\begin{aligned} w|_{y=\pm1} &= \frac{p}{48D}(x^4-6x^2+5) \\ &+ \sum_{\substack{n=1 \\ n:\text{ odd}}}^{\infty}\left[\frac{2B_n}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+b_n\right]\sin(1+x)\frac{n\pi}{2}\,. \end{aligned} \tag{53}$$

$$\begin{aligned} \left.\frac{\partial w}{\partial y}\right|_{y=\pm1} &= \frac{p}{6D} - \sum_{\substack{n=1 \\ n:\text{ odd}}}^{\infty}\left[B_n+\frac{n\pi}{2}b_n\left(\coth n\pi-\frac{1}{\sinh n\pi}\right)\right]\sin(1+x)\frac{n\pi}{2} \\ &+ \sum_{\substack{n=1 \\ n:\text{ odd}}}^{\infty}\left[\frac{A_n}{\sinh n\pi}\left(\cosh\frac{n\pi}{2}-\frac{2}{n\pi}\sinh\frac{n\pi}{2}\right)+\frac{n\pi a_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\right]\cosh\frac{n\pi}{2}x \\ &+ \sum_{\substack{n=1 \\ n:\text{ odd}}}^{\infty}\left[\frac{A_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\right]x\sinh\frac{n\pi}{2}x\,. \end{aligned} \tag{54}$$

It is necessary that Eqs. (53) and (54) are always zero for any value of $x$. In Eqs. (53) and (54), it suffices to consider the coefficients $B_n$ and $b_n$ for odd number of $n$. Because the constant terms, $x^4$, $x^2$, $\cosh\frac{n\pi}{2}x$ and $x\sinh\frac{n\pi}{2}x$ are even functions, $\sin(1+x)\frac{n\pi}{2}$ must be an even function or the coefficients must be zero if it is an odd function. When $n$ is an odd number, $\sin(1+x)\frac{n\pi}{2}$ is an even function and if $n$ is an even number, $\sin(1+x)\frac{n\pi}{2}$ is an odd function. Hence, from Eqs. (53) and (54),

$$\begin{aligned} &\frac{2B_n}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+b_n=0\,, \\ &B_n+\frac{n\pi}{2}b_n\left(\coth n\pi-\frac{1}{\sinh n\pi}\right)=0\,, \quad (n:\text{ even}) \end{aligned} \tag{55}$$

are obtained. To satisfy the Eq. (55), the conditions

$$\begin{aligned} (B_n)_{n:\text{even}}&=0\,, \\ (b_n)_{n:\text{even}}&=0 \end{aligned} \tag{56}$$

are necessary.

They are put, as before, as

$$\frac{2B_n}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+b_n=K_n\,, \tag{57}$$

where $K_n$ are the values shown in Table 2. Thus condition $w=0$ along $y=\pm1$ is satisfied.

By putting $\alpha$ the same as in Eq. (44) and also as

$$\beta_n'=B_n+\frac{n\pi}{2}b_n\left(\coth n\pi-\frac{1}{\sinh n\pi}\right). \tag{58}$$

Eq. (54) is reduced to

$$\left.\frac{\partial w}{\partial y}\right|_{y=\pm1}=\alpha+\sum_{\substack{n=1\\ n:\text{ odd}}}^{\infty}\beta_n'(-1)^{\frac{n-1}{2}}\cos\frac{n\pi}{2}\,. \tag{59}$$

If we denote

$$\int_0^1\left[\left(\frac{\partial w}{\partial y}\right)_{y=\pm1}\right]^2dx=I'\,, \tag{60}$$

differentiate $I'$ by $\beta_n'$ and make these derivatives zero as

$$\frac{\partial I'}{\partial\beta_n'}=0\,, \tag{61}$$

$\beta_n'$ can be determined which are the most suitable values for satisfying the zero condition of the slope along the boundary.

Calculating Eq. (61), we have

$$\beta_n'=2(-1)^{\frac{n-1}{2}}\int\alpha\cos\frac{n\pi}{2}x\,dx\,. \tag{62}$$

This is the same as the preceding Eq. (48).

Using these results, the constants $\beta_n'$, $B_n$ and $b_n$ are found as shown in Table 4.

Table 4.

| $n$ | 1 | 3 |
|---|---|---|
| $\beta_n'(\times p/D)$ | 0.055248 | 0.001272 |
| $A_n(\times p/D)$ | 0.258012 | 0.009556 |
| $a_n(\times p/D)$ | -0.217154 | -0.002243 |
| $B_n(\times p/D)$ | 0.554438 | 0.023384 |
| $b_n(\times p/D)$ | -0.346502 | -0.004866 |

The deflection at $x=0$, $y=0$ becomes:

$$w_0 = 0.030537\, p/D, \tag{63}$$

and also

$$w_0 = 0.001909\, pa^4/D. \tag{64}$$

5. The case of two neighbouring edges supported and the other two neighbouring edges built-in.

The plate supported along the edges $x=1$ and $y=1$ and clamped along the edges $x=-1$ and $y=-1$ is considered.

The relations

$$\begin{aligned} A_n &= B_n, \quad A_n' = B_n', \\ a_n &= b_n, \quad a_n' = b_n', \end{aligned} \tag{65}$$

are brought about as in cases (3) and (4).

By applying the values $A_n$ and $a_n$ shown in Table 1, the boundary conditions in which both the deflection and the moment are zero along the edges $x=1$, $y=1$, are satisfied.

Calculating the value of the deflection and the slope along the edge $x=-1$, we have

$$\begin{aligned} w|_{x=-1} &= \frac{p}{48D}(y^4-6y^2+5) \\ &\quad + \sum_{n=1}^{\infty}\left[\frac{2A_n'}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+a_n'\right]\sin(1+y)\frac{n\pi}{2}, \end{aligned} \tag{66}$$

$$\begin{aligned} \left.\frac{\partial w}{\partial x}\right|_{x=-1} &= -\frac{p}{6D}+\sum_{n=1}^{\infty}\left[A_n'+\frac{n\pi}{2}\left(\coth n\pi\cdot a_n'-\frac{a_n}{\sinh n\pi}\right)\right]\sin(1+y)\frac{n\pi}{2} \\ &\quad + \sum_{n=1}^{\infty}\frac{n\pi(-1)^n}{2\sinh n\pi}\left[\left\{(a_n+a_n')-\frac{2(A_n+A_n')}{(n\pi)^2}\right\}\sinh\frac{n\pi}{2}+\frac{A_n+A_n'}{n\pi}\cosh\frac{n\pi}{2}\right]\cosh\frac{n\pi}{2}y \\ &\quad - \sum_{n=1}^{\infty}\frac{n\pi(-1)^n}{2\sinh n\pi}\left[\left\{(a_n-a_n')-\frac{2(A_n-A_n')}{(n\pi)^2}\right\}\cosh\frac{n\pi}{2}+\frac{A_n-A_n'}{n\pi}\sinh\frac{n\pi}{2}\right]\sinh\frac{n\pi}{2}y \\ &\quad - \sum_{n=1}^{\infty}\frac{n\pi(-1)^n}{2\sinh n\pi}\left[\frac{A_n-A_n'}{n\pi}\cosh\frac{n\pi}{2}\right]y\cosh\frac{n\pi}{2}y \\ &\quad + \sum_{n=1}^{\infty}\frac{n\pi(-1)^n}{2\sinh n\pi}\left[\frac{A_n+A_n'}{n\pi}\sinh\frac{n\pi}{2}\right]y\sinh\frac{n\pi}{2}y. \end{aligned} \tag{67}$$

The deflection and the slope along $x=-1$ must be zero regardless of $y$.

If we put

$$\frac{2A_n'}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+a_n' = K_n \tag{68}$$

and adopt the values $K_n$ shown in Table 2, the deflection along $x=-1$ becomes zero. When $n$ is even number, $A_n$ and $a_n$ in Table 2 are zero hence $K_n$ are zero.

Expressing $a_n'$ by $A_n'$ according to Eq. (68) and substituting $a_n'$ into Eq. (67), we obtain Eq. (67) containing unknown $A_n'$ series.

Making minimum the integral $\int_0^1\left[\left(\frac{\partial w}{\partial x}\right)_{x=-1}\right]^2 dx$ and taking some approximation, we determine the constants $A_n'$ and $a_n'$. Table 5 shows these results.

Table 5.

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $A_n(\times p/D)$ | 0.258012 | 0.0000 | 0.009556 |
| $a_n(\times p/D)$ | -0.217154 | 0.0000 | -0.002243 |
| $B_n(\times p/D)$ | 0.52309 | 0.0438 | 0.005 |
| $b_n(\times p/D)$ | -0.33282 | 0.0117 | -0.001 |

The deflection $w_0$ becomes as follows;

$$w_0 = 0.03408\, p/D \tag{69}$$

and

$$w_0 = 0.00213\, pa^4/D\,. \tag{70}$$

6. The case of one edge supported and the other three edges built-in.

Suppose the plate is supported along one edge $x=1$. The values $A_n$ and $a_n$ shown in Table 1 are adopted. Then the condition that the plate is supported along $x=1$ is naturally satisfied. Then a consideration must be given about $A_n'$ and $a_n'$ where $n$ is an odd number, and $B_n(=B_n')$ and $b_n(=b_n')$ where $n$ takes both odd and even number.

The necessary conditions are as follows;

$$w|_{x=-1} = 0\,, \tag{71}$$

$$w|_{y=\pm1} = 0\,, \tag{72}$$

$$\left.\frac{\partial w}{\partial x}\right|_{x=-1} = 0\,, \tag{73}$$

$$\left.\frac{\partial w}{\partial y}\right|_{y=\pm1} = 0\,. \tag{74}$$

As for the conditions (71) and (72), it is sufficient if $A_n'$ and $a_n'$, $B_n$ and $b_n$ are combined with the next relations;

$$\frac{2A_n'}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+a_n'=K_n\,, \tag{75}$$

$$\frac{2B_n}{n\pi}\left(\coth n\pi-\frac{1}{n\pi}\right)+b_n=K_n\,. \tag{76}$$

Next, the conditions (73) and (74) must be considered. Here the forms are written

$$\begin{aligned}\left.\frac{\partial w}{\partial x}\right|_{x=-1} &= -\frac{p}{6D}+\sum_{\substack{n=1\\ n:\,\mathrm{odd}}}^{\infty}\left[A_n'+\frac{n\pi}{2}\left(a_n'\coth n\pi-\frac{a_n}{\sinh n\pi}\right)\right]\sin(1+y)\frac{n\pi}{2}\\ &\quad-\sum_{n=1}^{\infty}\left[\frac{B_n}{\sinh n\pi}\left(\cosh\frac{n\pi}{2}-\frac{2}{n\pi}\sinh\frac{n\pi}{2}\right)+\frac{n\pi b_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\right]\cosh\frac{n\pi}{2}y\\ &\quad-\sum_{n=1}^{\infty}\left[\frac{B_n}{\sinh n\pi}\sinh\frac{n\pi}{2}\right]y\sinh\frac{n\pi}{2}y\,,\end{aligned} \tag{77}$$

$$\begin{aligned}\left.\frac{\partial w}{\partial y}\right|_{y=\pm1} &= -\frac{p}{6D}+\sum_{n=1}^{\infty}\left[B_n+\frac{n\pi}{2}b_n\left(\coth n\pi-\frac{1}{\sinh n\pi}\right)\right]\sin(1+x)\frac{n\pi}{2}\\ &\quad-\sum_{\substack{n=1\\ n:\,\mathrm{odd}}}^{\infty}\frac{n\pi}{2\sinh n\pi}\left[\left\{(a_n+a_n')-\frac{2(A_n+A_n')}{(n\pi)^2}\right\}\sinh\frac{n\pi}{2}+\frac{A_n+A_n'}{n\pi}\cosh\frac{n\pi}{2}\right]\cosh\frac{n\pi}{2}x\\ &\quad+\sum_{\substack{n=1\\ n:\,\mathrm{odd}}}^{\infty}\frac{n\pi}{2\sinh n\pi}\left[\left\{(a_n-a_n')-\frac{2(A_n-A_n')}{(n\pi)^2}\right\}\cosh\frac{n\pi}{2}+\frac{A_n-A_n'}{n\pi}\sinh\frac{n\pi}{2}\right]\sinh\frac{n\pi}{2}x\\ &\quad+\sum_{\substack{n=1\\ n:\,\mathrm{odd}}}^{\infty}\frac{1}{2\sinh n\pi}\left[(A_n-A_n')\cosh\frac{n\pi}{2}\right]x\cosh\frac{n\pi}{2}x\\ &\quad-\sum_{\substack{n=1\\ n:\,\mathrm{odd}}}^{\infty}\frac{1}{2\sinh n\pi}\left[(A_n+A_n')\sinh\frac{n\pi}{2}\right]x\sinh\frac{n\pi}{2}x\,.\end{aligned} \tag{78}$$

$A_1'$, $a_1'$, $B_1$ and $b_1$ are numerically determined so as to sufficiently satisfy the conditions (77) and (78). The results of the calculation are

$$\left.\begin{aligned}A_1' &= 0.50097\ \times p/D\,,\\ a_1' &= -0.32337\times p/D\,,\\ B_1 &= 0.45492\ \times p/D\,,\\ b_1 &= 0.30308\ \times p/D\,.\end{aligned}\right\} \tag{79}$$

From Table 1, we have

$$\left.\begin{aligned}A_1 &= 0.258012\ \times p/D\,,\\ a_1 &= -0.217154\times p/D\,.\end{aligned}\right\} \tag{80}$$

Using the values (79) and (80), the deflection at midpoint is calculated as

$$w_0 = 0.02515\,p/D \tag{81}$$

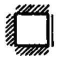
and

$$w_0 = 0.00157\,pa^4/D \tag{82}$$

## 5. Conclusions

The authors calculated the deflections of plates under various edge conditions supported and clamped. In Table 6, the results obtained are presented, together with the results calculated by S. Timoshenko[1] and by one of the authors, who calculated after rewriting to the form of finite differences equations[2].

Table 6.

| Boundary condition | $w_0 (\times pa^4/D)$ | | |
|---|---|---|---|
| | solved in this paper | solved by S. Timoshenko | solved replacing by finite differences equations |
|  | 0.004060 | 0.00406 | 0.00391 |
|  | 0.00124 | 0.00126 | 0.00126 |
|  | 0.002783 | 0.00278 | 0.00258 |
|  | 0.001909 | 0.00191 | 0.00182 |
|  | 0.00213 | — | — |
|  | 0.00157 | — | 0.00149 |

 represents edge supported,

 represents edge clamped.

Thus the authors make clear deflections under various edge conditions. Regarding the last case, quite a rough approximation is used but it is not likely that there is a large difference from the strict solution.

In this paper, the calculations are made for the simple case, where the plate has a square form and is loaded by uniform pressure. It is easy to extend the solution to the case of rectangular plate and it may be also applicable to the plate loaded not uniformly. The solution for the plate under sinusoidal load or the plate loaded in the form of a triangular prism can be reduced without difficulty.

The solution for the plate with free edges is excluded in this paper, the above solutions obtained however may be extended to other cases. In this case some suitable terms must be added to the right hand side of Eq. (5) so that the particular solution $W_1$ becomes zero at the four corners of plate ($x=\pm1$, $y=\pm1$). Generally, $W_1$ does not vanish at the four corners, except in the cases where the neighbouring two sides of free edge are both supported. For this reason $W_1$ takes a different form that in Eq. (5) and the solution obtained differs somewhat from that of Eq. (15).

It is easy to get the solution for the plate of which three edges are supported and one edge free. The authors obtained the result which shows the deflection at the midpoint of 0.01290 $pa^4/D$, while S. Timoshenko has shown the result of 0.01298 $pa^4/D$.

## References

1) S. Timoshenko: Theory of Plates and Shells (1940).
2) T. Nishihara and M. Yoshida: Trans. of the Japan Soc. Mech. Eng., Vol. 13, No. 45, (1947).