On the Forced Steady Oscillations in a Nonlinear Closed Loop Control System.

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1. Introduction.

In this paper, we treat the response of a closed loop control system under a periodic input, free oscillations of which are of the self-sustained type caused by nonlinear characteristic of the element of the control circuit, which consists of the nonlinear integral action controlling means with cubic characteristic, proportional measuring means and two-capacity process with self-regulation.

2. Control Equation.

The equation of the controlled system is:

$$b_2\ddot{\varphi}+b_1\dot{\varphi}+b_0\varphi=r_1\mu+r_2\sigma \tag{2.1}$$

where φ: controlled variable, μ: manipulated variable,
σ: disturbance, r_1, r_2, b_2, b_1, b_0: circuit constants.

For the proportional measuring means:

$$\xi=K\varphi \tag{2.2}$$

where ξ: measuring variable,
K: measuring sensitivity.

For the integral action controlling means with cubic characteristic:

$$-\dot{\mu}=k_1\xi-k_2\xi^3 \tag{2.3}$$

where k_1, k_2, : constants.

From (2.1), (2.2) and (2.3) we obtain the control equation for φ and σ.

$$b_2\dddot{\varphi}+b_1\ddot{\varphi}+b_0\dot{\varphi}+Kk_1r_1\varphi-K^3k_2r_1\varphi^3=r_2\dot{\sigma} \tag{2.4}$$

Here, we introduce new quantities:

$$\left.\begin{aligned}&\omega_0^2\equiv b_0/b_2\ , \qquad \tau\equiv\omega_0 t\ , \qquad s\equiv r_2/b_2\\ &R\equiv k_1K\frac{r_1b_2}{b_0b_1}\ , \qquad S\equiv\frac{b_1}{b_2}\sqrt{\frac{b_2}{b_0}}\ , \qquad T=k_2K^3\frac{r_1b_2}{b_0b_1}\end{aligned}\right\} \tag{2.5}$$

and represent the differentiation with respect to τ by dot, then we can transform (2.4) into the following equation.

$$\dddot{\varphi}+S\ddot{\varphi}+\dot{\varphi}+RS\varphi-TS\varphi^3=s\dot{\sigma} \tag{2.6}$$

3. Approximate Solution of Control Equation and Its Stability.

If the control circuit represented by control equation (2.6) is subjected to sinusoidal excitation $\sigma=\sigma_0 \sin u\tau$, we obtain as the steady solutions one with the same frequency as that of excitation and the other with a frequency different from that of excitation. The former is called the harmonic response and is represented by the form

$$\varphi=\varphi_1 \sin u\tau+\varphi_2 \cos u\tau\,; \tag{3.1}$$

and the latter is the so-called combined response which may approximately be expressed by

$$\varphi=\varphi_f \sin(u_f\tau+\theta_f)+\varphi_h \sin(u\tau+\theta_h) \tag{3.2}$$

a) Harmonic response.

In order to determine the stability of the solution (3.1) of equation (2.6), we assume that φ_1 and φ_2 are slowly varying function of the time τ, and differential quotients of order two or higher of φ_1 and φ_2 are negligible as compared with $\dot{\varphi}_1$ and $\dot{\varphi}_2$. Substituting (3.1) into (2.6) and taking the followings into consideration

$$\left.\begin{aligned}
&\varphi=\varphi_1(\tau)\sin u\tau+\varphi_2(\tau)\cos u\tau\;,\\
&\dot{\varphi}=\dot{\varphi}_1\sin u\tau+\dot{\varphi}_2\cos u\tau+\varphi_1 u\cos u\tau-\varphi_2 u\sin u\tau\;,\\
&\ddot{\varphi}=2\dot{\varphi}_1 u\cos u\tau-2\dot{\varphi}_2 u^2-\varphi_1 u^2\sin u\tau-\varphi_2 u^2\cos u\tau\;,\\
&\dddot{\varphi}=-\dot{\varphi}_1 u^2\sin u\tau-\dot{\varphi}_2 u^2\cos u\tau-\varphi_1 u^3\cos u\tau+\varphi_2 u^3\sin u\tau\;,\\
&\varphi^3=3/4(\varphi_1^2+\varphi_2^2)(\varphi_1\sin u\tau+\varphi_2\cos u\tau)+\cdots\cdots\cdots\cdots,\\
&\sigma=\sigma_0\sin u\tau\;,
\end{aligned}\right\} \tag{3.3}$$

and, in addition, putting

$$\varphi_0^2\equiv\varphi_1^2+\varphi_2^2 \tag{3.4}$$

$$\varphi_l^2\equiv\frac{R-1}{(3/4)T} \qquad \text{(we assume } R>1) \tag{3.5}$$

$$x\equiv\varphi_1/\varphi_l,\quad y\equiv\varphi_2/\varphi_l,\quad r=\varphi_0/\varphi_l,\quad F\equiv\frac{s\sigma_0}{\varphi_l}\;, \tag{3.6}$$

we obtain

$$\left.\begin{aligned}
&(1-3u^2)\dot{x}-2Su\dot{y}=-S(R-1)\Big(\frac{R-u^2}{R-1}-r^2\Big)x+u(1-u^2)y\\
&2Su\dot{x}+(1-3u^2)\dot{y}=uF-u(1-u^2)x-S(R-1)\Big(\frac{R-u^2}{R-1}-r^2\Big)y
\end{aligned}\right\} \tag{3.7}$$

In the steady state we have

$$\dot{x}=0 , \qquad \dot{y}=0 .$$

Therefore, if we represent the values of x, y, r and φ_0 in the steady state by x_0, y_0, r_0 and φ_{00} respectively, (x_0, y_0) shows singular point in x, y-plane.
In this state, equation (3.7) becomes

$$\left.\begin{aligned} &S(R-1)\left(\frac{R-u^2}{R-1}-r_0^2\right)x_0-u(1-u^2)y_0=0 \\ &u(1-u^2)x_0+S(R-1)\left(\frac{R-u^2}{R-1}-r_0^2\right)y_0=uF \end{aligned}\right\} \tag{3.8}$$

Hence,

$$\left.\begin{aligned} &\left[u^2(1-u^2)^2+S^2(R-1)^2\left(\frac{R-u^2}{R-1}-r_0^2\right)^2\right]r_0^2=u^2F^2 \\ &x_0=\frac{r_0^2(1-u^2)}{F} \\ &y_0=\frac{r_0^2S(R-1)\left(\frac{R-u^2}{R-1}-r_0^2\right)}{uF} \end{aligned}\right\} \tag{3.9}$$

Particularly, in the case of no external excitation to the control circuit, i. e. $F=0$, we obtain from (3.8) or (3.9)

$$u=1, \qquad r_0^2=\frac{R-u^2}{R-1}=1 ,$$

and, also, from (3.5) and (3.6)

$$\varphi_l^2=\varphi_{00}^2=\frac{R-1}{(3/4)T} . \tag{3.10}$$

Then we can see that φ_l represents the amplitude of the free nonlinear oscillation of the circuit. We further observe that, in the case of $R<1$, the circuit has no self-excited oscillation, and as the control sensitivity R increases, the amplitude of the free nonlinear oscillation increases, and also as the nonlinearity T increases, the amplitude decreases.

In Fig. 1 are shown response curves, taking F as a parameter, which represent the relation between the square of frequancy, u^2, and the square of amplitude, r_0^2, given by (3.9).

b) Stability of harmonic response.

In order to determine the stability of the solution, it is convenient to consider the solution curve of the following differential equation obtained from (3.7) in the x, y-plane;

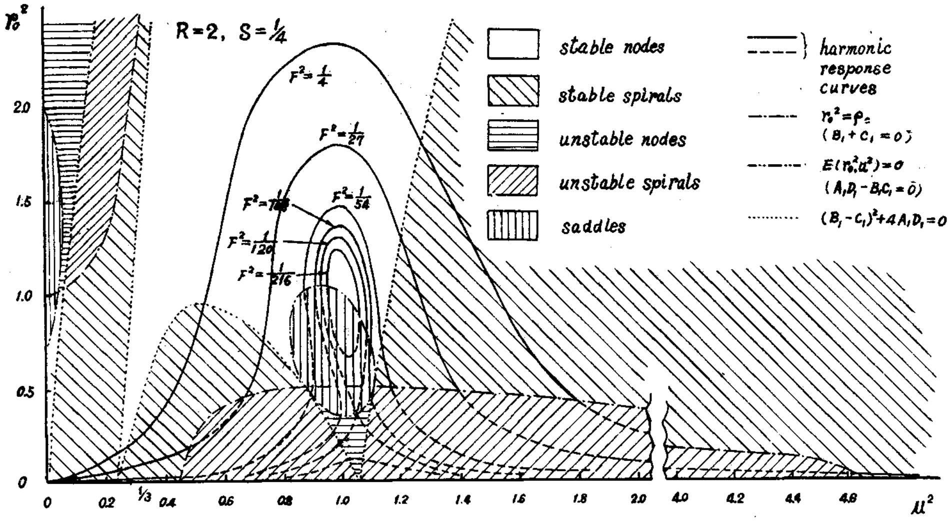

Fig. 1. Response curves for the harmonic oscillations and their stability.

$$\left.\begin{aligned}
&\frac{dy}{dx}=\frac{Ax+By+u(1-3u^2)F}{Cx+Dy+2su^2F}\equiv\frac{Y(x,y)}{X(x,y)}\\
&A=-u\left\{(1-u^2)(1-3u^2)-2S^2(R-1)\left(\frac{R-u^2}{R-1}-r^2\right)\right\}\\
&B=-S\left\{(R-1)(1-3u^2)\left(\frac{R-u^2}{R-1}-r^2\right)+2u^2(1-u^2)\right\}\\
&C=-S\left\{(R-1)(1-3u^2)\left(\frac{R-u^2}{R-1}-r^2\right)+2u^2(1-u^2)\right\}\\
&D=u\left\{(1-u^2)(1-3u^2)-2S^2(R-1)\left(\frac{R-u^2}{R-1}-r^2\right)\right\}
\end{aligned}\right\}\qquad(3.11)$$

For the purpose of investigating the character of the neighbourhood of singular point (x_0, y_0) as steady solution, we transform the origin to (x_0, y_0).
Putting

$$x=x_0+x_1\ ,\qquad y=y_0+y_1\ ,$$

we develop the numerator and denominator in the right hand side of (3.11) in powers of x_1 and y_1, and reject all but the linear terms in x_1 and y_1. The result is the following approximate differential equation in x_1 and y_1.
Putting

$$\left.\begin{aligned}
&\frac{dy_1}{dx_1}=\frac{A_1x_1+B_1y_1}{C_1x_1+D_1y_1}\\
&A_1=-(R-1)\left[(1-3u^2)\left\{\frac{u(1-u^2)}{R-1}-2Sx_0y_0\right\}-2S^2u\left\{\frac{R-u^2}{R-1}-(r_0^2+2x_0^2)\right\}\right]\\
&B_1=-S(R-1)\left[(1-3u^2)\left\{\frac{R-u^2}{R-1}-(r_0^2+2y_0^2)\right\}+2u\left\{\frac{u(1-u^2)}{R-1}+2Sx_0y_0\right\}\right]
\end{aligned}\right\}\qquad(3.12)$$

$$\left.\begin{aligned} C_1 &= -S(R-1)\Big[\Big\{(1-3u^2)\frac{R-u^2}{R-1}-(r_0^2+2x_0^2)\Big\}+2u\Big\{\frac{u(1-u^2)}{R-1}-2Sx_0y_0\Big\}\Big] \\ D_1 &= (R-1)\Big[(1-3u^2)\Big\{\frac{u(1-u^2)}{R-1}+2Sx_0y_0\Big\}-2S^2u\Big\{\frac{R-u^2}{R-1}-(r_0^2+2y_0^2)\Big\}\Big] \end{aligned}\right\}$$

By the theorem of Poincaré this differential equation has the same type of singularity as that of differential equation (3.11) in x and y, unless $A_1D_1-B_1C_1=0$. Hence, we can classify the types of the singularities by the forms of integral curves in the neighbourhood of the singularity as follows:

$$\left.\begin{aligned} (B_1-C_1)^2+4A_1D_1 &= 4(R-1)^2[S^2\{(3u^2-1)^2+4S^2u^2\}r_0^4 \\ &\quad -\{(3u^2-1)\frac{1-u^2}{R-1}+2S^2\frac{R-u^2}{R-1}-4S^2r_0^2\}^2u^2] \\ A_1D_1-B_1C_1 &= -(R-1)^2\{4S^2u^2-(3u^2-1)^2\}\Big[\frac{u^2(1-u^2)^2}{(R-1)^2}+S^2\Big\{\Big(\frac{R-u^2}{R-1}\Big)^2 \\ &\quad -4\frac{R-u^2}{R-1}r_0^2+3r_0^4\Big\}\Big] \\ &\equiv -\{4S^2u^2+(3u^2-1)^2\}E(r_0^2,\ u^2) \\ B_1+C_1 &= 4S(R-1)(3u^2-1)\Big[\frac{1}{2}\frac{R-u^2}{R-1}-\frac{u^2(1-u^2)}{(R-1)(3u^2-1)}-r_0^2\Big] \\ &\equiv 4S(R-1)(3u^2-1)[\rho_c-r_0^2] \end{aligned}\right\} \quad (3.13)$$

$$\left.\begin{aligned} &(B_1-C_1)^2+4A_1D_1>0\begin{cases} A_1D_1-B_1C_1<0\begin{cases} B_1+C_1<0 \cdots \text{stable nodes} \\ B_1+C_1>0 \cdots \text{unstable nodes} \end{cases} \\ A_1D_1-B_1C_1>0 \cdots\cdots\cdots \text{saddles} \end{cases} \\ &(B_1-C_1)^2+4A_1D_1=0\begin{cases} B_1+C_1<0 \cdots\cdots\cdots \text{stable nodes} \\ B_1+C_1>0 \cdots\cdots\cdots \text{unstable nodes} \end{cases} \\ &(B_1-C_1)^2+4A_1D_1<0\begin{cases} B_1+C_1<0 \cdots\cdots\cdots \text{stable spirals or centers} \\ B_1+C_1>0 \cdots\cdots\cdots \text{unstable spirals or centers} \end{cases} \end{aligned}\right\} \quad (3.14)$$

The domains in which these singularities exist on the u^2, r_0^2-plane are shown in Fig. 1 and the fact that the curve $E(r_0^2,\ u^2)=0$ coincides with the locus of vertical tangent of the response curves, is assured by calculating $\frac{d(u^2)}{d(r_0^2)}=0$ from (3.9).

c) Non-harmonic response

We see from Fig. 1 that for arbitrarily given F the oscillations are unstable in some range of u; that is, there exists no stable harmonic oscillation. In this case, what response will the system show? We can give a solution to this problem by investigating the possibility of occurrence of stable or unstable limit cycle $\{x(\tau),\ y(\tau)\}$ on x, y-plane. Then we consider the character of the integral curves of (3.11) for large values of x and y on the x, y-plane in order to determine the possibility of limit cycles. Now, if the values of x and y are large enough, we can approximately express (3.11) as follows;

$$\frac{dy}{dx}=\frac{-2Sux+(1-3u^2)y}{(1-3u^2)x+2Suy},$$

Putting

$$a=-2Su, \quad b=1-3u^2, \quad c=1-3u^2, \quad d=2Su,$$

we have

$$(b-c)^2+4ad=-4S^2u^2\leq 0, \qquad b+c=2(1-3u^2),$$

therefore, we may conclude that;

if $u^2>1/3$, ; stable spirals
if $u^2<1/3$, ; unstable spirals

Consequently, if we consider the large circle centered at the origin, in case $u^2<1/3$, the integral curves diverge out of the circle and tend to infinity as τ increases, and in case $u^2>1/3$, the integral curves converge into the circle. In this circle there exist several singuralities, numbers and types of which are different according as the values of u^2, so we cannot say anything concrete about the general property of the integral curves. Hence we must study the character of integral curves case by case. However, this is very difficult and we will not go into this here. We can only state as follows. In the domain where

$$u^2>1/3, \quad |1-u^2|>\varepsilon>0, \quad r_0^2<\rho_c, \quad E(r_0^2, u^2)>0,$$

as will be seen from equation (3.14), there exists a spiral point which is singular and for the large values of x and y, the integral curve converges into the circle as τ increases, so that there exists at least one stable limit cycle.

d) Combined response

When the integral curves of (3.11) have a limit cycle, the oscillation corresponding to the limit cycle $\varphi/\varphi_l=x\cos u\tau+y\sin u\tau$ may be considered to be affected by phase and amplitude modulation, since x and y would be periodic function of τ with large periods. To quantitatively investigate this oscillation it would seem reasonable to replace the general form (3.1) by a sum of two simple harmonic oscillations, as follows:

$$\varphi=\varphi_f\sin(u_f\tau+\theta_f)+\varphi_h\sin(u\tau+\theta_h) \tag{3.15}$$

The quantities φ_f, u_f, θ_f, φ_h and θ_h are all constants. The 1st and 2nd terms of the right hand side of (3.15) respectively show the free and harmonic components

of the combination oscillation.
Now, we have from (3.15)

$$\left.\begin{aligned}\varphi^3=&\frac{3}{4}\varphi_f^3\sin(u_f\tau+\theta_f)-\frac{3}{4}\varphi_f^3\sin 3(u_f\tau+\theta_f)+\frac{3}{4}\varphi_h^3\sin(u\tau+\theta_h)\\&-\frac{3}{4}\varphi_h^3\sin 3(u\tau+\theta_h)+\frac{3}{4}\varphi_f^2\varphi_h\{2\sin(u\tau+\theta_h)\\&-\sin[(2u_f\tau+u)\tau+2\theta_f+\theta_h]+\sin[(2u_f-u)\tau+2\theta_f-\theta_h]\}\\&+\frac{3}{4}\varphi_f\varphi_h^2\{2\sin(u_f\tau+\theta_f)-\sin[(2u\tau+u_f)\tau+2\theta_h+\theta_f]\\&+\sin[(2u-u_f)\tau+2\theta_h-\theta_f]\}+\cdots\cdots\end{aligned}\right\}\quad(3.16)\text{a}$$

If we neglect the terms of higher harmonics, we can obtain

$$\left.\begin{aligned}\varphi^3=&\frac{3}{4}\varphi_f(\varphi_f^2+2\varphi_h^2)\sin(u_f\tau+\theta_f)\\&+\frac{3}{4}\varphi_h(\varphi_h^2+2\varphi_f^2)\sin(u\tau+\theta_h)\end{aligned}\right\}\quad(3.16)\text{b}$$

From the above expression, we have the following sets of equations as before.

$$\left.\begin{aligned}&S\{R-u_f^2-\frac{3}{4}T(\varphi_f^2+2\varphi_h^2)\}\varphi_f=0\\&u_f(1-u_f^2)\varphi_f=0\\&[S\{R-u^2-\frac{3}{4}T(\varphi_h^2+2\varphi_f^2)\}\cos\theta_h-u(1-u^2)\sin\theta_h]\varphi_h=0\\&[S\{R-u^2-\frac{3}{4}T(\varphi_h^2+2\varphi_f^2)\}\sin\theta_h+u(1-u^2)\cos\theta_h]\varphi_h=s\sigma_0 u\end{aligned}\right\}\quad(3.17)$$

From the 1st and 2nd of the relations (3.17) we conclude that, if $\varphi_f\neq0$;

$$u_f=1,\qquad \varphi_f^2+2\varphi_h^2=\frac{R-1}{(3/4)T}=\varphi_l^2\qquad(3.18)$$

Here, if we put

$$\varphi_h/\varphi_l=x_h\ ,\qquad \varphi_f/\varphi_l=x_f\ ,\qquad r^2=x_h^2+x_f^2\ ,\qquad(3.19)$$

we have from the 3rd and 4th equation of (3.17),

$$x_h^2\Big[u^2(1-u^2)^2+S^2(R-1)^2\Big\{\frac{R-u^2}{R-1}-(x_h^2+2x_f^2)\Big\}^2\Big]=u^2F^2\qquad(3.20)$$

From (3.18) and (3.19), we find

$$\left.\begin{aligned}&x_h^2=1-r^2\ ,\quad x_f^2=1-2x_h^2=2r^2-1\\&1/2\leq r^2\leq1\ ,\quad 1/2\geq x_h^2\geq0\ ,\quad 0\leq x_f^2\leq1\ .\end{aligned}\right\}\quad(3.21)$$

In other words, r^2 can exist only between the values of $\frac{1}{2}$ and 1, and outside of this range there exists no combination oscillation. From (3.20) and (3.21) we obtain

$$x_h^2\left[u^2(1-u^2)^2 + S^2(R-1)^2\left\{\frac{R-u^2}{R-1}-(2-3x_h^2)\right\}^2\right]=u^2F^2\ , \tag{3.22}$$

From the above equation we can evaluate x_h^2, then r^2 and x_f^2. (3.21) is applicable only for sufficiently large values of F. Therefore, we have a better approximation for φ^3 in the vicinity of $u^2=1$, considering the last terms of both 1st and 2nd bracket { } of (3.16)a when the value of F is so small that a part of the response curves enter the saddle region. Transforming (3.16)a and omitting higher harmonics, we have the following expression.

$$\varphi^3=\frac{3}{4}(\varphi_f^2+\varphi_h^2)\{\varphi_f \sin(u_f\tau+\theta_f)+\varphi_h \sin(u\tau+\theta_h)\}\ . \tag{3.23}$$

In this case, we have the following sets of equations:

$$\left.\begin{array}{ll}
S\{R-u_f^2-\frac{3}{4}T(\varphi_f^2+\varphi_h^2)\}\varphi_f=0 & \text{a)}\\
u_f(1-u_f^2)\varphi_f=0 & \text{b)}\\
[S\{R-u^2-\frac{3}{4}\cdot T(\varphi_h^2+\varphi_f^2)\}\cos\theta_h-u(1-u^2)\sin\theta_h]\varphi_h=0 & \text{c)}\\
[S\{R-u^2-\frac{3}{4}\cdot T(\varphi_h^2+\varphi_f^2)\}\sin\theta_h+u(1-u^2)\cos\theta_h]\varphi_h=s\sigma_0 u & \text{d)}
\end{array}\right\} \tag{3.24}$$

In case $\varphi_f\neq 0$, we find from $(3.24)_a$ and $(3.24)_b$;

$$u_f=1\ ,\qquad \varphi_f^2+\varphi_h^2\equiv\varphi_{00}^2=\frac{R-1}{3/4\cdot T}=\varphi_l^2 \tag{3.25}$$

Hence, we obtain

$$\left.\begin{array}{l}
x_h^2+x_f^2=r^2=1\ ,\\
x_h^2=\dfrac{u^2F^2}{(1-u^2)^2(S^2+u^2)}
\end{array}\right\} \tag{3.26}$$

From (3.26) we find that x_h^2 increases as $u^2\to 1$ and the value exceeds 1 for some value of u, and in this range there exists no combination oscillation given by (3.10). Therefore, in case $x_h^2>1$, the 1st equation of (3.26) beoomes invalid. The range of non-existence of combination oscillation is given by the following expression.

$$1\leq\frac{u^2F^2}{(1-u^2)^2(S^2+u^2)}$$

The above expression is nothing but the range of u^2 in which the harmonic response curves given by (3.9) have the value $r_0^2>1$. Response curves for combination oscillations given by (3.21), (3.22) and (3.26) are shown in Fig. 2 with harmonic response curves.

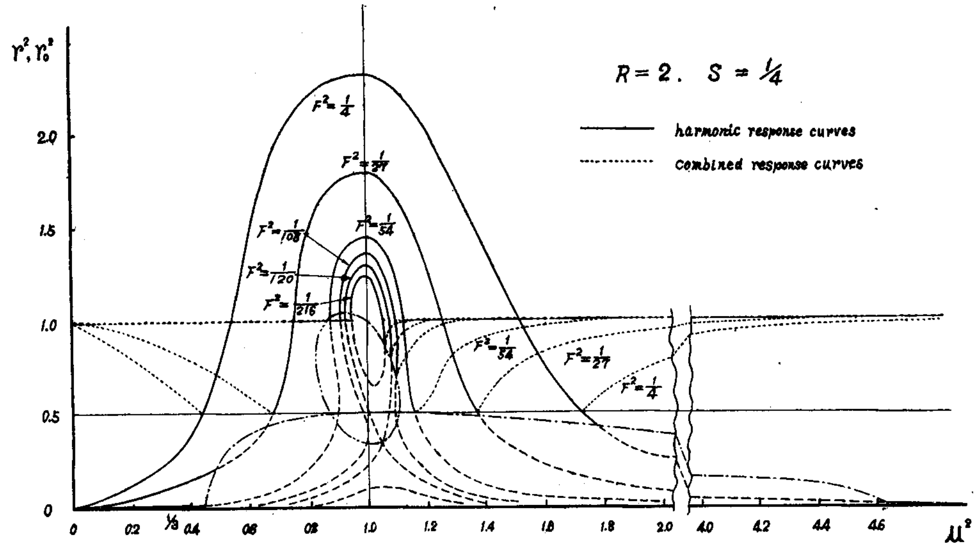

Fig. 2. Response curves for harmonic and combination oscillations.

4. Conclusion.

The response of the nonlinear closed loop control system under a periodic excitation, of which the control equation is a nonlinear differential equation of the third order represented by equation (2.6), is very similar to the solution of the well-known nonlinear differential equation(1) of the second order $\ddot{x}-\varepsilon(1-x^2)\dot{x}+x=P_0\sin\omega t$ investigated by Van der Pol, Andronow and Witt on the vacuum tnbe circuit. The response curves for both harmonic and combination oscillations of the system have the same appearance as in the case treated by Andronow and Witt. In addition, the domains of distribution of the singularities on r_0^2, u^2-plane are very similar to those in the case of Andronow and Witt within a comparatively large range which contains $u^2=1$.

We have found the difference of two cases at the critical curves of stability, i. e. there exist some stable harmonic oscillations in the range of both small and large values of u, even if the amplitude of oscillations is small.

References

(1) J. J. Stoker. Nonlinear Vibrations. 1950, p. 147.