Analytical Considerations on Cnoidal and Solitary Waves

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Synopsis

In this paper, the mathematical theory on progressive translation waves of the permanent type is developed.

The analysis of the second approximation based upon the momentum and energy approaches considering the effects of the vertical acceleration shows that the surface profile of waves is expressed in terms of an elliptic function which is designated as the cn-function, the name cnoidal being derived therefrom as analogous to sinusoidal, and the solitary wave is included as a special case of the cnoidal wave.

It should be noticed that the behaviour of cnoidal waves may be reduced to that of classical sinusoidal waves at a special limiting approximation.

1. Introduction

Though the accurate expression of translation waves, which travel in channels and rivers, is given by G. H. Keulegan,[1] it is usually said that the translation wave is in the category of gravity waves, with the surface wave, and has the permanent displacement in fluid particles as flood and tidal waves.

Up to the present time, many references on researches of the behaviour of translation wave have been made to. Especially, fruitful results have been obtained by P. Masse[2] and Dr. T. Hayashi[3] in the light of the unsteady problem of hydraulics and by J. Boussinesq,[4] G. H. Keulegan and G. W. Patterson[1] in the light of irrotational wave motions.

The classification of translation waves will be simply divided into the following two cases: one is the case in which the frictional force is negligible or in sloping channels the equilibrium between the gravity term and frictional term is assumed to hold as a special case, and the other is that in which the friction governs the wave characteristics to a pronounced degree. Although very long waves like floods belong to the latter case, these effects are considered to be negligible or in equilibrium for progressive translation waves. In this paper dealing with progressive waves of the permanent type, the former is exclusively treated.

In the first approximation concerned with waves of small amplitude, many comprehensive publications have been presented, and their results are very familiar to hydraulic engineers as the long wave theory. They propagate in channels without change of their form and their velocity of propagation is given by the well-known Lagrange velocity law or its similar formulas.

But, if the effects of the height above the undisturbed level of the fluid surface, the slope, and the curvature of the free surface are sufficiently appreciable, the first approximation is no longer satisfactory, and the second approximation considered by the existence of the vertical acceleration will become neccesary. This consideration was taken by Boussinesq, Keulegan and Patterson, on the assumption of irrotationality of progressive wave flows. Their hydraulic and mathematical treatment shows that the solution of translation waves of appreciable height is expressed in terms of an elliptic function which is designated as the cn-function, and they are called the cnoidal waves being derived therefrom as analogous to sinusoidal by Korteweg and deVries.

In this paper, another approach of progressive translation waves on running water, based upon the second approximate equation of open channel flows derived by the usual momentum and energy methods as F. Sérre[5] and the author did, is developed. Hence, the modification of the basic equation of flows governed by the hydrostatic law of pressure will be necessary. This aim is attained by the appropriate assumption of the local velocity distribution. Thus, the second approximation of translation waves of the permanent type can be easily derived, for the qualification "permanent" indicates that the progressive wave flows appear to be steady when observed with an appropriate velocity.

The analysis shows that the basic wave configurations of the second approximation are again those of cnoidal waves including the solitary waves as an extreme case. Of course, the behaviour of cnoidal waves is less familiar to hydraulic engineers, but it can be easily observed that translation waves expressed by this form are traveling in channels and seas, and the form of undular jump is also described by this expression.[6] Moreover, the configuration of surfs at near coasts is known as the same figure of cnoidal waves. Hence the application of the theory of cnoidal and solitary waves to the coastal engineering will be promoted with the development of this field. In fact, it should be noticed that the solution of cnoidal waves may be reduced to that of classical infinitesimal sinusoidal waves with a sufficient accuracy, at a special limiting approximation.

2. Fundamental Relations

In this paper, for the sake of mathematical simplicity, two dimensional fluid motions are concerned. If necessary, they will be easily extended to the case in which flows in wide channels are treated.

Taking the x-axis in the downstream direction along the channel bed, the z-axis vertically upward, t: the time, u_m: the mean velocity, and h: the depth, the equation of continuity is still in this case

$$\frac{\partial h}{\partial t}+u_m\frac{\partial h}{\partial x}+h\frac{\partial u_m}{\partial x}=0. \tag{1}$$

The following assumption of the local velocity in the x-direction, u, is introduced,

$$u(x, z, t)=u_m(x, t)\cdot f\left(\frac{z}{h}\right), \tag{2}$$

where $f\left(\frac{z}{h}\right)$ is an arbitrary function determined by the velocity distribution of flows. Provided the distribution of velocity in this unsteady case is assumed to be given by the same as in the uniform regime, the above expression would be acknowledged.

Under this assumption, in two dimensional flows, the stream function ψ is

$$\psi(x, z, t)=h\cdot u_m F\left(\frac{z}{h}\right)+\varphi(x, t), \tag{3}$$

where φ is an arbitrary function of variables x and t. Hence, considering the kinematic boundary condition, the vertical local velocity, w, is given by

$$w(x, z, t)=\frac{u\cdot z}{h}\frac{\partial h}{\partial x}+\left\{F\left(\frac{z}{h}\right)-F(0)\right\}\frac{\partial h}{\partial t}, \tag{4}$$

where $F\left(\frac{z}{h}\right)$ means the integral of $f\left(\frac{z}{h}\right)$, and the physical consideration indicates that $F(0)$ is zero. Boussinesq assumed that the local velocity was given by

$$w=\frac{u\cdot z}{h}\left(\frac{1}{u_m}\frac{\partial h}{\partial t}+\frac{\partial h}{\partial x}\right). \tag{5}$$

Considering equations (4) and (5), it will be recognized that only in the varied regime, Boussinesq's assumption is always sure, but in the unsteady regime, it is no longer satisfied except the true translation type of velocity distribution.

The pressure distribution is defined by the following expression,

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-g\cos\theta-\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right), \tag{6}$$

where p: the pressure, ρ: the density of fluid, g: the acceleration of gravity, and θ: the inclination angle of the channel bed.

Insertion of equations (2) and (4) into equation (6) and integration yield

$$p=\rho g\cos\theta\,(h-z)+\rho hu_m^2\frac{\partial^2 h}{\partial x^2}\int_n^1 mf^2(m)\,dm+\rho hu_m\frac{\partial^2 h}{\partial x\partial t}$$
$$\times\int_n^1\{mf(m)+F(m)f(m)\}\,dm+\rho h\frac{\partial^2 h}{\partial t^2}\int_n^1 F(m)\,dm+\rho h\frac{\partial u_m}{\partial t}\frac{\partial h}{\partial x}$$
$$\times\int_n^1 mf(m)\,dm+\rho u_m\frac{\partial h}{\partial x}\frac{\partial h}{\partial t}\int_n^1\{f(m)\,F(m)+mf'(m)\,F(m)-mf(m)$$
$$-m^2f'(m)-mf^2(m)\}\,dm-\rho u_m^2\left(\frac{\partial h}{\partial x}\right)^2\int_n^1 mf^2(m)\,dm+\rho\left(\frac{\partial h}{\partial t}\right)^2$$
$$\times\int_n^1\{f(m)\,F(m)-mf(m)\}\,dm,\qquad (7)$$

where m and n: z/h, and $f'(m)$: the derivative of $f(m)$. In the above equation the effect of surface tension expressed by $-T\dfrac{\partial^2 h}{\partial x^2}$ is neglected. Using equation (7), the momentum and energy approaches in hydrodynamics lead to the following fundamental equations.

Momentum:

$$h\frac{\partial u_m}{\partial t}+u_m\frac{\partial h}{\partial t}+g\frac{\partial M}{\partial x}=gh\sin\theta-\frac{\tau}{\rho},$$
$$gM=\alpha_m hu_m^2+\frac{gh^2\cos\theta}{2}+\gamma h^2u_m^2\frac{\partial^2 h}{\partial x^2}+(\beta+\varepsilon)\,h^2u_m\frac{\partial^2 h}{\partial x\partial t}$$
$$+\delta h^2\frac{\partial^2 h}{\partial t^2}+\beta h^2\frac{\partial u_m}{\partial t}\frac{\partial h}{\partial x}+(\varepsilon+\eta-\beta-\zeta-\gamma)\,hu_m\frac{\partial h}{\partial x}\frac{\partial h}{\partial t}$$
$$-\gamma hu_m^2\left(\frac{\partial h}{\partial x}\right)^2+(\varepsilon-\beta)\,h\left(\frac{\partial h}{\partial t}\right)^2,\qquad (8)$$

Energy:

$$\alpha_m\frac{\partial u_m}{\partial t}+\frac{\alpha_m-\alpha_e}{2}\;\frac{u_m}{h}\;\frac{\partial h}{\partial t}+g\frac{\partial H}{\partial x}=g\sin\theta-\frac{\tau}{\rho h},$$
$$gH=\frac{\alpha_e}{2}u_m^2+gh\cos\theta+\gamma' hu_m^2\frac{\partial^2 h}{\partial x^2}+(\beta'+\varepsilon')\,hu_m\frac{\partial^2 h}{\partial x\partial t}+\delta' h\frac{\partial^2 h}{\partial t^2}$$
$$+(\varepsilon'+\eta'+\alpha_e''-\beta'-\zeta'-\gamma')\,u_m\frac{\partial h}{\partial x}\frac{\partial h}{\partial t}+\beta' h\frac{\partial u_m}{\partial t}\frac{\partial h}{\partial x}$$
$$+\frac{\alpha_e'''+2\gamma'}{2}u_m^2\left(\frac{\partial h}{\partial x}\right)^2+\frac{\alpha_e'+2\varepsilon'-2\beta'}{2}\left(\frac{\partial h}{\partial t}\right)^2,\qquad (9)$$

where

$$\alpha_m=\int_0^1 f^2(m)\,dm,\quad \alpha_e=\int_0^1 f^3(m)\,dm,\quad \alpha_e'=\int_0^1 f(m)\,F^2(m)\,dm,$$
$$\alpha_e''=\int_0^1 mf^2(m)\,F(m)\,dm,\quad \alpha_e'''=\int_0^1 m^2f^3(m)\,dm,$$
$$\beta=\int_0^1\int_n^1 mf(m)\,dm\,dn,\qquad \beta'=\int_0^1 f(n)\int_n^1 mf(m)\,dm\,dn,$$
$$\gamma=\int_0^1\int_n^1 mf^2(m)\,dm\,dn,\qquad \gamma'=\int_0^1 f(n)\int_n^1 mf^2(m)\,dm\,dn,$$

$$\left\{\begin{array}{ll} \delta=\int_0^1\int_n^1 F(m)\,dm\,dn\,, & \delta'=\int_0^1 f(n)\int_n^1 F(m)\,dm\,dn\,, \\ \varepsilon=\int_0^1\int_n^1 f(m)\,F(m)\,dm\,dn\,, & \varepsilon'=\int_0^1 f(n)\int_n^1 f(m)\,F(m)\,dm\,dn\,, \\ \zeta=\int_0^1\int_n^1 m^2 f'(m)\,dm\,dn\,, & \zeta'=\int_0^1 f(n)\int_n^1 m^2 f'(m)\,dm\,dn\,, \\ \eta=\int_0^1\int_n^1 m f'(m)\,F(m)\,dm\,dn\,, & \eta'=\int_0^1 f(n)\int_n^1 m f'(m)\,F(m)\,dm\,dn\,. \end{array}\right. \tag{10}$$

Equation (10) shows the correction coefficients of velocity distribution including usual momentum and energy ones. And equations (8) and (9) are the fundamental relations derived by the momentum and energy approaches considered the vertical acceleration, respectively. Of course, they are reduced to the usual equations of open channel flows, if the effects of the vertical acceleration are neglected.

In usual open channel flows, the velocity varies with the depth to a considerable degree, and it is better that the above defined correction coefficients are introduced. But in true translation waves, the velocity remains its cross-sectional constancy at any section from the definition. Thus, $f(m)=1$, $F(m)=m$, and $f'(m)=0$, and equation (10) has the definite numerical values, as follows.

$$\left\{\begin{array}{l} \alpha_e=\alpha_m=1\,, \\ \alpha_e'=\alpha_e''=\alpha_e'''=\beta=\gamma=\delta=\varepsilon=\beta'=\gamma'=\delta'=\varepsilon'=\dfrac{1}{3}\,, \\ \zeta=\eta=\zeta'=\eta'=0\,. \end{array}\right. \tag{11}$$

Equations (8) and (9) in which correction coefficients are represented by equation (11) are the basic relations to discuss the present second approximation of wave profiles.

3. Cnoidal Waves

In developing the theory of cnoidal waves, consider the flows which are viewed from the moving coordinate system traveling with the constant velocity of propagation, because progressive wave flows of the permanent type will be appeared to be steady as shown in the previous section. The ascending waves are treated as well as the descending waves, but in this paper, the behaviour of descending waves are exclusively concerned.

Let v_w be this velocity of propagation in the x-direction, then the following transformations are established.

$$\left\{\begin{array}{l} u_m(x,\,t)=u_m(x-v_w t)=u_m(\zeta)\,, \\ h(x,\,t)=h(x-v_w t)=h(\zeta)\,. \end{array}\right. \tag{12}$$

Using the substitutions $\frac{\partial u_m}{\partial x}=\frac{du_m}{d\varsigma}$, $\frac{\partial u_m}{\partial t}=-v_w\frac{du_m}{d\varsigma}$ and likewise for h, the equation of continuity (1) becomes

$$(u_m - v_w)\frac{dh}{d\zeta} + h\frac{du_m}{d\zeta} = 0\,. \tag{13}$$

Hence, the relation of the progressive discharge rate is derived, as R. F. Dressler, the author and his colleagues did in discussing of hydraulic characteristics of roll-waves.

$$(v_w - u_m)\,h = K\,. \tag{14}$$

If the celerity of waves c is introduced, the following relationship may be considered in the direction of x-axis.

$$v_w = u_m + c\,, \qquad \text{and} \qquad c = \frac{K}{h}\,. \tag{15}$$

Substitution these quantities into equation (8) yields

$$\frac{K^2}{3}\frac{d}{d\zeta}\left[\frac{d^2h}{d\zeta^2} - \frac{1}{h}\left(\frac{dh}{d\zeta}\right)^2\right] + \left(gh\cos\theta - \frac{K^2}{h^2}\right)\frac{dh}{d\zeta} = gh\sin\theta - \frac{\tau}{\rho} \tag{16}$$

If it is assumed that at $h = h_0$, where h_0 is the normal depth of flows or the still water depth in the horizontal bed, the derivatives $\frac{dh}{d\zeta}$ and $\frac{d^2h}{d\zeta^2}$ are not necessarily vanished, equation (16) is transformed into the following expression, under the condition of $gh\sin\theta - \frac{\tau}{\rho} = 0$.

$$\frac{d^2h}{d\zeta^2} - \frac{1}{h}\left(\frac{dh}{d\zeta}\right)^2 = \frac{3c_m}{2K^2} - \frac{3}{h} - \frac{3g\cos\theta}{2K^2}h^2\,, \tag{17}$$

where c_m is a constant. Integration of equation (17) yields

$$\left(\frac{dh}{d\zeta}\right)^2 = -\frac{3g\cos\theta}{K^2}\left[h^3 - \frac{c_m' K^2}{g\cos\theta}h^2 + \frac{c_m}{g\cos\theta}h - \frac{K^2}{g\cos\theta}\right], \tag{18}$$

where c_m' is also a constant of integration.

The same procedure used the energy equation (9) follows

$$\left(\frac{dh}{d\zeta}\right)^2 = -\frac{3g\cos\theta}{K^2}\left[h^3 - \frac{c_e}{g\cos\theta}h^2 - \frac{c_e' K^2}{g\cos\theta}h - \frac{K^2}{g\cos\theta}\right], \tag{19}$$

where c_e and c_e' are both constants of integration.

In both cases, mathematical and physical considerations lead to the fact that the polynomial of equations (18) and (19) has three positive roots, say, h_1, h_2, and h_3, where $h_1 \geq h_2 \geq h_3$, and $h_3 = K^2/gh_1h_2\cos\theta$.

Hence they are transformed into

$$\left(\frac{dh}{d\zeta}\right)^2 = \frac{3g\cos\theta}{K^2}(h_1 - h)(h - h_2)\left(h - \frac{K^2}{gh_1h_2\cos\theta}\right), \tag{20}$$

Introducing the new variable as Keulegan did,

$$h = h_1\cos^2\chi + h_2\sin^2\chi\,, \tag{21}$$

equation (20) becomes

$$d\zeta = \varDelta \frac{d\chi}{\sqrt{1-k^2 \sin^2 \chi}}, \tag{22}$$

where

$$\varDelta = \frac{2K}{\sqrt{3g\cos\theta\left(h_1 - \dfrac{K^2}{gh_1h_2\cos\theta}\right)}}, \tag{23}$$

and

$$k^2 = \frac{h_1 - h_2}{h_1 - \dfrac{K^2}{gh_1h_2\cos\theta}} \leqq 1. \tag{24}$$

Selecting the origin of the moving coordinate system at the apex, $h=h_1$, $\zeta=0$, the wave forms of progressive waves are expressed by the following relation

$$\zeta = \varDelta \int_0^{\chi} \frac{d\chi}{\sqrt{1-k^2\sin^2\chi}} = \varDelta \cdot F(\chi, k), \tag{25}$$

where $F(\chi, k)$ is an incomplete elliptic integral of the first kind. Again introducing the Jacobi's elliptic function, since $\sin\chi = \mathrm{sn}\,\zeta$, $\cos\chi = \mathrm{cn}\,\zeta$, and $\mathrm{sn}^2\zeta + \mathrm{cn}^2\zeta = 1$, then

$$h = h_2 + (h_1 - h_2)\,\mathrm{cn}^2\left(\frac{\zeta}{\varDelta}, k\right), \tag{26}$$

or reverting to the original coordinate system,

$$h = h_2 + (h_1 - h_2)\,\mathrm{cn}^2\left[\frac{\sqrt{3g\cos\theta(h_1 - K^2/gh_1h_2\cos\theta)}}{2K}(x - v_w t),\ k\right]. \tag{27}$$

This equation represents an infinite number of undulations of identical size and shape, each symmetrical about a vertical plane passing through the apex. These waves moving without change of their form are known cnoidal waves as described in the introduction.

And h_1 and h_2 represent the depths from the channel bed to the crests and troughs, respectively.

If the variations from the undisturbed level are used as Keulegan did, equation (27) is transformed into the following expression, considering that $h=h_0+h'$, $h_1=h_0+h_1'$, $h_2=h_0-h_2'$, and $h_3=h_0-h_3'$.

$$h' = -h_2' + (h_1' + h_2')\,\mathrm{cn}^2\left[\sqrt{\frac{3(h_1'+h_3')}{4(h_0+h_1')(h_0-h_2')(h_0-h_3')}}(x - v_w t),\ \sqrt{\frac{h_1'+h_2'}{h_1'+h_3'}}\right]. \tag{28}$$

Deriving from the fundamental equation of irrotational translation waves with an appreciable wave height, Keulegan and Patterson showed the cnoidal wave expressible as follows.

$$h' = -h_2' + (h_1' + h_2')\,\mathrm{cn}^2\left[\sqrt{\frac{3(h_1'+h_3')}{4h_0^3}}(x - v_w t),\ \sqrt{\frac{h_1'+h_2'}{h_1'+h_3'}}\right]. \tag{29}$$

Hence, if the deviations from the undisturbed level are negligible compared with the uniform or original depth, equation (28) can be reduced to equation (29), but for higher cnoidal waves, equation (28) seems the more accurate expression. For it is caused by the fact that Keulegan and Patterson neglected the higher order terms of deviation.

4. Characteristics of Cnoidal Waves

(*A*) *Hydraulic condition for formation of cnoidal waves.* In the previous section, the wave profiles of cnoidal waves represented by equation (27) are derived by the assumption $h_1 \geqq h_2 \geqq K^2/gh_1h_2 \cos\theta$. Then, the hydraulic condition for formation of waves is easily obtained. Considering that $K^2=(v_w-u_1)^2 h_1^2=(v_w-u_2)^2 h_2^2=(v_w-u_3)^2 h_3^2$, the above relation is reduced to

$$\left\{\begin{array}{l} 1 \geqq (h_2/h_1)^2 \geqq \boldsymbol{F}_1^2=(v_w-u_1)^2/gh_1\cos\theta\,, \\ (h_1/h_2)^2 \geqq \boldsymbol{F}_2^2=(v_w-u_2)^2/gh_2\cos\theta \geqq (h_3/h_2)^2,\ (h_1/h_2)^2 \geqq 1,\ (h_3/h_2)^2 \leqq 1, \\ \boldsymbol{F}_3^2=(v_w-u_3)^2/gh_3\cos\theta \geqq (h_1/h_3)^2 \geqq 1, \end{array}\right. \tag{30}$$

where u_1, u_2 and u_3 are the mean velocities of flow at the points of h_1, h_2, and h_3, and $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ are the relative Froude Numbers expressed by the above notations.

Hence it follows that the flow regime of cnoidal waves at the crests is always appeared to be in the subcritical, that at the troughs in the subcritical or supercritical, and that at the hydraulically virtual depth h_3 in the supercritical when observed from the moving coordinate system traveling at the constant velocity of propagation v_w. Then cnoidal waves exist only when the above described fact is satisfied.

(*B*) *Wave characteristics.* Let u_0 and h_0 be the velocity of flows and the normal depth in the uniform regime. From equation (25), the wave length L of progressive waves is given by

$$L=2\cdot\Delta\cdot F_1\left(\frac{\pi}{2},\ k\right), \tag{31}$$

where $F_1\left(\frac{\pi}{2},\ k\right)$ is the complete elliptic integral of the first kind.

The second relationship is derived from the hydraulic fact that the relative mean displacement of waves is always zero, that is,

$$\int_0^L h\, d\zeta = L\cdot h_0\,, \tag{32}$$

or from equations (21) and (31)

$$(h_1-h_0)\cdot F_1\left(\frac{\pi}{2},\ k\right)=(h_2-h_1)\int_0^{\frac{\pi}{2}} \frac{1-\sin^2\chi}{\sqrt{1-k^2\sin^2\chi}}\, d\chi\,.$$

Hence this relation is expressed in terms of the first and second complete elliptic integrals as follows.

$$gh_1h_2\cos\theta\left[h_0F_1\left(\frac{\pi}{2},\ k\right)-h_1E_1\left(\frac{\pi}{2},\ k\right)\right]=K^2\left[F_1\left(\frac{\pi}{2},\ k\right)-E_1\left(\frac{\pi}{2},\ k\right)\right]. \tag{33}$$

The average discharge rate at a fixed point is given by

$$u_0h_0=\frac{v_w}{L}\cdot\int_0^L h\,d\zeta-K. \tag{34}$$

Then it is reduced to

$$v_w=u_0+\frac{K}{h_0}. \tag{35}$$

And in permanent wave flows which move without change of their form, the wave length is also showed by

$$L=v_w\cdot T, \tag{36}$$

where T is the wave period.

Thus the complete undular form of progressive cnoidal waves on running water and their characteristics are determined by the wave period, the maximum depth from the channel bed to the crest, and hydraulic quantities in the uniform regime, since the other quantities are determined by making use of above described relations. But because of difficulity of measuring the still water depth, it is rather better that the maximum depth h_1, the wave height h_1-h_2, the wave period T, and the flow velocity u_0 are considered as the fundamental quantities.

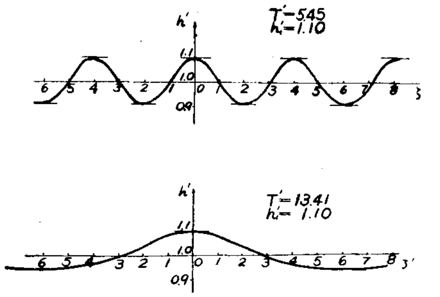

Fig. 1. Typical examples of cnoidal waves.

Figure 1 shows two typical cnoidal waves which are viewed from the moving system, where T' represents the dimensionless wave period expressed by $\frac{L}{v_w}\frac{\sqrt{gh_0\cos\theta}}{h_0}$ and h_1' is the dimensionless maximum depth h_1/h_0. In shorter waves, their forms are resembled the sinusoidal waves, but in longer waves, they are rather analogous to Gerstner's trochoidal waves. Experimental results which were performed at the Hydraulic Laboratory of the Engineering Research Institute, Kyoto University, as shown in Figure 2, were plotted. Figure 3 illustrates the relationship of dimensionless celerity $c'=c/\sqrt{gh_0\cos\theta}$ to dimensionless wave length $L'=L/h_0$ for the parametric values of h_1'. The dotted line shows the behaviour of shallow water waves of infinitesimal amplitude and $c'=1$ indicates the long wave celerity. As shown in Figure 2 the cnoidal waves are also dispersive, and their celerity approaches the celerity of solitary and long waves of the finite amplitude. The region of applicability of cnoidal waves will be limited by the essential assumption of translation waves. Experimental investigations by Keulegan verified the

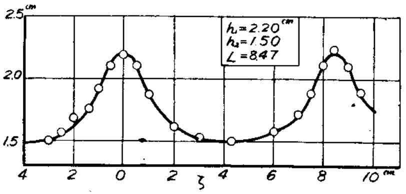

Fig. 2. Comparison of experimental results to the theoretical curve.

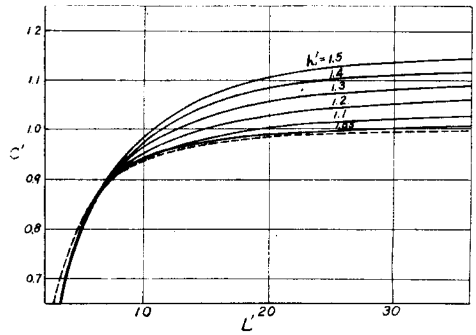

Fig. 3. Relation between the dimensio..less wave celerity and the dimensionless wave length.

analytical conclusion that the value 10 for L' could be considered the dividing line between the Stokian and the cnoidal wave types and for larger values of the ratio cnoidal waves would be applied.

(C) *Relation between cnoidal waves and shallow water waves of infinitesimal amplitude.* In above discussions, it is assumed that k varies in the range from 0 to 1, thus the configuration of cnoidol waves is derived. At the limiting value of k, the elliptic function takes the definite primary function, that is, at $k=1$, it becomes the hyperbolic, and at $k=0$, the circular.

In the present article, the latter case is concerned. If $K^2/gh_1h_2\cos\theta$ is not zero, from equation (33), $h_0=h_1$ and replacing this value in equation (24),

$$h_1=h_2=h_0\,.$$

Hence the wave length expressed by equation (31) becomes

$$L=\frac{4(v_w-u_0)\cdot h_0}{\sqrt{3gh_1\cos\theta-\dfrac{3(v_w-u_0)^2h_0^2}{h_1h_2}}}\cdot\int_0^{\frac{\pi}{2}}\left(1+\frac{k^2}{2}\sin^2\chi+\cdots\right)d\chi\,,$$

$$L=\frac{4(v_w-u_0)h_0}{\sqrt{3gh_0\cos\theta-3(v_w-u_0)^2}}\cdot\frac{\pi}{2}\,. \tag{37}$$

And if the wave length is relatively longer than the still water depth, i.e., $L'>3.626$, developing the above equation in power series yields

$$v_w=u_0+\sqrt{gh_0\cos\theta\left\{1-\frac{4\pi^2h_0^2}{3L^2}+\left(\frac{4\pi^2h_0^2}{3L^2}\right)^2-\cdots\right\}}\,, \tag{38}$$

hence, for longer waves, the above expression may be replaced by the following expression with a sufficient accuracy,

$$v_w\fallingdotseq u_0+\sqrt{\frac{g\cos\theta\cdot L}{2\pi}\tanh\frac{2\pi h_0}{L}}\,.$$

Equation (38) is also the velocity of propagation of waves of infinitesimal amplitude on running water. Thus, this limiting case may show the same result as in that of the infinitesimal waves. And it is easily recognized that for $L\to\infty$, it becomes the long wave celerity. The former limiting case of $k=1$ will be discussed in the following section.

5. Solitary Waves

In developing the theory of cnoidal waves, it is assumed that the fluid is still disturbed at infinity, that is, that the surface profile is in undulation at infinity together with all existences of the slope and the curvature of fluid surface. In this section, as a special case of developing of cnoidal waves, the assumption that the fluid is undisturbed at infinity is utilized.

At infinity $h=h_0$, both $\frac{dh}{d\zeta}$ and $\frac{d^2h}{d\zeta^2}$ are zero, then the constant of integration c_m is expressed as follows, from equation (17)

$$c_m=\frac{2K^2}{h_0}+gh_0^2\cos\theta\,.$$

Substituting this value in equation (18), integration of this equation shows

$$\left(\frac{dh}{d\zeta}\right)^2=-\frac{3g\cos\theta}{K^2}\left[h^3-\left(\frac{K^2}{gh_0^2\cos\theta}+2h_0\right)h^2+\left(\frac{2K^2}{gh_0\cos\theta}+h_0^2\right)h-\frac{K^2}{g\cos\theta}\right]. \quad (39)$$

Physical consideration leads that this equation is also transformed into

$$\left(\frac{dh}{d\zeta}\right)^2=\frac{3g\cos\theta}{K^2}(h-h_0)^2\left(\frac{K^2}{gh_0^2\cos\theta}-h\right). \quad (40)$$

And the surface profile varies from the normal depth h_0 to the maximum depth $K^2/gh_0^2\cos\theta$. Denoting this maximum depth by h_1, selecting the origin at the apex $h=h_1$, and integrating equation (40) yields the following relation

$$h=h_0+(h_1-h_0)\ \mathrm{sech}^2\frac{\sqrt{3g\cos\theta(h_1-h_0)}}{2K}\cdot(x-v_wt)\,. \quad (41)$$

This equation explains the configuration of solitary waves of the permanent type. The free surface is symmetrical about the normal at $\zeta=0$, and approaches uniformly the initial surface as h approaches h_0. Figure 4 illustrates the typical configuration of the solitary wave.

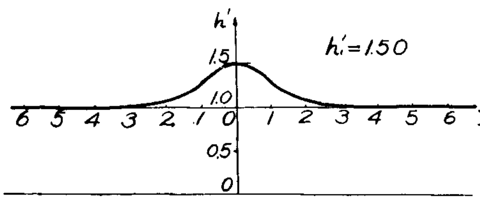

Fig. 4. Typical figure of solitary waves.

Again, the comparison between the result obtained here and the expression of Keulegan and Patterson is described. Equation (41) is also transformed into the following expression as in the cnoidal wave.

$$h'=h_1'\ \mathrm{sech}^2\left[\sqrt{\frac{3h_1'}{4h_0^2(h_0+h_1')}}\cdot(x-v_wt)\right]. \quad (42)$$

Keulegan's expression on the solitary wave is

$$h' = h_1' \operatorname{sech}^2\left[\sqrt{\frac{3h_1'}{4h_0^3}}\,(x - v_w t)\right]. \tag{43}$$

Hence it follows that the difference between equations (42) and (43) is due to the negligence of the maximum wave height above the undisturbed level in the hyperbolic function. Equation (42) is the Lord Rayleigh's type of solitary wave which is derived on the basis of irrotational wave motion, and equation (43) is the Boussinesq's. The experimental data which were obtained by Daily and Stephan of M. I. T. show that the wave figures of solitary waves are more precisely plotted by the expression of Boussinesq, Keulegan and Patterson than that of Rayleigh and the author. But the further fruitful experimental researches on the hydraulic behaviour of solitary waves by use of modern optical and electrical equipments are expected.

The wave form of solitary wave is derived by another approach which is considered as a limiting case of cnoidal waves described in the above section.

When $k \to 1$, in equation (33)

$$\lim_{k\to 1} \frac{K^2}{gh_1h_2\cos\theta} = \lim_{k\to 1} \frac{h_0 \cdot F_1\left(\frac{\pi}{2},\, k\right) - h_1 \cdot E_1\left(\frac{\pi}{2},\, k\right)}{F_1\left(\frac{\pi}{2},\, k\right) - E_1\left(\frac{\pi}{2},\, k\right)} = h_0 .$$

From equations (23) and (24) the following described results are obtained.

$$h_2 \to h_0, \qquad \text{and} \qquad \varDelta \to \frac{2K}{\sqrt{3g\cos\theta\,(h_1 - h_0)}} .$$

Thus equation (27) expressed the cnoidal wave is reduced to

$$h = h_0 + (h_1 - h_0)\operatorname{sech}^2 \frac{\sqrt{3g\cos\theta(h_1 - h_0)}}{2K}\,(x - v_w t). \tag{44}$$

The above equation is the same expression of equation (41). Hence it follows that the solitary wave appears as a special case of the cnoidal wave. It has been already mentioned by H. Lamb,[7] Keulegan and Patterson.[1]

Considering that $\frac{K^2}{gh_0^2\cos\theta} \geq h_0$, and $K^2 = (v_w - u_0)^2\,h_0^2$, the following relation is reduced

$$(v_w - u_0)^2 \geq gh_0\cos\theta ,$$

or

$$\boldsymbol{F}_0 = \frac{v_w - u_0}{\sqrt{gh_0\cos\theta}} \geq 1 . \tag{45}$$

where $\boldsymbol{F}_0$ is the relative Froude Number.

Then it follows that the flow regime of solitary waves is always appeared to be supercritical when observed from the moving coordinate system. Equation (45) is the necessary condition for formation of solitary waves, and that fact has been pointed out by K. O. Friedlich and D. H. Heyers[8] in discussing of the existence of irrotational solitary waves.

The velocity of propagation v_w and the wave celerity c are easily obtained from the relation of the maximum depth to the progressive discharge rate

$$K^2/gh_0^2 \cos\theta = h_1 ,$$

hence

$$v_w = u_0 + \sqrt{gh_1 \cos\theta}, \qquad \text{and} \qquad c = \sqrt{gh_1 \cos\theta}. \tag{46}$$

Equation (46) was adopted by S. Russell on empirical ground and confirmed theoretically by Boussinesq and Lord Rayleigh.

6. Conclusion

The theory and behaviour of permanent translation waves are developed by considering the effects of the vertical acceleration in open channel flows. The mathematical and hydraulic treatments lead that such wave profiles are expressed in terms of an elliptic function which is designated as the cn-function, the name cnoidal being derived therefrom as analogous to sinusoidal by Korteweg and deVries, and solitary waves expressed by equation (41) appear as an extreme case of cnoidal waves.

It is interesting that the behaviour of cnoidal waves has a close relation to the infinitesimal wave theory. Though these waves are, of course, less familiar to hydraulic engineers, they will be appeared to be essential waves often encountered in channels and seas, especially at near coasts.

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