Analysis on Creep of Statically Indeterminate Reinforced Concrete Structures

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1. Introduction

It is a well-known fact that the stress re-distribution caused by the creep and shrinkage of concrete will occur in the statically indeterminate reinforced concrete structure when it is subjected to sustained load for certain length of time.

Many theoretical and experimental studies have been made by many researchers such as Dischinger, Schwarz, Straub, Wilson and Ban. The author also has attempt at its theoretical solution by use of the slope-deflection method[1] and the moment-distribution method[2] modified for this particular problem.

In this paper, the variation in the redundant forces due to creep is theoretically analysed by using the fundamental elastic equations and "Theorem of Three or Four Moments," both of which are conventionally used for general statically indeterminate systems.

The effect of variation in the modulus of elasticity of concrete caused by age is so small that it is disregarded in the analysis[3]. Some numerical examples are also illustrated.

2. Variations in the redundant forces due to creep

The elastic equations of statically indeterminate systems are generally given by

$$\delta_{i0}+\delta_{i1}X_1+\delta_{i2}X_2+\cdots+\delta_{im}X_m+\cdots+\delta_{in}X_n=0 \quad (i=1,2,3,\cdots,n) \tag{1}$$

where δ is the displacement and X the statically indeterminate force.

Solving these equations, the unknown, X's, are obtained.

When the system is subjected to sustained loadings, there will be an increase in each of the displacements due to creep, resulting in changing δ_{i0} into δ_{i0t} and δ_{im} into δ_{imt}, respectively.

Since δ_{i0} and δ_{im} are the displacements of the fundamental system composed of various members, they can be expressed by

$$\begin{aligned}\delta_{i0} &= \delta_{i0a}+\delta_{i0b}+\delta_{i0c}+\cdots, \\ \delta_{im} &= \delta_{ima}+\delta_{imb}+\delta_{imc}+\cdots.\end{aligned} \tag{2}$$

The terms on the right side of the above equations correspond to the displacements of the component members, a, b, c and etc. of the system.

Now, assuming that each of the component members posseses its proper characteristics of $\nu_a\varphi_t$, $\nu_b\varphi_t\cdots$, where ν_a, $\nu_b\cdots$ are constants and φ_t the standard creep characteristics, the total displacement after the duration of sustained loadings will become

$$\begin{aligned}\delta_{i0t} &= \delta_{i0a}(1+\nu_a\varphi_t)+\delta_{i0b}(1+\nu_b\varphi_t)+\cdots \\ &= \delta_{i0}+(\nu_a\delta_{i0a}+\nu_b\delta_{i0b}+\cdots)\varphi_t \\ &= \delta_{i0}+\bar{\delta}_{i0}\varphi_t,\end{aligned} \tag{3}$$

where

$$\bar{\delta}_{i0} = \nu_a\delta_{i0a}+\nu_b\delta_{i0b}+\nu_c\delta_{i0c}+\cdots.$$

Similarly

$$\begin{aligned}\delta_{imt} &= \delta_{im}+\bar{\delta}_{im}\varphi_t \\ \bar{\delta}_{im} &= \nu_a\delta_{ima}+\nu_b\delta_{imb}+\nu_c\delta_{imc}+\cdots.\end{aligned} \tag{3'}$$

In case the influences of axial forces and of shears besides those of bending-moments are taken into consideration in calculating δ_i and assuming that each member has a different value of creep characteristics for the axial force, shear, and bending moment, as is usually expected in the reinforced concrete members, δ_{ioa}, for example, must be computed as follows:

$$\delta_{i0a} = \delta^B_{i0a}+\delta^D_{i0a}+\delta^Q_{i0a}, \tag{4}$$

where the terms on the right side correspond to the displacements caused by bending moments, direct forces, and shears, respectively; and the suffixes B, D, Q of δ_{i0a} also mean the influences of bending moment, direct force, and shear.

Therefore, $\delta_{i0a}(1+\nu_a\varphi_t)$ in Eq. (3) must be transformed into

$$\begin{aligned}&\delta^B_{i0a}(1+\nu^B_a\varphi_t)+\delta^D_{i0a}(1+\nu^D_a\varphi_t)+\delta^Q_{i0a}(1+\nu^Q_a\varphi_t) \\ &= \delta_{i0a}+(\nu^B_a\delta^B_{i0a}+\nu^D_a\delta^D_{i0a}+\nu^Q_a\delta^Q_{i0a})\varphi_t \\ &= \delta_{i0a}+\bar{\delta}_{i0a}\varphi_t,\end{aligned} \tag{5}$$

where $\bar{\delta}_{i0a} = \nu^B_a\delta^B_{i0a}+\nu^D_a\delta^D_{i0a}+\nu^Q_a\delta^Q_{i0a}$.

A similar relationship holds for each of the other δ_{i0}, $\delta_{im}\cdots$, and Eqs. (3) and (3′) will come

$$\begin{aligned}\delta_{i0t} &= \delta_{i0}+(\bar{\delta}_{i0a}+\bar{\delta}_{i0b}+\bar{\delta}_{i0c}+\cdots)\varphi_t \\ &= \delta_{i0}+\bar{\delta}_{i0}\varphi_t \\ \delta_{imt} &= \delta_{im}+\bar{\delta}_{im}\varphi_t,\end{aligned} \tag{6}$$

where
$$\begin{cases} \bar{\delta}_{i0} = \bar{\delta}_{i0a}+\bar{\delta}_{i0b}+\bar{\delta}_{i0c}+\cdots \\ \bar{\delta}_{im} = \bar{\delta}_{ima}+\bar{\delta}_{imb}+\bar{\delta}_{imc}+\cdots . \end{cases}$$

Thus, when each of the displacements, δ_i, increases as given above, this tends to cause discontinuity of member displacement and for that reason the redundant forces are assumed to vary as creep is produced in order to satisfy the equilibrium conditions of displacements.

Now, let Y_m be the variation of redundant force X_m. Neglecting the variation in the modulus of elasticity of concrete, and assuming that the basic creep characteristics φ_t has the same properties as those of concrete, Y_m will cause a secondary displacement as shown below: [4][5]

$$\int_0^t \{\delta_{im}+(\varphi_t-\varphi_\tau)\bar{\delta}_{im}\}\frac{dY_m}{d\tau}\cdot d\tau$$
$$=\delta_{im}Y_m+\bar{\delta}_{im}\int_0^t Y_m\frac{d\varphi_\tau}{d\tau}d\tau . \tag{7}$$

Consequently, the required conditions for continuity of the displacement will be given by using Eqs. (3 or 6) and (7) as follows:

$$(\delta_{i0}+\bar{\delta}_{i0}\varphi_t)+(\delta_{i1}+\bar{\delta}_{i1}\varphi_t)X_1+(\delta_{i2}+\bar{\delta}_{i2}\varphi t)X_2+\cdots$$
$$+(\delta_{im}+\bar{\delta}_{im}\varphi_t)X_m+\cdots+(\delta_{in}+\bar{\delta}_{in}\varphi_t)X_n$$
$$+\int_0^t\{\delta_{i1}+(\varphi_t-\varphi_\tau)\bar{\delta}_{i1}\}\frac{dY_1}{d\tau}d\tau+\cdots+\int_0^t\{\delta_{im}+(\varphi_t-\varphi_\tau)\bar{\delta}_{im}\}\frac{dY_m}{d\tau}d\tau+\cdots$$
$$+\int_0^t\{\delta_{in}+(\varphi_t-\varphi_\tau)\bar{\delta}_{in}\}\frac{dY_n}{d\tau}d\tau=0. \quad (i=1,2,3,\cdots,n) \tag{8}$$

Substituting Eq. (1) into the above,

$$\int_0^t\{\delta_{i1}+(\varphi_t-\varphi_\tau)\bar{\delta}_{i1}\}\frac{dY_1}{d\tau}d\tau+\cdots+\int_0^t\{\delta_{im}+(\varphi_t-\varphi_\tau)\bar{\delta}_{im}\}\frac{dY_m}{d\tau}d\tau+\cdots$$
$$+\int_0^t\{\delta_{in}+(\varphi_t-\varphi_\tau)\bar{\delta}_{in}\}\frac{dY_n}{d\tau}d\tau$$
$$+\bar{\delta}_{i0}\varphi_t+\bar{\delta}_{i1}X_1\varphi_t+\cdots+\bar{\delta}_{im}X_m\varphi_t+\cdots+\bar{\delta}_{in}X_n\varphi_t=0.$$
$$(i=1,2,3,\cdots,n). \tag{9}$$

Differentiating the above by φ_t,

$$\left(\delta_{i1}\frac{dY_1}{d\varphi_t}+\bar{\delta}_{i1}Y_1\right)+\cdots+\left(\delta_{im}\frac{dY_m}{d\varphi_t}+\bar{\delta}_{im}Y_m\right)+\cdots+\left(\delta_{in}\frac{dY_n}{d\varphi_t}+\bar{\delta}_{in}Y_n\right)$$
$$+\bar{\delta}_{i1}X_1+\cdots+\bar{\delta}_{im}X_m+\cdots+\bar{\delta}_{in}X_n+\bar{\delta}_{i0}=0.$$
$$(i=1,2,3,\cdots,n). \tag{10}$$

These are the fundamental differential equations for obtaining the variation of the redundant forces due to creep.

3. Solution of the simultaneous differential equations

The simultaneous equation (10) is solved in the following manner. Examples will be shown for three reduntant forces X_1, X_2 and X_3.

(a) General solution

For $i=1, 2, 3$, Eq. (10) becomes as follows.

$$\begin{aligned}(\delta_{11}\dot{Y}_1+\bar{\delta}_{11}Y_1)+(\delta_{12}\dot{Y}_2+\bar{\delta}_{12}Y_2)+(\delta_{13}\dot{Y}_3+\bar{\delta}_{13}Y_3)\\ +X_1\bar{\delta}_{11}+X_2\bar{\delta}_{12}+X_3\bar{\delta}_{13}+\bar{\delta}_{10}=0\,,\\ (\delta_{21}\dot{Y}_1+\bar{\delta}_{21}Y_1)+(\delta_{22}\dot{Y}_2+\bar{\delta}_{22}Y_2)+(\delta_{23}\dot{Y}_3+\bar{\delta}_{23}Y_3)\\ +X_1\bar{\delta}_{21}+X_2\bar{\delta}_{22}+X_3\bar{\delta}_{23}+\bar{\delta}_{20}=0\,,\\ (\delta_{31}\dot{Y}_1+\bar{\delta}_{31}Y_1)+(\delta_{32}\dot{Y}_2+\bar{\delta}_{32}Y_2)+(\delta_{33}\dot{Y}_3+\bar{\delta}_{33}Y_3)\\ +X_1\bar{\delta}_{31}+X_2\bar{\delta}_{32}+X_3\bar{\delta}_{33}+\bar{\delta}_{30}=0\,,\end{aligned} \tag{11}$$

where $$\dot{Y}_i=\frac{dY_i}{d\varphi_t}. \qquad (i=1, 2, 3)$$

Solving the above equations, we obtain

$$Y_1=y_1+\eta_1\,,\quad Y_2=y_2+\eta_2\,,\quad Y_3=y_3+\eta_3\,, \tag{12}$$

where $$y_i=-X_i-\frac{\Delta_{i0}}{\Delta_0}\,,\qquad (i=1, 2, 3) \tag{13}$$

and Δ is a determinant obtained by the coefficients of y_1, y_2 and y_3 in Eq. (11). And

$$\eta_i=C_1M_{1i}(\lambda_1)e^{\frac{\varphi_t}{\lambda_1}}+C_2M_{1i}(\lambda_2)e^{\frac{\varphi_t}{\lambda_2}}+C_3M_{1i}(\lambda_3)e^{\frac{\varphi_t}{\lambda_3}}\,, \tag{14}$$

λ_1, λ_2 and λ_3 are the roots of determinant equation (15)

$$M(\lambda)=\begin{vmatrix}\delta_{11}+\lambda\bar{\delta}_{11} & \delta_{12}+\lambda\bar{\delta}_{12} & \delta_{13}+\lambda\bar{\delta}_{13}\\ \delta_{21}+\lambda\bar{\delta}_{21} & \delta_{22}+\lambda\bar{\delta}_{22} & \delta_{23}+\lambda\bar{\delta}_{23}\\ \delta_{31}+\lambda\bar{\delta}_{31} & \delta_{32}+\lambda\bar{\delta}_{32} & \delta_{33}+\lambda\bar{\delta}_{33}\end{vmatrix}=0\,, \tag{15}$$

and $M_{1i}(\lambda_k)$ is the minor of element $\delta_{1i}+\lambda\delta_{1i}$ in determinant (15) for $\lambda=\lambda_k$.

C_1, C_2 and C_3 are the constants which are determined by the initial conditions of $Y_i=0$ when $\varphi_t=0$. That is, C's are obtained by

$$C_1M_{1i}(\lambda_1)+C_2M_{1i}(\lambda_2)+C_3M_{1i}(\lambda_3)=-y_i=X_i+\frac{\Delta_{i0}}{\Delta_0}\,, \qquad (i=1, 2, 3) \tag{16}$$

(b) Solution for the special case of $\bar{\delta}_{ik}=\delta_{ik}$

In case all of the creep characteristics of all the members are the same as those in a plain concrete statically indeterminate system, in other words, an assumption of $\nu_a=\nu_b=\nu_c=\cdots=1$ can be made in Eq. (3 or 6), $\bar{\delta}_{ik}$ is equall to δ_{ik}. Then, each of the expressions in the parenthesis in Eq. (11) will be as follows,

$$(\delta_{ik}\dot{Y}_k+\bar{\delta}_{ik}Y_k)=\delta_{ik}(\dot{Y}_k+Y_k)\,,$$

and
$$X_1\bar{\delta}_{i1}+X_2\bar{\delta}_{i2}+X_3\bar{\delta}_{i3}+\bar{\delta}_{i0}=X_1\delta_{i1}+X_2\delta_{i2}+X_3\delta_{i3}+\delta_{i0}=0\,.$$

Accordingly,
$$\dot{Y}_k+Y_k=0 \qquad \therefore\; Y_k=0\,. \tag{17}$$

If δ_{i0} contains the displacements δ_{iw} at $t=0$ of the support and δ_{is} caused by shrinkage in addition to the displacement δ_{ig} caused by sustained loads, then δ_{is} can be assumed to grow in such away as

$$\delta_{is}(t)=\frac{\varphi_t}{m}\delta_{is}=\frac{\varphi_t}{m}\bar{\delta}_{is}\,,$$

where m is the final value of φ_t,

Then, the solution of Eq. (11) gives the following three independent differential equations of first order:

$$\frac{dY_i}{d\varphi_t}+Y_i+\left(X_{iw}-\frac{1}{m}X_{is}\right)=0\,,\qquad (i=1,2,3) \tag{18}$$

in which X_{iw}, X_{is} indicate the redundant forces due to the support displacement and shrinkage effect, respectively, and both of which can be computed by the elastic theory.

Thus, the following is obtained:

$$Y_i=\left(\frac{X_{is}}{m}-X_{iw}\right)(1-e^{-\varphi_t})\,. \tag{19}$$

Eqs. (17) and (19) are the solutions obtained by Fr. Dischinger, and they mean that the creep of concrete does not affect the stress distribution in the structure so far as the displacements of the support and shrinkage do not take place.

(c) Example

For example, a symmetrical, two hinged frame having a steel tie, which is subjected to a uniformly distributed vertical load as shown schematically in Fig. 1, is considered.

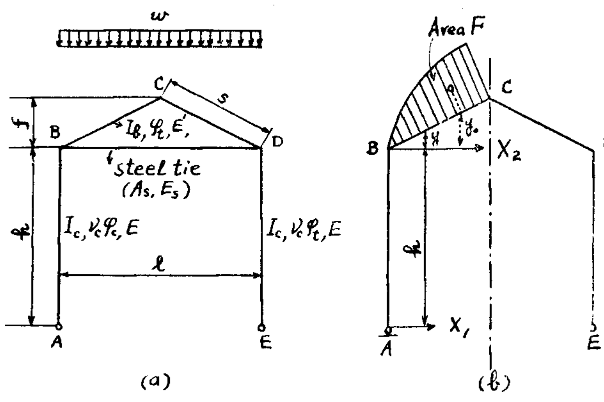

Fig. 1. Analysis cf symmetrical, two hinged frame with steel tie.

The basic statically determinate system is chosen as shown in Fig. 1 (b), in which A indicates a roller and E a hinge. (I_c, I_b) and $(\nu_c\varphi_t, \varphi_t)$ mean the moments of inertia and the creep characteristics of column AB and beam BC, respectively.

The fundamental equations for the redundant forces X_1, X_2 are:

$$\begin{aligned} -\delta_{10}+X_1\delta_{11}+X_2\delta_{12} &= 0 \\ -\delta_{20}+X_1\delta_{21}+X_2\delta_{22} &= 0\,, \end{aligned} \tag{20}$$

in which δ's can be calculated as follows. (The effect of direct force is neglected except for the steel tie.[6])

$$\begin{aligned} \delta_{10} &= \int\frac{M_0(h+y)}{EI}ds = \frac{2F(y_0+h)}{EI_b} \\ \delta_{20} &= \int\frac{M_0 y}{EI}ds = \frac{2Fy_0}{EI_b} \\ \delta_{11} &= \int\frac{(h+y)^2}{EI}ds = \frac{2h^3}{3EI_c}+\frac{2s}{3EI_b}\{h^2+h(h+f)+(h+f)^2\} \\ &= \delta_{11c}+\delta_{11b}\,, \\ \delta_{12} &= \delta_{21} = \int\frac{(h+y)}{EI}ds = \frac{fs}{EI_s}\left(h+\frac{2}{3}f\right) \\ \delta_{22} &= \int\frac{y^2}{EI}ds+\frac{l}{E_sA_s} = \frac{2}{3}\frac{f^2s}{EI_b}+\frac{l}{E_sA_s} \\ &= \delta_{22b}+\delta_{22T}\,. \end{aligned} \tag{21}$$

F is the area of the moment diagram as drawn in Fig. 1 (b), and y_0 is the height of the centroid of the moment diagram, measured from BD level. For a uniform load ω on the horizontally projected surface, are obtained

$$F = \frac{1}{3}\frac{1}{8}\omega l^2 s\,, \quad y_0 = \frac{5}{8}f\,.$$

Under the consideration that the steel does not creep, the followings are obtained from Eqs. (3) and (21);

$$\begin{aligned} \delta_{10t} &= \delta_{10}+\bar{\delta}_{10}\varphi_t = \delta_{10}+\delta_{10}\varphi_t\,, && (\bar{\delta}_{10}=\delta_{10}) \\ \delta_{20t} &= \delta_{20}+\bar{\delta}_{20}\varphi_t = \delta_{20}+\delta_{20}\varphi_t\,, && (\bar{\delta}_{20}=\delta_{20}) \\ \delta_{11t} &= \delta_{11}+\bar{\delta}_{11}\varphi_t\,, && (\bar{\delta}_{11}=\delta_{11b}+\delta_{11c}\nu_c) \\ \delta_{12t} &= \delta_{21t} = \delta_{12}+\bar{\delta}_{12}\varphi_t = \delta_{12}+\delta_{12}\varphi_t\,, && (\bar{\delta}_{12}=\delta_{12}) \\ \delta_{22t} &= \delta_{22}+\bar{\delta}_{22}\varphi_t\,, && (\bar{\delta}_{22}=\delta_{22b}). \end{aligned} \tag{22}$$

When $t=0$, $(\varphi_t=0)$, X_1 and X_2 are given by the elastic theory.

$$X_1 = \frac{\delta_{10}\delta_{22}-\delta_{12}\delta_{20}}{\delta_{11}\delta_{22}-\delta_{12}\delta_{21}}\,, \quad X_2 = \frac{\delta_{11}\delta_{20}-\delta_{21}\delta_{10}}{\delta_{11}\delta_{22}-\delta_{12}\delta_{21}}\,. \tag{23}$$

The simultaneous differential equations for additional redundant Y_1, Y_2 due to the creep are obtained from Eq. (11), as below:

$$\begin{aligned} (\delta_1\dot{Y}_{11}+\bar{\delta}_{11}Y_1)+(\delta_{12}\dot{Y}_2+\bar{\delta}_{12}Y_2)+X_1\bar{\delta}_{11}+X_2\bar{\delta}_{12}-\bar{\delta}_{10} &= 0 \\ (\delta_{21}\dot{Y}_1+\bar{\delta}_{21}Y_1)+(\delta_{22}\dot{Y}_2+\bar{\delta}_{22}Y_2)+X_1\bar{\delta}_{21}+X_2\bar{\delta}_{22}-\bar{\delta}_{20} &= 0\,. \end{aligned} \tag{24}$$

Solving the above, from Eq. (12),

$$
\begin{aligned}
Y_1 &= y_1 + C_1(\delta_{12}+\lambda_1\bar{\delta}_{12})e^{\frac{\varphi_t}{\lambda_1}} + C_2(\delta_{12}+\lambda_2\bar{\delta}_{12})e^{\frac{\varphi_t}{\lambda_2}} \\
Y_2 &= y_2 - C_1(\delta_{11}+\lambda_1\bar{\delta}_{11})e^{\frac{\varphi_t}{\lambda_1}} - C_2(\delta_{11}+\lambda_2\bar{\delta}_{11})e^{\frac{\varphi_t}{\lambda_2}},
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
y_1 &= -X_1 + \frac{\bar{\delta}_{10}\bar{\delta}_{22}-\bar{\delta}_{12}\bar{\delta}_{20}}{\bar{\delta}_{11}\bar{\delta}_{22}-\bar{\delta}_{12}\bar{\delta}_{21}}, \\
y_2 &= -X_2 + \frac{\bar{\delta}_{11}\bar{\delta}_{20}-\bar{\delta}_{21}\bar{\delta}_{10}}{\bar{\delta}_{11}\bar{\delta}_{22}-\bar{\delta}_{12}\bar{\delta}_{21}},
\end{aligned}
\tag{26}
$$

and λ_1, λ_2 are the roots of

$$
M(\lambda) = \begin{vmatrix} \delta_{11}+\lambda\bar{\delta}_{11} & \delta_{10}+\lambda\bar{\delta}_{12} \\ \delta_{21}+\lambda\bar{\delta}_{21} & \delta_{22}+\lambda\bar{\delta}_{22} \end{vmatrix} = 0\,. \tag{27}
$$

C_1, C_2 are determined by Eq. (28) from the initial conditions that $Y_1 = Y_2 = 0$ at $\varphi_t = 0$.

$$
\begin{aligned}
C_1 &= \frac{-1}{(\lambda_1-\lambda_2)(\delta_{11}\bar{\delta}_{12}-\bar{\delta}_{11}\delta_{12})}\{y_1(\delta_{11}+\lambda_2\bar{\delta}_{11})+y_2(\delta_{12}+\lambda_2\bar{\delta}_{12})\}, \\
C_2 &= \frac{1}{(\lambda_1-\lambda_2)(\delta_{11}\bar{\delta}_{12}-\bar{\delta}_{11}\delta_{12})}\{y_1(\delta_{11}+\lambda_1\bar{\delta}_{11})+y_2(\delta_{12}+\lambda_1\bar{\delta}_{12})\}.
\end{aligned}
\tag{28}
$$

For assumed values: $l = 10\text{ m}$, $h = 5\text{ m}$, $s = 5.85\text{ m}$,

$I_c = 0.0016\text{ m}^4$, $E = 21\times10^8\text{ kg/m}^2$

$I_b = 0.00313\text{ m}^4$, $E_s = 21\times10^9\text{ kg/m}^2$

$A_s = 0.0008\text{ m}^2$

$\omega = 2000\text{ kg/m}$, $M_0 = \frac{1}{8}\omega l^2 = 25000\text{ kgm}$

$\nu_c = 2$, $\varphi_t = 1$,

δ's are computed by Eqs. (21) and (22)

$\delta_{10} = \bar{\delta}_{10} = 0.0918\text{ m}$, $\delta_{20} = \bar{\delta}_{20} = 0.025\text{ m}$, $\delta_{11} = 1.016\times10^{-4}\text{ m}$

$\delta_{12} = \bar{\delta}_{12} = 0.187\times10^{-4}\text{ m}$, $\delta_{22} = 0.0595\times10^{-4}\text{ m}$,

and $\bar{\delta}_{11} = 1.264\times10^{-4}\text{ m}$, $\bar{\delta}_{22} = 0.0535\times10^{-4}\text{ m}$.

Thus from Eq. (23): $X_1 = 311\text{ kg}$, $X_2 = 3220\text{ kg}$

from Eq. (26): $y_1 = -239\text{ kg}$, $y_2 = 1200\text{ kg}$

from Eq. (27): $\lambda_1 = -0.6819$, $\lambda_2 = -1.144$.

Then after determining C_1, C_2 by Eq. (28), the followings are obtained:

$$Y = -176\text{ kg}, \qquad Y = 793\text{ kg}.$$

Professor S. Ban gives the followings for the same problem as the above by using the approximate method proposed by himself.[6)]

$Y = -190\,\text{kg}, \quad Y = 842\,\text{kg}$ as the first approximation

and $Y = -168\,\text{kg}, \quad Y = 761\,\text{kg}$ as the improved one.

Similar analysis can be made for a tied arch with steel tie on which R. Schwarz discussed.[7]

4. Analysis by "Theorem of Three Moments or Four Moments"

(a) Application of the theorem

Theorem of three or four moments is frequently used very effectively in the analysis of continuous beams and rigid frames.

Therefore, the application of the theorem to the creep problems as described in the preceding chapters will be developed in the following.

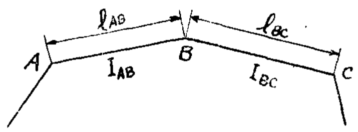

Fig. 2. Two members framed into joint B.

Select any two adjacent members AB and BC framing into joint B, as shown in Fig. 2, and the end tangent deflection angles θ at B of both members AB, BC can be given by[8]

$$\text{for } AB: \quad \theta_B = -\frac{1}{E_{AB} I_{AB}} \left\{ \frac{l_{AB}}{6} (M_A + 2M_B) + \mathfrak{B}_{AB} \right\} + \mathfrak{D}_{AB}$$

$$\text{for } BC: \quad \theta_B' = \frac{1}{E_{BC} I_{BC}} \left\{ \frac{l_{BC}}{6} (2M_B + M_C) + \mathfrak{A}_{BC} \right\} + \mathfrak{D}_{BC}\,. \tag{29}$$

Where $\mathfrak{B}_{AB}$ is equal to the reaction at B of a simple beam AB due to an imagined loading equivalent to the moment diagram M_{0AB} obtained for the load on beam AB, and $\mathfrak{A}_{BC}$ the reaction at B of a simple beam BC for a load equal to the moment diagram M_{0BC} for the loading on beam BC: $\mathfrak{D}_{AB}$ and $\mathfrak{D}_{BC}$ are the member rotation angles at B of members AB and BC. Angle rotation is assumed positive when clockwise.

Since B is a rigid joint, θ_B must be equal to θ_B', consequently the well-known fundamental equation of "theorem of three moments" can be derived by the condition that

$$\theta_B = \theta_B'\,. \tag{30}$$

When the creep of each member is taken into account, the above expressions have to be modified as follows:

Let φ_{AB} and φ_{BC} be the creep characteriatics of member AB and BC, then θ_B and θ_B' will increase with the duration of sustained loads to $\theta_B(1+\varphi_{AB})$ and to $\theta_B'(1+\varphi_{BC})$. These variations in θ tend to disrupt the condition of equal angles, Eq. (30), so that the additional bending moment M_{At}, M_{Bt} will be caused to hold the equilibrium of angles; consequently, the fundamental equation is derived as follows:

$$\theta_B\varphi_{AB}-\frac{1}{E_{AB}I_{AB}}\frac{l_{AB}}{6}\left\{(M_{At}+2M_{Bt})+\int_0^t(M_{At}+2M_{Bt})\frac{d\varphi_{AB}}{dt}dt\right\}$$
$$=\theta_B'\varphi_{BC}+\frac{1}{E_{BC}I_{BC}}\frac{l_{BC}}{6}\left\{(2M_{Bt}+M_{Ct})+\int_0^t(2M_{Bt}+M_{Ct})\frac{d\varphi_{BC}}{dt}dt\right\}. \tag{31}$$

Differentiating by φ_t under the assumptions that $\varphi_{AB}=\nu_{AB}\varphi_t$, $\varphi_{BC}=\nu_{BC}\varphi_t$ and ν_{AB}, ν_{BC} are constants,

$$\frac{l_{AB}}{E_{AB}I_{AB}}(\dot{M}_{At}+\nu_{AB}M_{At})+2\left\{\frac{l_{AB}}{E_{AB}I_{AB}}(\dot{M}_{Bt}+\nu_{AB}M_{Bt})+\frac{l_{BC}}{E_{BC}I_{BC}}(\dot{M}_{Bt}+\nu_{BC}M_{Bt})\right\}$$
$$+\frac{l_{BC}}{E_{BC}I_{BC}}(\dot{M}_{Ct}+\nu_{BC}M_{Ct})=6(\theta_B\nu_{AB}-\theta_B'\nu_{BC}), \tag{32}$$

where $$\dot{M}_t=\frac{dM_t}{d\varphi_t}.$$

Multiplying each term by the standard value $\frac{EI}{l}$ and expressing $\frac{EI}{E_{AB}I_{AB}}\frac{l_{AB}}{l}$, $\frac{EI}{E_{BC}I_{BC}}\frac{l_{BC}}{l}$ by l'_{AB} and l'_{BC}, we get

$$l'_{AB}(\dot{M}_{At}+\nu_{AB}M_{At})+2\{(l'_{AB}+l'_{BC})\dot{M}_{Bt}+(\nu_{AB}l'_{AB}+\nu_{BC}l'_{BC})M_{Bt}\}$$
$$+l'_{BC}(\dot{M}_{Ct}+\nu_{BC}M_{Ct})=6\frac{EI}{l}(\theta_B\nu_{AB}-\theta_B'\nu_{BC}). \tag{33}$$

And, similarly, the expression corresponding to "theorem of four moments" can be given by

$$l'_{AB}\{(\dot{M}_t^r+2\dot{M}_t^l)+\nu_{AB}(M_t^r+2M_t^l)\}$$
$$+l'_{BC}\{(2\dot{M}_t^r+\dot{M}_t^l)+\nu_{BC}(2M_t^r+M_t^l)\}=6\frac{EI}{l}(\theta_B^r\nu_{AB}-\theta_B^l\nu_{BC}). \tag{34}$$

Using the above equations in place of the elastic fundamental equations, the simultaneous differential equations, as given in the preceding chapter, are obtained and they can be solved in the same manner.

(b) Example

An example will be given for a continuous beam of three spans having no support displacements, as shown in Fig. 3, for which $M_0=M_3=0$.

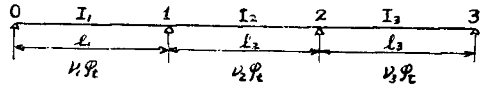

Fig. 3. Analysis of continuous beam of three spans.

The fundamental elastic equations for support 1 and 2 are

$$\begin{aligned}2(l_1'+l_2')M_1+l_2'M_2&=-K_1\\ l_2'M_1+2(l_2'+l_3')M_2&=-K_2,\end{aligned} \tag{35}$$

where $$K_1=6\left(\frac{l_1'}{l_1}\mathfrak{B}_1+\frac{l_1'}{l_2}\mathfrak{A}_2\right),\quad K_2=6\left(\frac{l_2'}{l_2}\mathfrak{B}_2+\frac{l_3'}{l_3}\mathfrak{A}_3\right).$$

Two unknown bending moments M_1 and M_2 can be obtained by the above equations. In the creep analysis, Eqs. (31) and (33) are used as follows:

$$\begin{aligned} 2\{(l_1'+l_2')\dot{M}_{1t}+(l_1'\nu_1+l_2'\nu_2)M_{1t}\}+(l_2'\dot{M}_{2t}+l_2'\nu_2M_{2t}) &= -K_1' \\ (l_2'\dot{M}_{1t}+l_2'\nu_2M_{1t})+2\{(l_2'+l_3')\dot{M}_{2t}+(l_2'\nu_2+l_3'\nu_3)M_{2t}\} &= -K_2' , \end{aligned} \tag{36}$$

in which

$$K_1' = \{2(\nu_1 l_1'+\nu_2 l_2')M_1+l_2'\nu_2M_2\}+k_1$$
$$K_2' = \{\nu_2 l_2'M_1+2(\nu_2 l_2'+\nu_3 l_3')M_2\}+k_2$$
$$k_1 = 6\left(\frac{l_1'}{l_1}\nu_1\mathfrak{B}_1+\frac{l_2'}{l_2}\nu_2\mathfrak{A}_2\right)$$
$$k_2 = 6\left(\frac{l_2'}{l_2}\nu_2\mathfrak{B}_2+\frac{l_3'}{l_3}\nu_3\mathfrak{A}_3\right),$$

or, rewritten as

$$\begin{aligned} 2(L_{12}\dot{M}_{1t}+\bar{L}_{12}M_{1t})+(L_2\dot{M}_{2t}+\bar{L}_2M_{2t}) &= -K_1' \\ (L_2\dot{M}_{1t}+\bar{L}_2M_{1t})+2(L_{23}\dot{M}_{2t}+\bar{L}_{23}M_{2t}) &= -K_2 , \end{aligned} \tag{37}$$

in which

$$L_{12}=l_1'+l_2' , \qquad L_2=l_2' , \qquad L_{23}=l_2'+l_3'$$
$$\bar{L}_{12}=l_1'\nu_1+l_2'\nu_2 , \qquad \bar{L}_2=l_2'\nu_2 , \qquad \bar{L}_{23}=l_2'\nu_2+l_3'\nu_3$$

Eq. (37) has the similar form as Eq. (24) and can be solved exactly the same way.

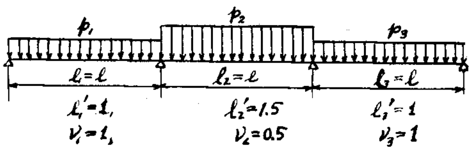

Fig. 4. An example of continuous beam bridge.

If the continuous beam structure and the loading conditions are symmetrical with respect to the beam center, as shown in Fig. 4, then $M_1=M_2$, $M_{1t}=M_{2t}$, $K_1'=K_2'$, $k_1=k_2$. Consequently, the solution can be easily determined by the following differential equation

$$(2L_{12}+L_2)\dot{M}_{1t}+(2\bar{L}_{12}+\bar{L}_2)M_{1t} = -K_1' . \tag{38}$$

Solving the above,

$$\begin{aligned} M_{1t} &= -\frac{K_1'}{2\bar{L}_{12}+\bar{L}_2}(1-e^{-\alpha\varphi_t}) \\ &= -\left(M_1+\frac{k_1}{2\bar{L}_{12}+\bar{L}_2}\right)(1-e^{-\alpha\varphi_t}) , \end{aligned} \tag{39}$$

where

$$\alpha = \frac{2\bar{L}_{12}+\bar{L}_2}{2L_{12}+L_2} .$$

For example, in the continuous beam bridge shown in Fig. 4,

$$\mathfrak{B}_1=\frac{p_1 l^3}{24} , \quad \mathfrak{A}_2=\frac{p_2 l^3}{24}$$
$$K_1=K_2=6(\mathfrak{B}_1+1.5\mathfrak{A}_2) .$$

Moment M_1 calculated by the elastic theory is

$$M_1 = \frac{-6}{6.5}(\mathfrak{B}_1 + 1.5\mathfrak{A}_2)\,.$$

Next, for $\nu_1 = \nu_3 = 1$, $\nu_2 = 0.5$ [ν is a constant relating to the creep characteristics as assumed in Eq. (3)], by using Eqs. (36)~(39),

$$M_{1t} = \frac{36}{221}(3\mathfrak{B}_1 - 2\mathfrak{A}_2)(1 - e^{-\frac{17}{26}\varphi_t})\,.$$

For $\varphi_t = 2$

$$M_{1t} = \frac{36}{221} \times 0.7296(3\mathfrak{B}_1 - 2\mathfrak{A}_2)\,.$$

Therefore, M_1 will become, after the duration of sustained loading (at $\varphi_t = 2$), as follows,

$$\begin{aligned}(M_1)_t &= M_1 + M_{1t} \\ &= -(0.5664\mathfrak{B}_1 + 1.6223\mathfrak{A}_2)\,.\end{aligned}$$

Then

for $p_2 = p_1$ $(M_1)_t/M_1 = 0.949$ and

for $p_2 = 2p_1$ $(M_1)_t/M_1 = 1.029$.

Thus, the end moment M_1 will vary by about 5% due to the creep difference between the two adjacent beams. Fig. 5 shows how much M_t varies with the creep difference and the loading conditions. In the actual reinforced concrete continuous

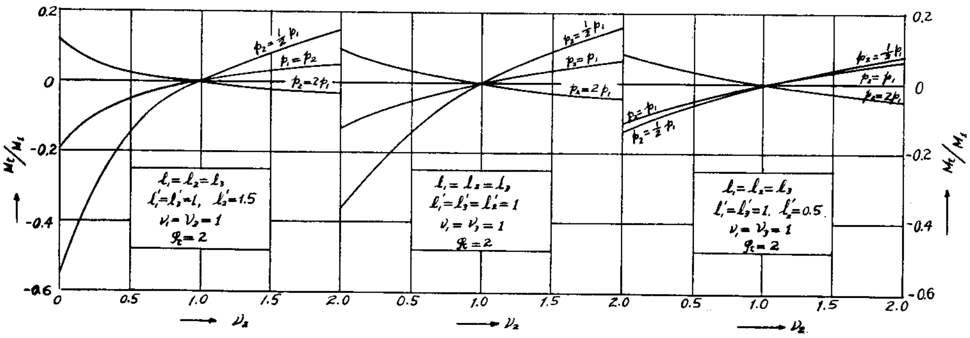

Fig. 5. Variation of moment dependent of creep difference and loading conditions.

beam bridge, the creep characteristics of a beam over each span and the magnitude of sustained load over each span beam are, in general, not so different in value from each other so that the variation in the moment distribution may not be large; the variation may possibly 10 to 15 percent, except for special cases in which large differences exist between the creep characteristics of the adjacent beams on the intermediate support.

5. Summary

The creep characteristics of the members composing the actual reinforced concrete buildings or bridge structures of statically indeterminate system may be quite identical in some special cases but rather very different in many other cases.

In this paper the effect of the difference in the creep characteristics of the members on the stress distribution is analysed theoretically. The studies show that the variation in the redundant forces is considerably large especially in the structure composed of reinfoced concrete and steel members.

The modified "Theorem of Three or Four Moments" may be conveniently applied in the creep analysis of the continuous beam bridges or frame structures.

These analysis developed here, if still slightly modified, can also be used for the similar problems of the structures of composite or prestressed members.

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