Monte Carlo Calculation of the Multiple Scattering of the Electron

By

Tunahiko SIDEI, Takenobu HIGASIMURA
and Katuhiro KINOSITA

Department of Applied Physics

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Abstract

The behavior of an electron in a thick Al layer is studied by the Monte Carlo method, assuming that the track of the electron is simplified by an approximate model, and the penetration and backscattering probability are obtained from a set of hundred histories. The consistency of the value of the practical range of the electron obtained by this calculation with that of the empirical formula proved that the method is quite adequate for solving the pentration and backscattering problem of the electron through a thick layer.

1. Introduction

On the behavior of the high energy electron which is incident on a layer of a material, various calculations were performed in the past to obtain the angular distribution of the penterating electron through a thin layer. These calculations were based on the assumption that the electron does not lose its kinetic energy along the path in this layer[1]. On the other hand, some calculations were carried out pertaining to the loss of the kinetic energy suffered by the electron or its range without regard for the deflection of the penetrating electrons[2]. Empirical formulae were also found on the practical range of the electron[3]. When the layer is thick, it is quite difficult to obtain the pentration probability, the energy and the angular distribution of the electron because both the energy loss and the angular deflection must be considered, and no calculating method has yet been found. The authers, therefore, applied the Monte Carlo method to the problem of thick layer of Al by dividing the path of the electron properly to many segments and calculating the angular distribution and the energy loss of the electron for each segments and obtaining the whole track by connecting them.

The procedure of the calculation is principally the same as in the case of the multiple scattring of gamma-ray. But it is impossible to follow completely each collision because the electron has much larger scattering probability than the gamma-ray. Therefore, a single scattering is assumed in place of the multiple scattering suffered by the electron in the segment. To use the same history for various geometries, the electron is traced in an infinite homogeneous medium (aluminium) and thereafter various boundary conditions are applied to test if the history satisfies them. Some results were obtained by calculating and testing 100 histories.

2. Calculation Procedure

2.1 Simplified model of the problem

Let an electron of the energy E_1 be incident at the point P_1 in an infinite homogeneous medium (Fig. 1). The electron will collide so many times with the atoms of the medium and it will travel degrading its energy and varying its direction of motion. Now, however, the electron is assumed to deflect at the point P_1 through the angle Θ_1, Φ_1 and to travel a certain distance t_1 in a straight line without losing its energy; and, after arriving at the point P_2, it suddenly loses its energy by an amout of ΔE_1. The electron which has reduced its energy to $E_2=E_1-\Delta E_1$ is scattered by the agle Θ_2, Φ_2 at P_2 and travels straight the distance t_2 and the energy loss ΔE_2 occurs at P_3, and so on. The electron, whose energy is sufficiently degraded after (n−1) times of scatterings, suffers the last scattering at the point P_n by the angle Θ_n, Φ_n and after traveling the distance t_n it ceases at the point P_{n+1}. Thus the whole history is terminated.

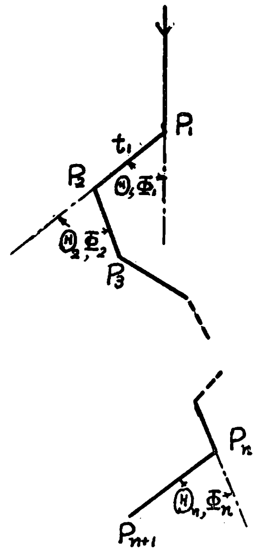

Fig. 1 Simplified model of the track of the elctron in the medium.

If the track of the electron is simplified as above mentioned, it is natural that the distribution of each scattering angle Θ_i, Φ_i $(i=1, 2, 3, \cdots, n)$ may be taken to be equal to the angular distribution of the multiple scattering suffered by the electron of the energy E_i in the course of passage through a thin foil of thickness t_i and that the distribution may be represented by the following Molière's formula which accurately describes the angular distribution of the multiple scattering of the electron[4].

$$\left.\begin{aligned} f(\Theta)\Theta d\Theta &= \left[2e^{-\delta^2}+\frac{f^{(1)}(\delta)}{B}+\frac{f^{(2)}(\delta)}{B^2}\right]\delta d\delta \\ \delta &= \frac{\Theta}{\chi\sqrt{B}} \\ \chi &= \frac{22.9Z}{pc\beta}\sqrt{\frac{t}{A}} \quad \text{(degree)} \end{aligned}\right\} \tag{1}$$

where the left hand side is the probability of scattering which may occur in the angle $\Theta \sim \Theta + d\Theta$, t is thickness expressed by g/cm^2, p is the momentum of the incident electron and β, c, A and N have their usual meanings. B is a constant which depends on the energy of the electron, the atomic weight and the atomic number of the medium, and the thickness of the layer, and its value, as well as $f^{(1)}$ and $f^{(2)}$, are given in the paper of Molière.

In order to calculate the energy loss ΔE_i, ΔE_i is taken to be the sum of the average ionization loss and the average radiation loss suffered by the electron while passing t_i, i.e.:

$$\Delta E = 0.153 \frac{Z}{A} \frac{t}{\beta^2}\left[\ln \frac{E(E+mc^2)^2}{2I^2mc^2}\beta^2 + (1-\beta^2) - (2\sqrt{1-\beta^2}-1+\beta)\ln 2 + \frac{1}{8}(1-\sqrt{1-\beta^2})^2\right]$$
$$+3.44\times 10^{-4}\frac{Z}{A}t(E+mc^2)\left[4\ln\frac{2(E+mc^2)}{mc^2}-\frac{4}{3}\right] \tag{2}$$

and, the value of the ionization energy of Al $I=150\,\text{eV}$ is used[5] and its straggling is neglected.

For the application of the Molière's formula, t_i must be taken thin enough so as to satisfy $\chi\sqrt{B}=40°$ which is the half angle of Molière's distribution. Molière's formula is recognized to be accurate within the half angle $\chi\sqrt{B}=20°$. However, the formula was extended tentatively beyond this limit because the calculation becomes far laborious if t_i is decided from $\chi\sqrt{B}=20°$. When the electron penetrates a layer of thickness t, the actual path of the electron scattered in large angle becomes considerably larger than t and this fact attributes to the main couse which prevents the Molière's formula from being accurate for larger scattering angles. In this case, if the path length is put equal to $t/\cos\Theta$ in the first approximation where Θ is the scattering angle and if $t/\cos\Theta$ is inserted into the Molière's formula instead of t, the equation may be used beyond $\chi\sqrt{B}=20°$. In our model, the thickness of the thin layer is taken to be equal rather to $t/\cos\Theta$ and it is presumed that the extention to $\chi\sqrt{B}=40°$ will not commit any material error.

Before the history of the electron is followed by this simplified model, it is necessary to calculate previously the value of each t_i and E_i. The energy of the incident electron is taken to be 2 MeV and t_1 is calculated from the condition $\chi\sqrt{B}=40°$ by the formula (1). ΔE_1 is calculated by inserting t_1 into the formula (2). Then, for the electron with the energy $E_2=E_1-\Delta E_1$, t_2 is calculated by applying the condition $\chi\sqrt{B}=40°$. In such a manner, a set of t_i and E_i is obtained. Their values are shown in the table.

t_i decreases steeply as the electron loses its energy as can be seen in the table. When the energy of the electron becomes smaller enough, the electron does not move actively and its motion thereafter does not give a large contribution to the result of the problem. Therefore, the above mentioned calculation is ceased at the 15th segment. For the 16th segment, t_{16} is put equal to the practical range of the electron of the energy E_{16} and is calculated by using the following formula[3].

$$\left.\begin{aligned} t &= 412\,E^{n} \\ n &= 1.265-0.0954\ln E \end{aligned}\right\} \qquad (3)$$

where E is expressed in MeV and t in mg/cm^2. Also the angular distribution of the 16th scattering is considered to be $\cos^2\Theta$ distribution. The value of t_{16} thus determined is much smaller than the path length that the electron has traveled before as can be easily seen from the table (i.e. $t_{16} \ll \sum_{i=1}^{15} t_i = 1117$ mg/cm^2).

Table

i	E_i (MeV)	t_i (mg/cm^2)
1	2.000	281
2	1.557	191
3	1.262	138
4	1.051	105
5	0.891	82
6	0.766	65
7	0.666	53
8	0.583	44
9	0.514	36
10	0.456	30
11	0.406	25
12	0.363	21
13	0.326	18
14	0.293	15
15	0.265	13
16	0.240	56

2.2 Random sampling

By employing a table of random numbers of uniform distribution, the angle Θ_i of the i-th scattering is picked at random from the distribution represented by the Molière's formula (1) as follows. At first, a random number R_{Θ} is picked from the uniform distribution from 0 to 1 and the equation $R_{\Theta}=\int_0^{\delta_i'} 2\delta e^{-\delta^2} d\delta$ is solved and then a set of $\frac{100}{M}$ random numbers δ_i', which is distributed as $2\delta e^{-\delta^2}$, are produced, where $M=\int_0^{\infty} f(\Theta)\Theta d\Theta=\int_0^{\infty}\left[2e^{-\delta^2}+\frac{f^{(1)}(\delta)}{B}+\frac{f^{(2)}(\delta)}{B^2}\right]\delta d\delta$. Next, a δ_i'' is picked at random from the distribution $\frac{|f^{(1)}(\delta)|}{B}\delta$ and is added to the set of random numbers δ_i' if $f^{(1)}(\delta_i'')>0$, or a number in this set, which has the nearest value to δ_i'', is removed from δ_i' if $f^{(1)}(\upsilon_i'')<0$ and this procedure is repeated $\frac{100}{M}\int_0^{\infty}\frac{|f^{(1)}(\delta)|}{B}\delta d\delta$ times. Also, by picking $\frac{100}{M}\int_0^{\infty}\frac{|f^{(2)}(\delta)|}{B^2}\delta d\delta$ random numbers from the distribution $\frac{f^{(2)}(\delta)}{B^2}\delta$ and testing to determine as before whether they are to be added or removed, 100 random numbers δ_i that obey Molière's distribution $\left[2e^{-\delta^2}+\frac{f^{(1)}(\delta)}{B}+\frac{f^{(2)}(\delta)}{B^2}\right]\delta d\delta$ can be obtained completely and the scattering angle Θ_i is expressed as $\Theta_i=x\sqrt{B}\,\delta_i=40°\times\delta_i$. After this procedure is repeated for 15 scatterings and each 100 random numbers are determined, 100 Θ_{16}'s for the last segment are determined by solving the equation $R_{16}=\int_0^{\Theta_{16}}\cos^2\Theta\sin\Theta d\Theta$ because Θ_{16} was assumed to be $\cos^2\Theta$ distribution.

It is obvious that the azimutal angle of the scattering Φ is uniformly distributed from 0° to 360°. 1600 Φ's are determined from $\Phi = 360° \times R_\Phi$ where R_Φ is a random number.

When all scattering angle Θ_i, Φ_i are thus determined for 100 histories, it is necessary to calculate the coordinate of each scattering point P_i. A cartesian coordinate is taken with its origin at the point of incidence P_1 and its z-axis on the incident ray, and the coordinate of P_i is expressed as (x_i, y_i, z_i). The direction of the electron after i-th scattering is determined by the angle θ_i from the z-axis and the azimuth ϕ_i relative to x-axis. θ_i, ϕ_i are determined from the well-known trigonometric equations :

$$\cos\theta_i = \cos\theta_{i-1}\cos\Theta_i + \sin\theta_{i-1}\sin\Theta_i\cos\Phi_i$$
$$\sin\theta_i \sin\Delta\phi_i = \sin\Theta_i \sin\Phi_i$$
$$\phi_i = \phi_{i-1} + \Delta\phi_i$$

But it is very laborious to solve these equations analytically. We found it very facile for this purpose to use a special slide rule[6] which can be produced by employing a stereogram. The coordinate of P_i can be calculated from

$$x_i = \sum_{i=1}^{i-1} t_i \sin\theta_i \cos\phi_i\,, \quad y_i = \sum_{i=1}^{i-1} t_i \sin\theta_i \sin\phi_i\,, \quad z_i = \sum_{i=1}^{i-1} t_i \cos\theta_i\,.$$

Thus the histories of 100 electrons that is incident with the energy of 2 MeV upon a infinitely extended homogeneous medium of Al are completely described.

2.3 Tests for various boundary conditions

Solutions of various problems can be obtained when these histories are tested and determind whether or not they satisfy each boundary conditions of the problems. We have chosen the following three problems.

(a) Relation between the penetration probability and the thickness of the Al layer when the electron of 2 MeV is perpendicularly incident.

Let z_M be the maximum of the z_i's of a certain history, then this electron can penetrate an Al layer of thickness $t < z_M$. But there must be no scattering point as $z_j < 0$ for $j < M$ because an electron can not penetrate if it is backscattered before the M-th scattering point is reached. Therefore, picking such z_M's from each 100 histories and plotting the number of histories $N(t)$ which satisfy $z_M > t$, the estimate of the penetration probability is given by $N(t)/100$. The results obtained by these tests are shown in Fig. 2.

(b) Relation between the backscattering probability and the angle of incidence when the electron of 2 MeV is incident obliquely upon a Al slabs having sufficient thickness.

When the angle of incidence α is equal to zero (perpendicular incidence), an electron can be considered as backscattered if there is such i as to fulfill $z_i < 0$ in the history. Let N_{back} be the number of such histories, the estimate of the backscattering probability is given by $N_{\mathrm{back}}/100$.

When α is not equal to zero and the z-axis is taken along the incident ray as before, then the angle between the surface of the Al slab and the z-axis is also α. Infinite numbers of such surfaces can be made by rotating the slab around the z-axis and, since the expected value of the backscattering probability is the same for all the surfaces, a history can be tested for all of these infinite number of slabs. However, only four tests were applied for the following special cases where the calculation is easily carried out. Namely, when the surface of the Al slab is parallel to x-axis, the backscattering occurs if such P_i that satisfies the following inequality exists:

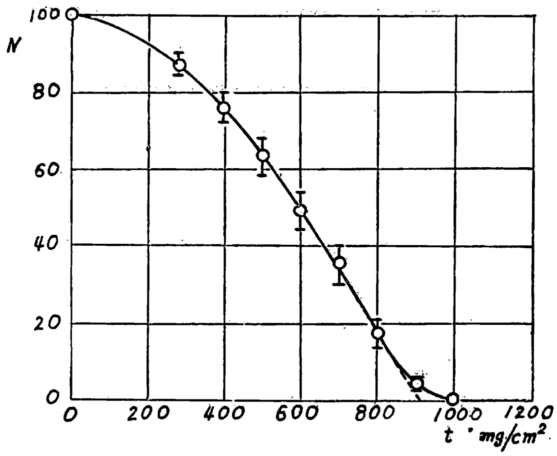

Fig. 2 The number N of penetrating electrons of 2.00 MeV. as a function of the thickness of Al layer (mg/cm^2).

$$y_i > z_i \cot\alpha \quad \text{or} \quad -y_i > z_i \cot\alpha .$$

Wheh the surface is parallel to y-axis, the condition is

$$x_i > z_i \cot\alpha \quad \text{or} \quad -x_i > z_i \cot\alpha .$$

If the N_{back} scccesses are obtained as a result of 400 such tests, the estimate of the backscattering probability is given by $N_{\text{back}}(\alpha)/100$. The variance of the estimate is not $\sqrt{P(1-P)/N}$ as in usual case where P is the estimate obtained and N is the independent history tested. In our case, although 400 histories are tested, each four histories are not independent of each other, therefore, the variance of this case is larger than the value given for $N=400$ but smaller than for $N=100$. The results obtained are shown in Fig. 3. The error represented on the curve corresponds to the case in which $N=100$.

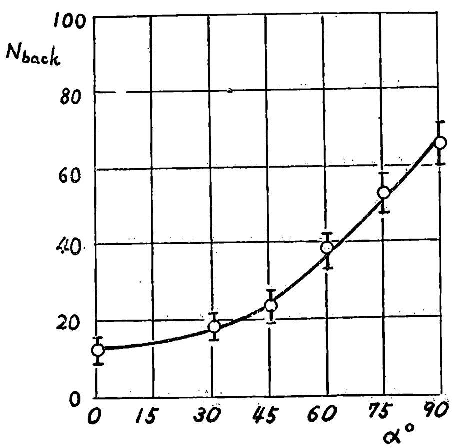

Fig. 3 The number N_{back} of the backscattering electrons of 2.000 MeV as a function of the angle of incidence α.

(c) Relation between the penetration probability of the electron and the thickness of the Al layer when the electron with smaller energy than 2 MeV is incident perpendiculary upon it.

In the above approximate model, all electrons scattered i times have the same energy E_i. Therefore, the motion of the electron, which is incident with the energy E_i, can be discussed by using the history of 2 MeV after the i-th scattering. Therefore, we tested their histories in the cases where the electron of 1.56 MeV, 1.05 MeV and 0.51 MeV is incident perpendicularly upon the Al layer by rewriting as follows.

In the histories of the electron of 2 MeV described above, the second segments ($i=2$) correspond to the energy of 1.56 MeV. Therefore, a coordinate transformation is carried out in such a manner that the electron corresponding to the first segment be perpendicularly incident at the origin. Let the direction cosine of the segment $\overline{P_1P_2}$ be (l, m, n), then P_i has its coordinate for the new system

$$\begin{aligned} z_i' &= l(x_i-x_2)+m(y_i-y_2) \\ &\quad +n(z_i-z_2) \\ &= lx_i+my_i+nz_i-z_2' \end{aligned}$$

where z_2' is the coordinate of P_2. Thus, calculating z_i' of the each scattering points of all histories and testing as (a), the penetration probability shown in Fig. 4 is obtained.

For the electron of 1.05 MeV and 0.51 MeV, similar calculations are carried out and each curve is obtained as shown in Fig. 5 and Fig. 6.

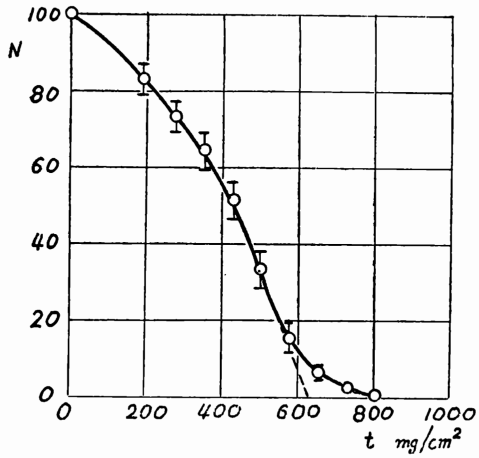

Fig. 4 The number N of the penetrating electrons of 1.56 MeV as a function of the thickness of Al layer (mg/cm²).

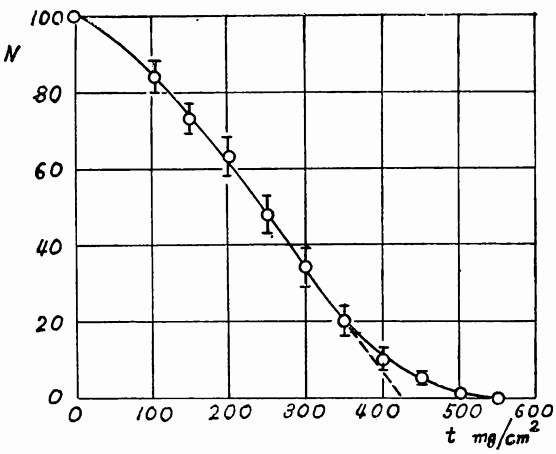

Fig. 5 The number N of the penetrating electrons of 1.05 MeV as a function of the thickness of Al layer (mg/cm²).

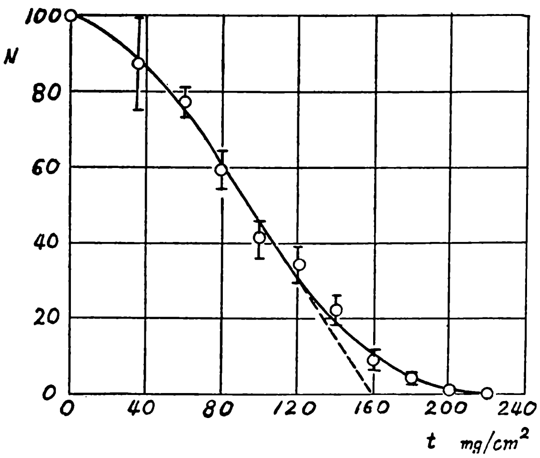

Fig. 6 The number N the penetrating electrons of 0.514 MeV as a function of the thickness of Al layer (mg/cm²).

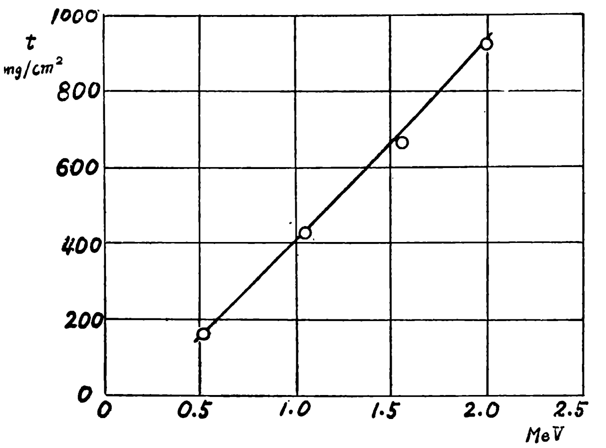

Fig. 7 The curve shows the pratical range of the electron in mg/cm² calculated by katz's experimental formula (3). Circles in the figure are the values obtained by this calculation.

3. Discussion

Many assumptions made in the above calculations naturally brings about some error in the results. But it is very difficult to estimate it analytically. We estimated the practial range of the electron by extrapolating the penetration curve and obtained the values 920, 663, 425 and 160 mg/cm² for the electron of the energy 2.00, 1.56, 1.05 and 0.51 MeV respectively. Katz's empirical formula about the practical range (3) gives the values 946, 708, 439 and 170 mg/cm². The consistency is rather good as can be seen from Fig 7. This means that our simplified model is sufficiently adequate to discuss the behavior of high energy electron in thick medium.

Although the size of our sample is too small to obtain the rigourous results, it has been confirmed that it is possible to obtain some useful results if sufficiently large sample is calculated by this method on an automatic computer. The penetration and the backscattering coefficient of various beta-ray, the efficiency of the G-M counter and of the ionization chamber for various gamma-ray can easily be obtained by using same histories. However, the history in high Z materials becomes more troublesome to calculate because the electron is much more scattered in high Z materials than in low Z materials and the number of collisions needed to form a whole history becomes larger. In order to obtain the results of the same accuracy in the case of Pb, as in the case of Al, the history must be composed of about one hundred segments.

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