Boundary Layer Growth of Open Channel Flows on a Smooth Bed and its Contribution to Practical Application to Channel Design

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Abstract

The hydraulic behavior of the boundary layer growth, which has been investigated in connection with the practical application to the air-entrainment into flows by turbulence, is concerned in the light of mathematical and physical interpretation made possible by the combination of three basic equations of boundary layer, main flow and discharge in hydrodynamics and of experimental verification conducted at the Hydraulics Laboratory, Engineering Research Institute, Kyoto University, with results of the clear difference between the behavior of boundary layer of open channel flows and that along the flat plate in the unconfined flow.

Furthermore, the application to design problem of steep chutes and the hydraulic behavior of flows in transition to the fully developed turbulent flow near an entrance of channel are discussed.

Introduction

When a fluid flows from a reservoir into a conduit or over a spillway and a chute, the fluid initially given by the nearly constant velocity profile is retarded by the tangential shear along the boundaries and the velocity gradient is quite large, within a thin layer near the boundaries, if the distance the fluid has travelled along the channel is not so long. This retarding effect of the boundary shear gradually spreads farther and farther, so that the thickness of the layer of retarded fluid continually increases to the free surface of flow. It is well known that this zone of retarded fluid is called the boundary layer.

Although it is a matter of common observation that even in open channel flows there exists the boundary layer near the inlet from a reservoir, the analysis of the behavior of open channel flows over a spillway in hydraulics, hitherto, was made in common by the following two procedures of approach: the evaluation of flow patterns by means of describing the stream line in classical hydrodynamics, and the one-dimensional analysis of the hydraulic equation in which some suitable empirical formulas like Chézy's or Manning's in fully developed turbulent flows are used, with the aid of numerical integration.

The dynamical process of motion in the fluid is governed by the hydraulic equation of motion based on the Newtonian principle of motion, and the solution obtained from the Navier-Stokes equation of motion under certain suitable boundary and initial conditions associated with a problem under consideration indicates the behavior of flows. However, until the present day no general methods have been become available for the integration of the Navier-Stokes equation owing to the great mathematical difficulties. Furthermore, solutions which are valid for all values of viscosity are known only for some particular cases.

In the turbulent flow, which is of more importance from the point of view of practical application to engineering problems, O. Reynolds introduced the fundamentally important concept of virtual turbulent friction as far back as 1880. However, this concept was in itself insufficient to make the theoretical analysis of turbulent flows possible. Great progress was achieved with the introduction of L. Prandtl's mixing length theory in 1904, which, together with systematic experiments conducted by many investigators, paved the way for the theoretical approach of this flow in every field of applied aero- and hydrodynamics.

The significance of the behavior of boundary layer growth in open channel flow is that the open channel flow is a confined flow, which contacts with the free surface, an additional unknown function, readily sensible for the influence of boundaries. This is quite different from the boundary layer growth in the unconfined flow, and therefore, the interdependency of the boundary layer equation to the main flow outside the boundary layer should be considered, while the behavior of the flow around an airfoil or near an obstacle can be practically calculated without any modification of its original equation. The existence of free surface gives a rise of difficulty of the analysis of boundary layer growth in open channel flows, owing to the shear friction between the fluid and the atmosphere and the doubtfulness of clear formulation of boundary layer. Neverthless, it seems the actual hydraulic behavior of open channel flows like the high velocity flow on steep chutes suddenly changes at the critical point where the boundary layer intersects the free surface.

The history of the investigation of boundary layer growth in open channel flows was initially related to the study of air-entrainment process into the fluid flow over steep chutes suggested by E. A. Lane[1]. Then, many investigators studied the process associated with the practical application to spillway design and almost verified with field observations, especially it should be remembered that this subject was one of special topics to study in the International Hydraulic Convention held at Minneapolis in 1953[2]. However, all procedures of approach have ever been studied are only related to the empirical development based on the well known relationship of turbulent boundary layer growth on a flat plate in the unconfined flow. In 1952, G. Halbronn[3] first investigated the interdependency between both flows of boundary layer and main flow, and in the same year, A. E. Craya and J. W. Delleur[4] also studied this subject associated with the problem in horizontal and slightly sloping channels, with results of refined interpretation of boundary layer growth near the critical regime. Both of them are highly evaluated as the outstanding theoretical results revealed the significance of interdependency between the boundary layer flow and main flow. More recently, in 1953, W. J. Bauer[5] conducted the systematic experimentation of velocity distribution of the turbulent boundary layer under accelerative flow, resulted in the interpretation of applicability to the process of air-entrainment in the high velocity flow.

In this paper, the present purpose is to reveal the general behavior of boundary layer growth under steady regime in the light of hydrodynamic interpretation made possible by the mathematical analysis of boundary layer equation and of experimental verification conducted in a smooth bed at the Hydraulics Laboratory, Engineering Research Institute, Kyoto University. However, the final stage of clear formulation of the subject, resulted in the practical application contributed to the establishment of modern hydraulics of high velocity flow in open channels and design problem of channels, stands still far away owing to the great complicated phenomena combined with turbulence and non-linear mechanics related to the existence of pressure gradient as well as the unknown function of fluid depth.

Basic Principles of Boundary Layer in Open Channel Flows

The prerequisite requirement to investigate the exact behavior of boundary layer growth in open channel flows is to establish the basic relationships of flows in both of main and boundary layer zones under accelerative flows made the interpretation of interdependency possible. This is attained by combining two equations of boundary layer and main flow together through the principle of constant discharge under steady regime.

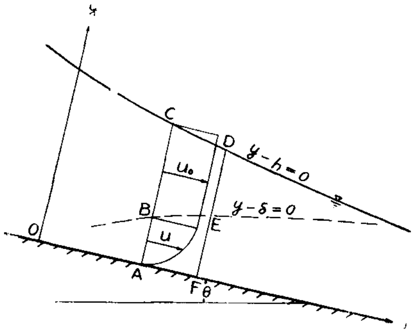

Fig. 1. Schematic diagram of open channel flows.

(1) *Equation of momentum in main flow*

Taking x-axis in the downstream direction along the channel bed, y-axis vertically upward, and denoting u and v: components of velocity in both of x and y directions, p:

pressure, h: depth of flow, δ: thickness of boundary layer, g: acceleration of gravity, θ: inclination angle of channel bed, ρ: density of fluid and subscript o: values in the main flow, the law of momentum conservation yields the following relationship, considering the water column BCDE of unit width on a virtual boundary acting no tangential shear.

$$\int_\delta^h (\rho u_0^2+p)dy-\left\{\int_\delta^h (\rho u_0^2+p)dy+\frac{\partial}{\partial x}\int_\delta^h (\rho u_0^2+p)dydx\right\}$$

$$=-u_0\frac{\partial}{\partial x}\int_0^\delta \rho u\,dydx+\int_\delta^{\delta+\frac{\partial\delta}{\partial x}dx}\left(p+\frac{1}{2}\frac{\partial p}{\partial x}dx\right)dy-\rho g\sin\theta(h-\delta)dx\,. \tag{1}$$

Usually the vertical acceleration of fluid particle in open channel flows is considered to be practically negligible at hydraulic ordinary points, so that the pressure distribution is given by the hydrostatic law,

$$p=\rho g\cos\theta(h-y)\,. \tag{2}$$

Differentiating Eq. (2) with respect to x, inserting into Eq. (1), integrating Eq. (1) between two limits of zone under consideration and dividing both terms by dx, Eq. (1) is, finally, reduced to the following momentum equation in the main flow.

$$\frac{d}{dx}\int_\delta^h u_0^2dy-u_0\frac{d}{dx}\int_0^\delta u\,dy=g(h-\delta)\left(\sin\theta-\cos\theta\frac{dh}{dx}\right). \tag{3}$$

Using the constant discharge principle under steady regime as seen in (4),

$$q=\int_0^h u\,dy=\int_0^\delta u\,dy+u_0(h-\delta)=\text{const.}$$

Eq. (3) is reduced to

$$u_0\frac{du_0}{dx}=g\left(\sin\theta-\cos\theta\frac{dh}{dx}\right). \tag{4}$$

This equation of momentum in the main flow is also obtained by integrating the Euler equation of motion with respect to y throughout the whole zone of main flow from δ to h, that is,

$$\int_\delta^h\left(u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y}\right)dy=g(h-\delta)\left(\sin\theta-\cos\theta\frac{\partial h}{\partial x}\right). \tag{5}$$

Considering the following relation derived from the equation of continuity,

$$|v_0|_\delta^h=-\int_\delta^h\frac{\partial u_0}{\partial x}dy\,,$$

Eq. (5) becomes

$$\int_\delta^h\frac{\partial u_0^2}{\partial x}dy-u_0\int_\delta^h\frac{\partial u_0}{\partial x}dy=g(h-\delta)\left(\sin\theta-\cos\theta\frac{\partial h}{\partial x}\right).$$

Finally, it follows

$$u_0\frac{du_0}{dx}=g\left(\sin\theta-\cos\theta\frac{dh}{dx}\right).$$

Hence, both procedures of method lead the basic equation to the same momentum equation.

It can be reduced to the well known Bernoulli equation, that is, Eq. (4) is readily integrable with respect to x and it follows

$$\frac{u_0{}^2}{2g}+h\cos\theta-x\sin\theta=C.$$

Thus, under the boundary condition that $u_0=u_{00}$ and $h=h_0$ at $x=0$, the above equation becomes

$$E=\frac{u_0{}^2}{g}+h\cos\theta-x\sin\theta, \tag{6}$$

where E is the specific energy expressed by $(u_{00}^2/2g)+h_0\cos\theta$. This equation is the well known Bernoulli equation of motion in the main flow characterized by the non-friction and constancy of energy.

(2) *Boundary layer equation for momentum*

The application of law of conservation of momentum for the water column ABEF in Fig. 1 in the boundary layer yields the following relationship as same as in the previous section.

$$-\frac{d}{dx}\int_0^\delta(\rho u^2+p)dy=-u_0\frac{d}{dx}\int_0^\delta\rho u\,dy-\frac{d}{dx}\int_\delta^{\delta+\frac{d\delta}{dx}dx}\left(p+\frac{1}{2}\frac{\partial p}{\partial x}dx\right)dy$$
$$-\rho g\delta\sin\theta+\tau_0, \tag{7}$$

where τ_0: shearing stress along the channel bottom. Again, under the assumption of hydrostatic pressure of distribution, the integration of Eq. (7) leads to

$$\frac{\tau_0}{\rho}=g\delta\left(\sin\theta-\cos\theta\frac{dh}{dx}\right)-\frac{d}{dx}\int_0^\delta u^2dy+u_0\frac{d}{dx}\int_0^\delta u\,dy. \tag{8}$$

Similarly, the above equation may also be obtained by integrating the Navier-Stokes or Boussinesq equation of motion with respect to y between 0 and δ. Thus, considering that

$$v_\delta=-\int_0^\delta\frac{\partial u}{\partial x}dy,$$

the equation of motion

$$\int_0^\delta\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)dy=g\delta\left(\sin\theta-\cos\theta\frac{\partial h}{\partial x}\right)-\binom{\nu}{\varepsilon}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{9}$$

where ν: kinematic viscosity and ε: kinematic eddy viscosity, becomes

$$\frac{\tau_0}{\rho}=g\delta\left(\sin\theta-\cos\theta\frac{dh}{dx}\right)-\frac{d}{dx}\int_0^\delta u^2dy+u_0\frac{d}{dx}\int_0^\delta udy\,.$$

Hence it may be understood that the above equation is equal to Eq. (8).

(3) *Modification of boundary layer equation by means of displacement and momentum thicknesses*

The boundary layer equation (8) has been derived by the assumption that the thickness of boundary layer was certainly definite. It is, however, more desired to express the boundary layer equation in terms of certain suitable length capable of precise definition, and this is also done by introducing the displacement and momentum thicknesses defined by

$$u_0\delta_*=\int_0^\delta(u_0-u)dy \quad \text{and} \quad u_0{}^2\vartheta=\int_0^\delta u(u_0-u)dy\,. \tag{10}$$

As Eq. (8) is reduced to

$$\frac{\tau_0}{\rho}=g\delta\left(\sin\theta-\cos\theta\frac{dh}{dx}\right)+\frac{d}{dx}\int_0^\delta u(u_0-u)dy-\frac{du_0}{dx}\int_0^\delta udy\,, \tag{11}$$

and Eq. (4) in the main flow is also transformed to

$$g\delta\left(\sin\theta-\cos\theta\frac{dh}{dx}\right)=\frac{du_0}{dx}\int_0^\delta u_0dy\,, \tag{12}$$

so it follows, from these two equations of (11) and (12),

$$\frac{\tau_0}{\rho}=\frac{d}{dx}\int_0^\delta u(u_0-u)dy+\frac{du_0}{dx}\int_0^\delta(u_0-u)dy\,. \tag{13}$$

Thus, inserting two relations of the displacement and momentum thicknesses above defined into Eq. (13) yields

$$\frac{\tau_0}{\rho u_0{}^2}=\left(\frac{u_*}{u_0}\right)^2=\frac{C_f}{2}=\frac{d\vartheta}{dx}+\frac{1}{u_0}\frac{du_0}{dx}(2\vartheta+\delta_*)\,, \tag{14}$$

where u_*: frictional velocity and C_f: local skin friction coefficient defined by $\tau_0/(\rho u_0{}^2/2)$. Eq. (14) is the well known equation seen in many publications related to the boundary layer theory.

Designating the ratio δ_*/ϑ by H, Eq. (14) becomes

$$\frac{C_f}{2}=\frac{d}{dx}\left(\frac{\delta_*}{H}\right)+\frac{2+H}{H}\frac{\delta_*}{u_0}\frac{du_0}{dx}\,. \tag{15}$$

Although H is a variable dependent on the flow characteristics in both of constant pressure flow and flow with pressure gradient, it is customary that it is assumed constant for a given problem without large range of Reynolds numbers as a first approximation of engineering problem.

(4) *Relationship of constant discharge principle*

As already described in the previous section, the steady flow is characterized by the constant discharge principle at any section of channel, which is expressed in terms of displacement thickness as follows.

$$q=\int_0^h u dy=\int_0^\delta u dy+u_0(h-\delta)=u_0(h-\delta_*) \,. \tag{16}$$

(5) *Interrelationship among thickness of boundary layer, displacement and momentum thicknesses*

The hydraulic significance of open channel flows with the growth of boundary layer is commonly associated with the mathematical behavior of the boundary layer thickness. However, as the displacement and momentum thicknesses are substantially confirmed by experimentation, so consequently, it is more desired to replace the ill-defined thickness by the more precisely defined displacement and momentum thicknesses. Neverthless, the boundary layer thickness plays a primary role in hydraulics on the transitional behavior of open channel flows. Therefore, it is of practical convenience to derive the mutual correlation among these thicknesses.

In fully developed laminar boundary layer, the velocity profile is practically in a form of

$$\frac{u}{u_0}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2 . \tag{17}$$

Hence, the displacement and momentum thicknesses and H become

$$\delta_*=\frac{1}{3}\delta\,,\quad \vartheta=\frac{2}{15}\delta \quad \text{and} \quad H=\frac{5}{2} . \tag{18}$$

In fully developed turbulent boundary layer, when the volocity profile is assumed of the power type

$$\frac{u}{u_0}=\left(\frac{y}{\delta}\right)^n, \quad \text{where} \quad 0\leqq n<1 , \tag{19}$$

δ_*, ϑ and H become, respectively,

$$\delta_*=\frac{n}{1+n}\delta\,,\quad \vartheta=\frac{n}{(1+n)(1+2n)}\delta \quad \text{and} \quad H=1+2n . \tag{20}$$

As in the power law the power varies with the increase of Reynolds numbers, so these values in Eq. (20) are not constant but variables of given flow characteristics. Table 1 describes the interrelationships among δ_*, ϑ and H as a parametric expression of power. Evidently, the law in which the power is one seventh is the famous Blasius 7th law.

If the velocity profile of turbulent layer forms logarithmic,

Table 1. Interrelationship among boundary layer, displacement and momentum thicknesses.

n	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
δ_*	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$
ϑ	$\frac{2}{15}$	$\frac{5}{42}$	$\frac{3}{28}$	$\frac{7}{72}$	$\frac{4}{45}$	$\frac{9}{110}$
H	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{4}{3}$	$\frac{9}{7}$	$\frac{5}{4}$	$\frac{11}{9}$

$$\frac{u}{u_*} = A_s + \frac{1}{\kappa}\log\frac{u_* y}{\nu}, \tag{21}$$

therefore, δ_*, ϑ and H become

$$\delta_* = \frac{1}{p}\delta, \quad \vartheta = \frac{p-2}{p^2}\delta$$

and $H = \frac{p}{p-2}$, (22)

where $p = \kappa u_0/u_*$.

(6) *Consistency between boundary layer approach and usual hydraulic method of approach*

The reduction of basic relationship in both of boundary layer and main flows is based on the concept that both flows in the open channel flow are nominally divided by the assumed boundary layer thickness In hydraulics, the common practice is to analyze the hydraulic equation derived by the one-dimensional approach of momentum or energy conservation principles. Therefore, it will be desired to establish the consistency between both approaches.

Integrating Eq. (4) with respect to y throughout the zone of main flow,

$$\int_\delta^h u_0 \frac{du_0}{dx} dy = g(h-\delta)\left(\sin\theta - \cos\theta\frac{dh}{dx}\right). \tag{23}$$

Adding Eq. (23) to Eq. (8) yields

$$\begin{aligned}\frac{\tau_0}{\rho} &= gh\left(\sin\theta - \cos\theta\frac{dh}{dx}\right) - \frac{d}{dx}\int_0^\delta u^2 dy + u_0\frac{d}{dx}\int_0^\delta u dy - \int_\delta^h u_0\frac{du_0}{dx}dy \\ &= gh\left(\sin\theta - \cos\theta\frac{dh}{dx}\right) - \frac{d}{dx}\int_0^h u^2 dy + \frac{d}{dx}\{u_0^2(h-\delta)\} \\ &\quad + u_0\frac{d}{dx}\int_0^h u dy - u_0\frac{d}{dx}\{u_0(h-\delta)\} - u_0(h-\delta)\frac{du_0}{dx}.\end{aligned}$$

Finally, it becomes

$$\frac{\tau_0}{\rho} = gh\sin\theta - gh\cos\theta\frac{dh}{dx} + \frac{\beta q^2}{h^2}\frac{dh}{dx}, \tag{24}$$

where β is the momentum correction factor. Transforming Eq. (24) into the usual form in hydraulics of open channel flows with the aid of normal depth h_0 and critical depth h_c leads to

$$\frac{dh}{dx} = \tan\theta\frac{h^3 - h_0^3}{h^3 - h_c^3}. \tag{25}$$

This equation is the basic equation of open channel flows usually derived by the one-dimensional approach of analysis, and it follows that the consistency of both procedures of approach is established.

Velocity Distribution in Boundary Layer

In engineering problems contributable to practical calculation of the skin friction and the boundary layer thickness with the growth of boundary layer, the usual procedure of approach is to solve the one-dimensional equation like the von Kármán integral equation. Therefore, for the analysis of boundary layer growth of open channel flows, the prior knowledge of the shape of velocity profile across any section is required.

Although many investigators have studied this subject, it stands far away to reach the final stage of clear formulation of velocity profile and especially a rational theory for fully developed turbulent flow is still non-existent and in view of the extreme complexity of such flows it will remain so for a considerable time. Only the way possible to approach the fruitful success in this problem is to derive the solution or formula of steady uniform flows. Thus, for many years, engineers and scientists have been enforced to establish the velocity distribution in pipes, conduits and open channels under such flow characteristics. And the analysis of the behavior of boundary layer by the application of momentum equation to assumed families of velocity profile remains popular, since engineers have to make computations relating to the turbulent skin friction and the like with the aid of some development of empirical methods.

In this section, velocity profile and related skin friction of turbulent layer obtained from the experiment conducted at the Hydraulics Laboratory, Kyoto University, are concerned, with the aid of theoretically derived results by many investigators. The experimental flume used is of length of 36 ft. and its slope is variable from 0 to 30 degree. The side walls and bed consist of a very smooth lucite, and as seen in the later section the Manning roughness is 0.0084 (m-sec) in average. At the entrance to channel there is a reach where the hypothesis of flow without curvature is not valid and which makes the velocity near the bottom faster than that in the upper flow. Therefore, a lucite guide vane was set up to make the velocity at the entrance reach uniform.

The problems of laminar layer of boundary layer flow are largely mathematical, since it is certain that the fundamental mechanical principles are fully understood and that the equation adequately describe the phenomena when the flow may be regarded as incompressible. It is well known that the shape of velocity profile in unconfined flows of constant pressure is the Blasius profile and in the steady uniform regime of open channel flows, T. Ishihara, Y. Iwagaki and T. Goda[6] proved that it became parabolic as a close approximation, though the laminar layer has of less importance in engineering problems owing to its appearance only for small Reynolds numbers.

In fully developed turbulent layer, which is mainly subjected to study, Reynolds has refined the method of appropach with insufficient results to make the theoretical analysis possible. A somewhat different method of simplifying the Navier-Stokes relation was proposed by Boussinesq, who reasoned that the similarity between the molecular motion in laminar flow and the eddy motion in turbulent flow should permit the shearing stress resulting from the two types of motion to be described in a similar fashion. However, great progress of the subject was achieved with the introduction of mixing length theory by Prandtl, which distinguishly formulated the actual interrelationship of the Reynolds and Boussinesq parameters and in the same period G. I. Taylor developed his statistical theory of turbulence, which was so fruitful in treating the problem of isotropic turbulence. Von Kármán extended the theory, clothed it in more elegant mathematical form, and attempted, with incomplete success, to treat the problem of shear flow. More recently, F. H. Clauser[7] divided the velocity distribution in turbulent layer into the two parts of inner and outer, resulted from the detailed analysis of velocity profile of flows under both of constant pressure and pressure gradient. On the other hand, the effort in establishment of the power law still continues from the practical point of view of dimensional analysis in classical hydraulics.

In the past the popular form has been of the power type $u_0 \propto y^{\frac{1}{7}}$ which is known as the Blasius 7th law, i.e.

$$C_f = \frac{2\tau_0}{\rho u_0^2} = 0.0450\ R_\delta^{-\frac{1}{4}}, \tag{26}$$

and

$$\frac{u}{u_0} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}. \tag{27}$$

The power law is directly related to the Reynolds number, resulted from the empirical analysis of the experimental data of steady uniform flows. Nikuradse investigated the correlation between the power of power law and the Reynolds number in pipe flows and obtained the variation of powers with the increase of Reynolds numbers. Fig. 2 indicates the correlation of the power to the Reynolds number in steady unifom regime. Fig. 3 describes the skin friction coefficient as a function of Reynolds numbers and the trend of plotting of data is slightly large compared with the Blasius curve.

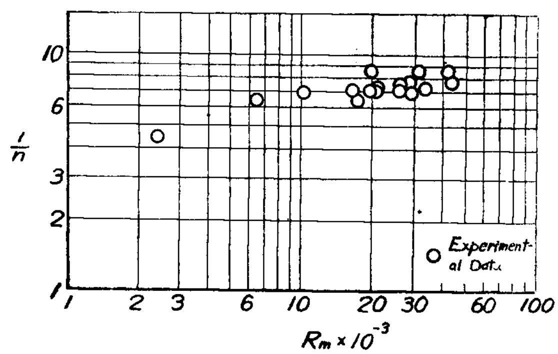

Fig. 2. Correlation of power to Reynolds numbers in steady uniform regime.

As the problem of boundary layer of open channel flows is related to the flow with pressure gradient, so consequently, it has to be dealt with the velocity profile and skin friction law under the existence of pressure gradient. These flows have been studied with the application of the von Kármán integral equation for momentum and the use of empirical relations obtained from a few experimental studies of flow in convergent and divergent passages by E. Gruschwitz[8]), A. E. von Doenhoff and N. Tetervin[9]) and other investigators, while the earliest procedure was suggested by Buri. All such procedures rest on empirical assumption like the Squire–Young formula and are considered the practical approach of method in engineering application not contributable to the construction of a rational theory founded on knowledge of the underlying physical phenomena.

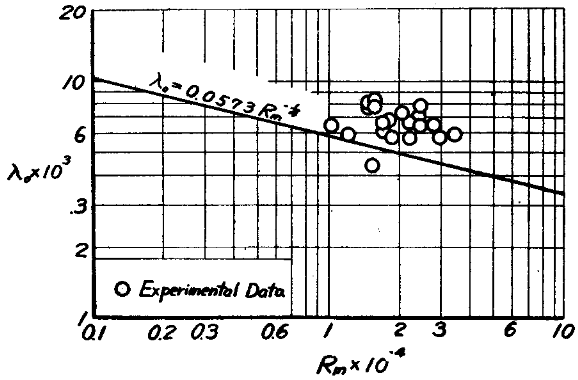

Fig. 3. Skin friction coefficient in steady uniform regime.

Bauer studied the velocity profile of turbulent layer in open channel flows with systematic experiments and concluded that the shape of velocity profile associated with the development of turbulent layer on steep slopes was better approximated in most reaches by an expression of the power type than one of the form of logarithmic function and especially it is true at large values of the Reynolds number. However, it is natural that the power law with the adequately determined power describes the actual profile of velocity as a very close approximation and the plotting shown by Bauer has been compared with the results theoretically obtained from the logarithmic law, with results of obscure conclusion in clear difference between two types of flow shape. Fig. 4 indicates the behavior of local skin friction coefficient to the Reynolds number in terms of boundary layer thickness for a smooth bed. It indicates that the coefficient throughout the whole

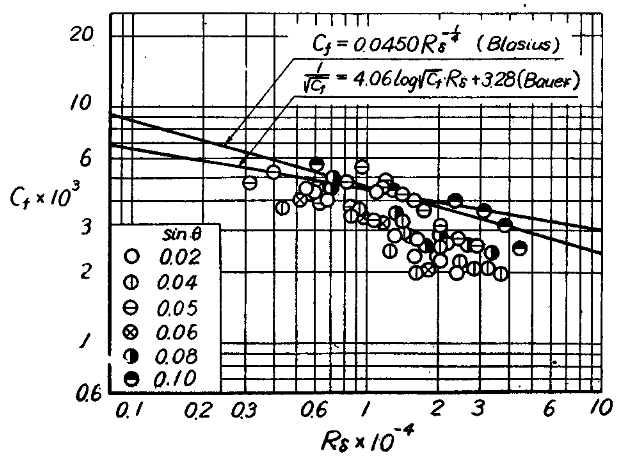

Fig. 4. Local skin friction coefficient for a smooth bed.

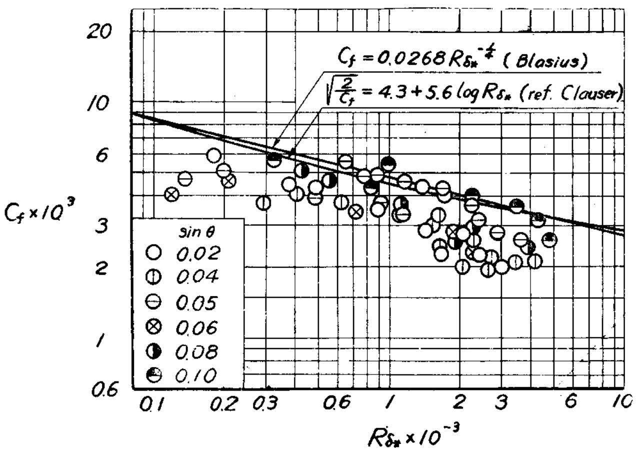

Fig. 5. Relation between skin friction coefficient and Reynolds number of displacement thickness.

region of experimentation has slightly less values than those derived by curves of Blasius and Bauer. Fig. 5 also described the same coefficient for the Reynolds number of displacement thickness. Within the applicability of Reynolds numbers obtained from the experiment, the following empirical formula will be proposed, predicting the validity of the 7th power law.

$$C_f = 0.0225\, R_{\delta*}^{-\frac{1}{4}}.$$

As in the power law the skin friction coefficient is assumed to be proportional to the m th power of Reynolds numbers, so between m and H, it is found that

$$m(1+H)+2(H-1)=0, \tag{29}$$

from the dimensional analysis. It means that the application of Eqs. (26) or (28) to a given problem under investigation involves that H is not a variable but a constant of 9/7 without the range of Reynolds numbers. Fig. 6 indicates the growth of H with the increase of Reynolds numbers of displacement thickness. Evidently in inspection, H is not constant and has a trend of increase to a constant value determined by flow characteristics given. Therefore, it may be understood that there is a contradiction between the physical interpretation of empirical formula of Eq. (28) and the actual flow indicated in Fig. 6. However, it should be expected that the guide vane set up at the entrance to channel to protect the separation of flow by the negative pressure gradient and the free surface may influence the shape of velocity profile of turbulent layer to some extent, and

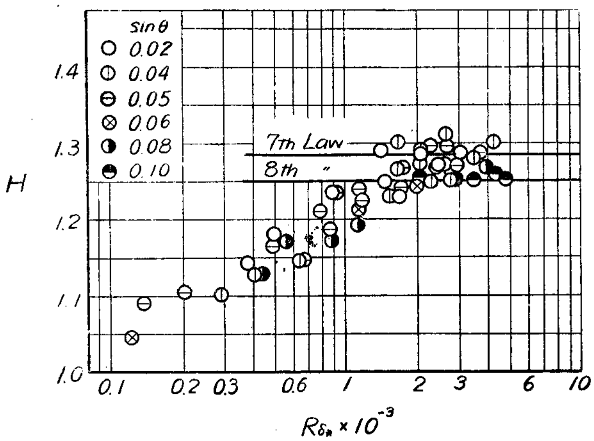

Fig. 6. The growth of H with the increase of Reynolds numbers.

it is required the further study of this effect.

As the power law is essentially empirical, based on the dimensional analysis of data obtained, so the adequate consequence involved in the law may describe the actual phenomena in a very simple mathematical form and apply to the similar problem within the limit of applicability. However, as the substantial characters of mechanical principle of turbulence in flow itself is not concerned in this approach, so it is classified as a procedure in classical hydraulics. Under such a historic background the establishment of dynamical principle of turbulence has been required as one of the fundamental subject in modern hydrodynamics, and many outstanding scientists and engineers have been enforced to study, and thus, the superiority of logarithmic law has also been acknowledged. More recently, Clauser concluded that the very simple assumption of a constant eddy viscosity which was proportional to $u_0\delta_*$ accurately predicted the behavior of the outer 80 to 90% of turbulent layer in both flows of constant and pressure gradient, and when it was combined with an inner eddy viscosity proportional to u_*y in the inner 10 to 20% zone, a complete remarkably accurate shape of the turbulent velocity profile was obtained.

For the application of logarithmic law to the velocity profile in open channel flows, many investigators have studied and especially G. H. Keulegan[10], R. W. Powell[11] and Y. Iwagaki[12] have obtained so fruitful conclusion on the turbulent characteristics of flow. Fig. 7 indicates the velocity profile of fully developed turbulent flow in steady uniform regime with the expression of logarithmic form and it is proved that these plotting tends to the theoretical study of Iwagaki. In 1952, Halbronn analyzed the turbulent boundary layer with the aid of logarithmic law of von Kármán and $\kappa=0.4$. Although Bauer concluded the power law of velocity profile in turbulent layer had a superiority over the logarithmic law from his experimental study, his plottings were compared with the theoretical results from logarithmic law. It means that the actual shape of velocity profile is expressed approximately in terms of both power and logarithmic laws, though the essential character of both approaches should be distinguishly divided. Fig. 8 describes the logarithmic expression of velocity profile of turbulent boundary layer. At an earlier reach of development of boundary layer the velocity has a nearly constant value and it becomes logarithmic with the increase of boundary layer growth. Two curves

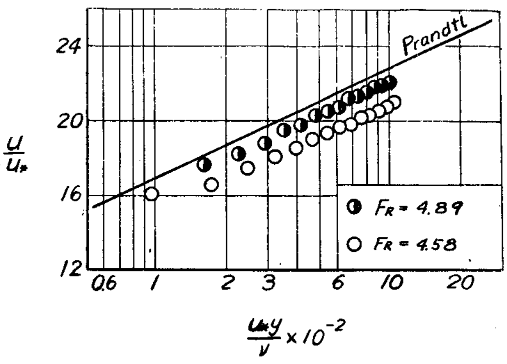

Fig. 7. Velocity profile of fully developed turbulent flow in terms of logarithmic expression.

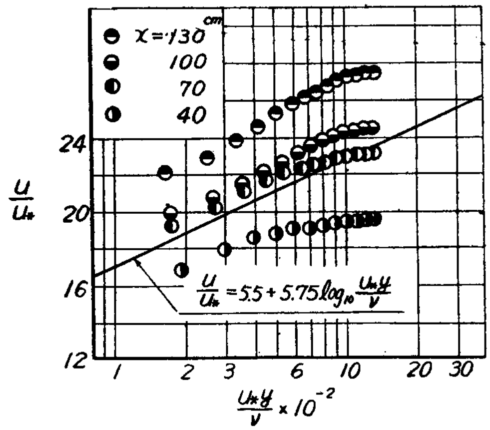

Fig. 8. Velocity profile of turbulent boundary layer in terms of logarithmic expression.

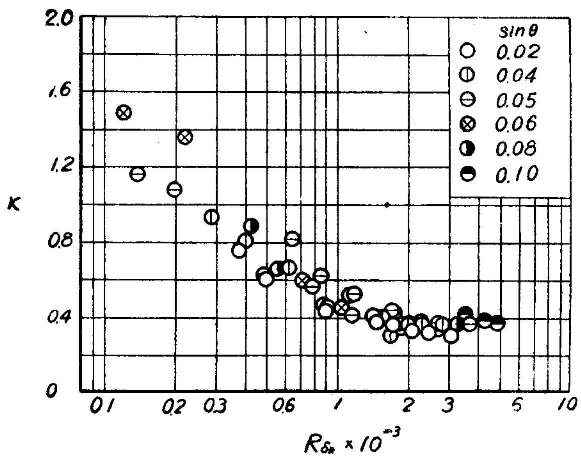

Fig. 9. Kármán constant in turbulent boundary layer.

in Fig. 5 are of Blasius and old law described by Clauser. It is also understood that the plotting has less value compared with these two curves. The Kármán constant κ is determined 0.38 to 0.40 from the experiments of steady uniform regime. With the aid of Prandtl form of logarithmic law, κ is also expressed in a form of

$$\kappa = \sqrt{2C_f}\frac{H}{H-1}. \qquad (30)$$

Fig. 9 indicates the behavior of κ with the increase of Reynolds numbers of displacement thickness. Evidently in inspection in Fig. 9, the trend of behavior of κ describes a rapid decrease from a large value to a constant of 0.40 determined by many investigators with the increase of Reynolds numbers. This fact is not explained and the further study of the behavior of κ is also required if the conclusion of Bauer on the unvalidity of logarithmic law to the shape of velocity profile of turbulent layer means the above described fact.

Mathematical Analysis of Boundary Layer Growth

As often described in the previous section, the growth of boundary layer in open channel flows is readily influenced by the free surface and characteristics of boundaries, so the required solution is simultaneously solved by the basic equations (6), (15) and (16) together. Before discussing their mathematical behavior, considered is the boundary condition permitted by the suitable hydraulic conditions of flow. It will be practically acknowledged at the entrance of channel there is a disturbed reach where the hypothesis of flow without curvature is not valid, and which makes it difficult to locate the origin of the development of boundary layer. However, as it is always possible to start the calculation of the boundary layer growth at a point located some distance downstream from the entrance, where the displacement and momentum thicknesses can eventually be determined by the experiment, so it is of practical convenience to assume the origin at the point where the boundary layer will be developed.

Hence, at $x=0$ and $\delta_*=0$, the velocity in the main flow is given by the solution of

$$E=\frac{u_0^2}{2g}+\frac{q\cos\theta}{u_0},$$

for given specific head E and discharge q.

The above equation has two physically possible roots: tranquil and rapid. The critical velocity is $u_0=\sqrt[3]{gq\cos\theta}$ and specific energy is $E=(3/2)H_c\cos\theta$, so consequently, for the sake of simplicity of analysis as Craya and Delleur did, the following dimensionless forms are introduced.

$$\begin{aligned}&\frac{h}{H_c}=\pi, \quad \frac{x}{H_c}=\xi, \quad \frac{\delta}{H_c}=i, \quad \frac{\delta_*}{H_c}=\eta, \\ &\frac{u_0}{\sqrt{gH_c\cos\theta}}=r, \quad \frac{q}{H_c\sqrt{gH_c\cos\theta}}=\sigma, \quad \frac{\nu}{H_c\sqrt{gH_c\cos\theta}}=\nu',\end{aligned} \tag{31}$$

Hence, the basic relations of boundary layer growth are dimensionlessly expressed as follows.

$$\frac{3}{2}=\frac{r^2}{2}+\pi-\xi\tan\theta, \tag{32}$$

$$\frac{C_f}{2}=\frac{d}{d\xi}\left(\frac{\eta}{H}\right)+\frac{2+H}{H}\frac{\eta}{r}\frac{dr}{d\xi}, \tag{33}$$

$$\sigma=r(\pi-\eta). \tag{34}$$

Eliminating π from Eqs. (32) and (34) yields

$$\frac{3}{2}+\xi\tan\theta=\frac{r^2}{2}+\frac{\sigma}{r}+\eta. \tag{35}$$

Differentiating Eq. (35) with respect to ξ, it leads to

$$d\xi\cdot\tan\theta=\left(r-\frac{\sigma}{r^2}\right)dr+d\eta. \tag{36}$$

From Eqs. (33) and (36), finally

$$\left(\frac{1}{H}-\frac{C_f}{2\tan\theta}\right)\frac{d\eta}{dr}+\left(\frac{2+H}{H}\frac{1}{r}-\frac{1}{H^2}\frac{dH}{dr}\right)\eta=\frac{C_f}{2\tan\theta}\left(r-\frac{\sigma}{r^2}\right). \tag{37}$$

If the skin friction coefficient and H are assumed constant throughout the whole region under investigation, Eq. (37) becomes linear and the solution is readily obtainable as a function of velocity of main flow. The boundary condition for solving equation is dimensionlessly expressed as follows.

$$2\sigma=3r-r^3. \tag{38}$$

(1) *Growth of laminar boundary layer*

Assuming the velocity profile in laminar layer is given by Eq. (17) indicated in steady uniform regime, the basic equation becomes from Eq. (37)

$$\frac{d\eta}{dr}=\left\{27\eta^2+\frac{10\nu'}{\tan\theta}\left(\frac{\sigma}{r^2}-r\right)\right\}\Big/\left(\frac{10\nu'}{\tan\theta}-6\eta r\right). \tag{39}$$

Evidently, Eq. (39) is non-linear and the solution has to be obtained by numerical integration. However, at the point where is satisfied the following relation of

$$\eta r=\frac{5}{3}\nu'\cot\theta, \tag{40}$$

$d\eta/dr$ becomes infinite and thus it is understood that Eq. (39) is not adequate for laminar layer at this point. However, this point is very near the origin of development, and nelgecting this unreliability of basic equation and considering the flow concerned is a flow over a relatively steep chute, Eq. (36) is approximated by

$$d\xi\cdot\tan\theta=\left(r-\frac{\sigma}{r^2}\right)dr. \tag{41}$$

For case of initially supercritical flow, Eq. (39) also becomes singular in the topological sense and as the singular point is proved a saddle point, so it is seen that there are two curves of integration passed the saddle point.

With the use of Eq. (41), the basic equation is linearized as follows.

$$\frac{d\eta^2}{dr}+\frac{9}{r}\eta^2=\frac{10\nu'}{3\tan\theta}\left(1-\frac{\sigma}{r^3}\right). \tag{42}$$

In Eq. (42), it means the singular point expressed in Eq. (40) is transformed to the origin of the coordinate system. Under a given boundary condition of Eq. (38), the solution is

$$\frac{3\tan\theta}{\nu'}\eta^2=r\left\{1-\left(\frac{r_0}{r}\right)^{10}\right\}-\frac{10\sigma}{7}\frac{1}{r^2}\left\{1-\left(\frac{r_0}{r}\right)^7\right\}. \tag{43}$$

The boundary layer thickness, flow depth and location are also calculated.

Halbronn integrated the laminar boundary layer equation, neglecting the slope of free surface and obtained

$$\left(\frac{g\sin\theta}{3\nu}\delta^2-u_0\right)\Big/\left(\frac{g\sin\theta}{3\nu}\delta_i{}^2-u_{0i}\right)=\left(\frac{u_{0i}}{u_0}\right)^9. \tag{44}$$

Putting $\delta_i=0$ and $u_{0i}=u_{00}$ for comparison with the result of Eq. (43), the dimensionless form of Eq. (44) becomes

$$\frac{3\tan\theta}{\nu'}\eta^2=r\left\{1-\left(\frac{r_0}{r}\right)^{10}\right\}. \tag{45}$$

Therefore, it follows that Eq. (45) obtained by Halbronn is more approximate, compared with the result of Eq. (43).

In accelerative flows, as the thickness of boundary layer spreads farther and farther, so $(r_0/r)^{10}$ and $(r_0/r)^7$ decrease rapidly and the approximate behavior of Eq. (43) is described in a manner of

$$\frac{3\tan\theta}{\nu'}\eta^2 = r - \frac{10\sigma}{7}\frac{1}{r^2}. \tag{46}$$

(2) *Turbulent boundary layer growth in terms of power law*

As seen in the previous section, the skin friction is assumed to be proportional to the m th power of Reynolds numbers, so the basic equation for turbulent boundary layer growth in terms of power law becomes

$$\lambda\left(\frac{r\eta}{\nu'}\right)^m = \frac{d}{d\xi}\left(\frac{\eta}{H}\right) + \frac{2+H}{H}\frac{\eta}{r}\frac{dr}{d\xi}. \tag{47}$$

Hence, eliminating ξ from Eqs. (36) and (47) yields the basic relation of turbulent layer of open channel flows.

$$\frac{d\eta}{dr} = \frac{\dfrac{\lambda}{\tan\theta}\dfrac{\eta^m r^m}{\nu'^m}\left(r - \dfrac{\sigma}{r^2}\right) - \dfrac{2+H}{H}\dfrac{\eta}{r} + \dfrac{\eta}{H^2}\dfrac{dH}{dr}}{\dfrac{r}{H} - \dfrac{\lambda}{\tan\theta}\dfrac{\eta^m r^m}{\nu'^m}}. \tag{48}$$

Eq. (48) has the same unreliability, and therefore, neglecting such a fact and transforming the singular point to the origin, the solution will be readily obtained by numerical integration as an approximate behavior of turbulent layer growth. However as H is not constant, so an additional function for H is required to calculate. Neverthless, it will be understood from Figs. 4 or 5 that the skin friction coefficient is assumed to be proportional to the m th power of Reynolds numbers. Therefore, H is assumed constant in the contrast with the experimental results. It will be acknowledged from the rapid increase of H with the increase of displacement thickness. Under such an approximate condition, Eq. (48) becomes the following linear equation

$$\frac{d\eta^{1-m}}{dr} + \frac{(2+H)(1-m)}{r}\eta^{1-m} = \frac{\lambda H(1-m)}{\nu'^m\tan\theta}r^m\left(r - \frac{\sigma}{r^2}\right). \tag{49}$$

Hence, the solution is

$$\frac{\tan\theta\,\nu'^m}{\lambda H}\eta^{1-m} = \frac{1-m}{3+(1-m)(1+H)}r^{2+m}\left\{1-\left(\frac{r_0}{r}\right)^{3+(1-m)(1+H)}\right\} - \frac{\sigma}{1+H}r^{-1+m}\left\{1-\left(\frac{r_0}{r}\right)^{(1-m)(1+H)}\right\}. \tag{50}$$

As it is so difficult to determine the location of the origin of boundary layer growth by experiments, so preferable is that the boundary condition is $\eta=\eta_0$ at $r=r_0$. Therefore, the solution becomes, considering the specific energy is measured from the virtual channel bed stands at the level of displacement thickness.

$$\frac{\tan\theta\nu'^m}{H\lambda}\eta^{1-m}\left\{1-\left(\frac{\eta_0}{\eta}\right)^{1-m}\left(\frac{r_0}{r}\right)^{(1-m)(2+H)}\right\}=\frac{1-m}{3+(1-m)(1+H)}r^{2+m}\left\{1-\left(\frac{r_0}{r}\right)^{3+(1-m)(1+H)}\right\}$$
$$-\frac{\sigma}{1+H}r^{-1+m}\left\{1-\left(\frac{r_0}{r}\right)^{(1-m)(1+H)}\right\}. \quad (51)$$

In the same manner as in laminar layer, the boundary layer thickness, flow depth and location are also calculated. At the downstream region, the approximate behavior is described by

$$\frac{\tan\theta\nu'^m}{\lambda H}\eta^{1-m}=\frac{1-m}{3+(1-m)(1+H)}r^{2+m}-\frac{\sigma}{1+H}r^{-1+m}. \quad (52)$$

As the above procedure is applied until the boundary layer reaches the free surface, so the critical point of the theory will be concerned. It is known this point is the location of initial air-entrainment. Denoting values at the critical point by subscript c, the critical condition is

$$\pi_c=i_c. \quad (53)$$

Hence, from Eq. (34),

$$r_c\eta_c=\frac{H-1}{2}\sigma. \quad (54)$$

Therefore, the velocity at the critical point is

$$\frac{\tan\theta\nu'^m}{\lambda H}\frac{(H-1)^{1-m}\sigma^{1-m}}{2^{1-m}}=\frac{1-m}{3+(1-m)(1+H)}r_c^3\left\{1-\left(\frac{r_0}{r_c}\right)^{3+(1-m)(1+H)}\right\}$$
$$-\frac{\sigma}{1+H}\left\{1-\left(\frac{r_0}{r_c}\right)^{(1-m)(1+H)}\right\}. \quad (55)$$

Also, in accelerative flows the approximate critical velocity is

$$\frac{1-m}{3+(1-m)(1+H)}r_c^3=\frac{\tan\theta\nu'^m}{\lambda H}\frac{(H-1)^{1-m}\sigma^{1-m}}{2^{1-m}}+\frac{\sigma}{1+H}, \quad (56)$$

and the critical depth and location are

$$\pi_c=i_c=\frac{H+1}{2}\frac{\sigma}{r_c}, \quad (57)$$

$$\xi_c\tan\theta=\frac{H+1}{2}\frac{\sigma}{r_c}+\frac{r_c^2-3}{2}. \quad (58)$$

(3) *Turbulent boundary layer growth in terms of logarithmic law*

It is of practical convenience that another variable of p expressed by $\kappa u_0/u_*$ as von Kármán and Halbronn did is applied to the boundary layer equation, since the velocity in the main flow is commonly associated with the frictional velocity as seen in the Prandtl expression. Therefore, the displacement thickness is

$$\eta=\frac{C_1\nu'}{\kappa}\frac{e^p}{r}, \quad (59)$$

where $C_1=e^{-A_s}$ and for steady uniform flows in pipes $A_s=5.5$. Tranforming Eq. (33) with the use of Eq. (22) yields

$$\kappa^2 d\xi = p(p-2)d\eta + 2\eta dp + p(3p-4)\frac{\eta}{r}dr\,. \tag{60}$$

Eliminating ξ and η with the aid of Eqs. (36) and (59), the relation between the dimensionless velocity and parameter p becomes

$$\frac{dr}{dp}=\frac{\frac{\kappa^2}{\tan\theta}-p^2+2p-2}{\left\{\frac{\kappa^2}{\tan\theta}+2p(p-1)\right\}\frac{1}{r}-\frac{\kappa^3}{C_1\nu'\tan\theta}\left(r^2-\frac{\sigma}{r}\right)e^{-p}}\,. \tag{61}$$

Eq. (61) is the basic equation of turbulent boundary layer in terms of logarithmic form, and the displacement thickness, flow depth and location are calculated.

Before starting the integration by numerical method, the boundary condition for this case has to be considered. Taking the origin of coordinate system at the point where the boundary layer starts, it is intuitively assumed that the friction acts on the fluid at a very large amount and therefore, p becomes very small and substantially zero. It proves a contradiction of boundary condition at the origin, since η is not zero when p is zero. It is due to an essential character of logarithmic law which is applied to the upper zone from laminar sublayer. However, as the magnitude of ν' is quite small, 10^{-4} to 10^{-6}, compared with other quantities, it is practically assumed that $\eta=0$ at the origin of growth. Evidently in calculation, the initial value of p has less effect on the boundary layer growth in numerical integration, though it is not certain that $p=0$ at the origin or not.

The approximate equation neglected the displacement thickness becomes also

$$\frac{dr}{dp}=\frac{re^p(p^2-2p+2)}{\frac{\kappa^3}{C_1\nu'\tan\theta}(r^3-\sigma)-2e^p p(p-1)}\,. \tag{62}$$

However, for all cases of the logarithmic law, the mathematical behavior is obtained only by numerical integration.

Hydraulic Significance of Boundary Layer Growth and its Contribution to the Practical Application to Design Problem

In this section, hydraulic significance of the boundary layer growth of open channel flows is concerned in the light of mathematical analysis and experimental verification mainly conducted at the Hydraulics Laboratory, Kyoto University. Furthermore, its contribution to the practical application to engineering purpose of channel design in terms of interpretation made possible by the present analysis will be accented.

(1) *Growth of boundary layer*

The turbulent boundary layer of open channel flows is of more importance from the practical point of view and it has been associated with the air-entrainment by turbulence in the high velocity flow. The mathematical behavior of turbulent boundary layer growth is described as the solution of Eq. (48) for flows of the power type and Eq. (61) for flows of logarithmic form. Until the present day, however, as seen in many publications, most solutions are associated with the well known solution of unconfined flows with zero pressure gradient, that is, for the Blasius 7th law,

$$\frac{\delta}{x}=0.371\,R_x^{-\frac{1}{5}}, \qquad (63)$$

where R_x is the Reynolds number in terms of distance. V. Michels and M. Lovely[13] obtained the empirical relation in a similar fashion of Eq. (63) to compare with the field observation of air-entrained flow at the Glenmaggie Dam and the Werribee Diversion Weir. Fig. 10 describes some examples of the boundary layer growth of open channel flows and theoretical curves calculated by various approaches. The curve for logarithmic law is calculated under the assumption of $\kappa=0.4$ and $A_s=5.5$ valid for pipe flows. For the Blasius 7th law, the curve becomes

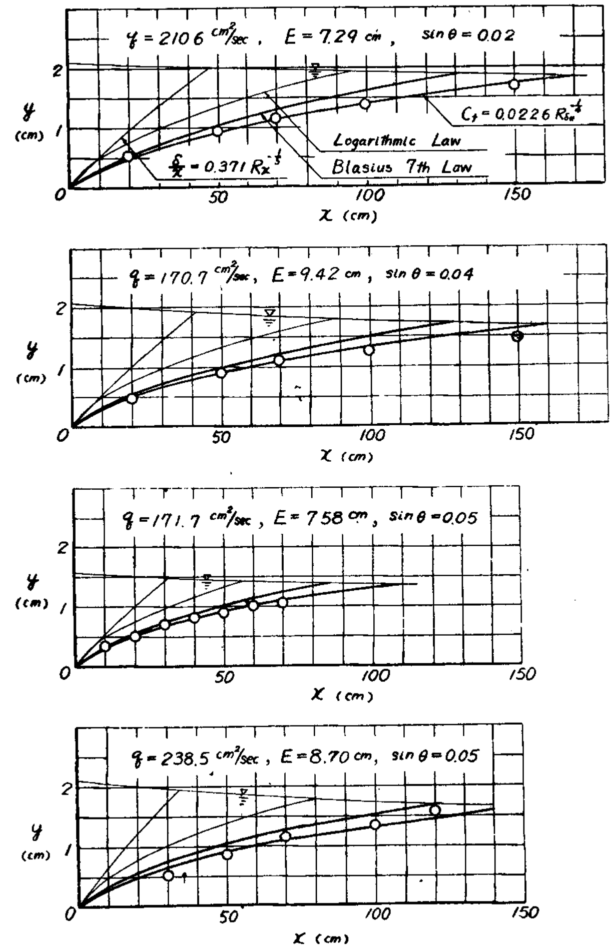

Fig. 10. Examples of boundary layer growth of open channel flows.

$$\frac{\tan\theta}{\nu^{\frac{1}{4}}}\eta^{\frac{5}{4}}=0.00367\,r^{\frac{7}{4}}\left\{1-\left(\frac{r_0}{r}\right)^{\frac{41}{7}}\right\}-0.00753\,\sigma r^{-\frac{5}{4}}\left\{1-\left(\frac{r_0}{r}\right)^{\frac{20}{7}}\right\}, \qquad (64)$$

with the use of $m=-0.25$, $H=(9/7)$ and $\lambda=0.0225$ for the thickness of boundary layer. Applying the empirical formula of Eq. (28) to Eq. (48) for comparison with experimental data, the equation of boundary layer growth is slightly changed to

$$\frac{\tan\theta}{\nu'^{\frac{1}{4}}}\eta^{\frac{5}{4}} = 0.00309\, r^{\frac{7}{4}}\left\{1-\left(\frac{r_0}{r}\right)^{\frac{41}{7}}\right\} - 0.00633\,\sigma r^{-\frac{5}{4}}\left\{1-\left(\frac{r_0}{r}\right)^{\frac{20}{7}}\right\}. \tag{65}$$

It is seen that the behavior of boundary growth described by Eq. (63) is so rapid and the 7th law or similar formula express the closest approximation compared with the experimental data. A. T. Ippen, R. S. Tankin and F. Raichlen[14] conducted the experiment of turbulent boundary layer growth during the measurement of flow turbulence and proved that the growth is rather mild compared with the result of Eq. (63). The development indicated by Bauer is milder than the results described in the figure owing to his assumption that the thickness of boundary layer is given by the amount of 99% of maximum velocity.

(2) *Critical point*

The boundary layer theory will be applied until the thickness of boundary layer spreads farther to the free surface of open channel flow. Denoting this point the critical point, the location of critical point has been studied as the initiation of air-entrainment into flows by turbulence since Lane. It is readily calculated with the use of previously equations. Fig. 11 describes the location of critical point as a parametric expression of discharge with the aid of Eq. (55).

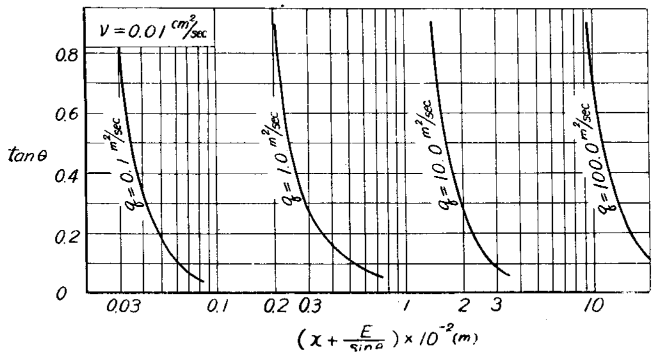

Fig. 11. Location of critical point for various slopes and discharge.

(3) *Influence of hydraulic and channel characteristics on the boundary layer growth*

It will be predicted that the hydraulic characteristics like discharge and channel characteristics like slope and roughness have a significant effect on the growth of boundary layer. Fig. 12 indicates the boundary layer thickness as a function of distance in the direction of flow, resulted from experimental data of Ippen, Tankin and Raichlen, Bauer, and the author for smooth boundaries. Although data plotted are scattering, in inspection, it may be seen that points corresponding to the same boundaries, but with much different rate of discharge and slope, fall along the same line and it follows that the hydraulic characteristics have less significant effect, as Bauer concluded. However, as seen in Fig. 11, the effect of the hydraulic characteristics on boundary layer growth will be readily recognized. That is, at the same slope the location of critical point becomes farther with the increase of discharge and same

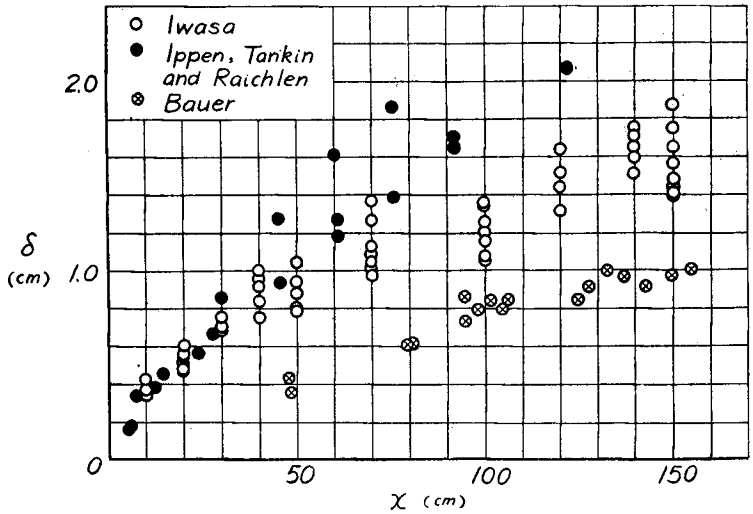

Fig. 12. Relation between boundary layer growth and distance.

conclusion for variable slopes is evident for constant discharge. It seems the interrelationship between main and boundary layer flows has an influence on the boundary layer growth of open channel flows to a considerable degree.

It is also understood from the dynamical and mathematical principles of boundary layer that the growth of boundary layer is feasibly influenced by the velocity profile and especially skin friction law plays a significant effect on the growth as seen in the later section.

(4) *X-wise distribution of shear and Manning roughness*

The theory of boundary layer is concerned with not only the application to practical problems of air-entrained flow but also the transitional behavior from an uniformly distributed flow near an entrance to fully developed turbulent flow, so it will be also considered one of the fundamental subjects contributable to the design of channel in moden hydraulics. Up to the present day, the channel was designed to make the safe-pass of design discharge possible under the assumption of fully developed turbulent flow ; with the use of given slope and channel alignment determined by the topographical and geological circumstances, the momentum or energy approaches of one-dimensional analysis was to be solved for the design of open channels. The basic concept previously mentioned is considered an extention of the theory established in steady uniform regime. One of the purposes of the present study is to make the dynamical principle and hydraulic behavior of steady but non-uniform flow in open channels clear, by means of the boundary layer theory, and applicability of the usual concept in classical hydraulics to channel design should be concerned. Fig. 13 describes the x-wise distribution of frictional velocity resulted from the author's experiment as a function of distance from the entrance of channel and a curve shows the theoretical one calculated with the boundary layer theory. It is seen that there is a close agreement between the experimental results and the theoretical curve, though numbers of plottings are so few.

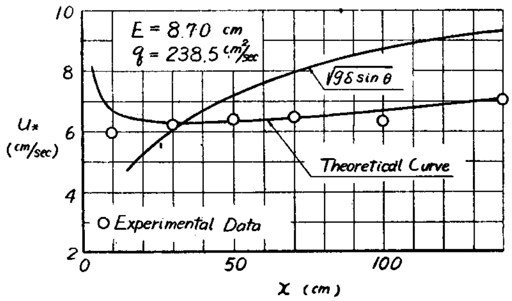

Fig. 13. Frictional velocity in turbulent boundary layer.

Fig. 14 indicates the distribution of Manning roughness as a function of distance. At the earlier stage of boundary layer growth, values of roughness is very large and they tend rapidly to the constant value which is equal to that in steady uniform regime. Therefore, it is concluded that the Manning roughness is determined only by the channel characteristics consists of the bed and side wall of channel and for the practical application to problems of channel design the usual approach is adequate as approximation.

(5) *Surface turbulence associated with boundary layer growth*

As the high velocity flow over a very steep chute has the entrainment of air and forms an air-water mixture, it is virtually observed that the discharge of high velocity flow bulks to some extents and for the practical design of spillway the virtual increase of discharge is frequently considered. Although there is still unsolved for the clear formulation of dynamics of air-entrainment, as suggested by Lane, the necessary condition to carry the air bubbles into flows is assumed that the thickness of boundary layer will be reached the free surface of flow. The above concept is almost verified by many investigators. Evidently, it is so difficult to produce the self air-entrainment

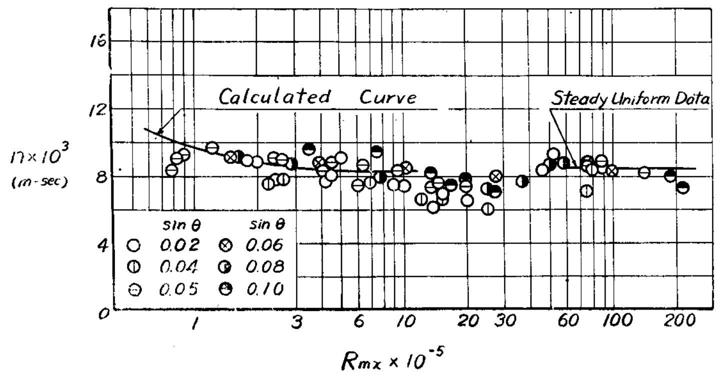

Fig. 14. Manning roughness of a very smooth channel.

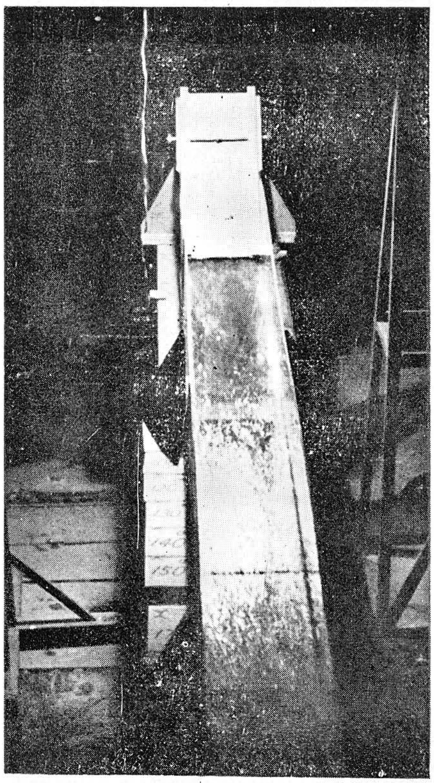

Photo. 1. Flow near critical point.

in experimental flumes of the laboratory. However, it is of common observation that the flow has a sudden transition from the flow with a smooth plane surface to the flow with very small shuggy and also the color of free surface suddenly changes from transparent to opaque as seen in Photo. 1. However, the final establishment of formulation of air-entrained flow still stands far away.

(6) *Contribution of critical point to channel design*

So far as the application to the design of steep chutes like a spillway, the problem requires the location of the critical point for given discharge and head on a given chute, to determine the location of initiation of bulked discharge. It has been recognized that the location was determined by calculation of previous relations. Fig. 15 indicates the locus of location at which the flow becomes critical under the assumption of $E=(q/2)^{2/3}$ for a very

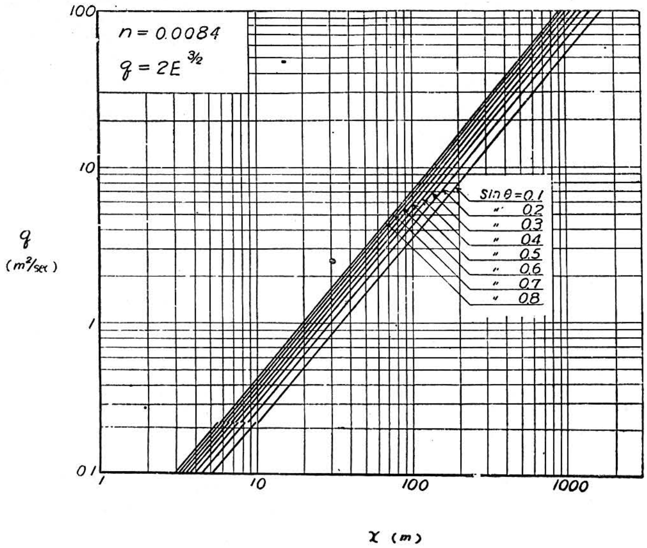

Fig. 15. Location of critical point for a very smooth channel.

smooth channel like a lucite. Evidently, it is seen that the location is so far compared with the actual measurements by many investigators. It is due to the so small value of roughness. Fig. 16 also indicates the location of critical point for a concrete channel under the assumption of Manning roughness 0.013. It is readily understood that the theoretical results obtained in the present study indicates an outstanding agreement with data of field observation by G. H. Hickox[15], and Michels and Lovely[13].

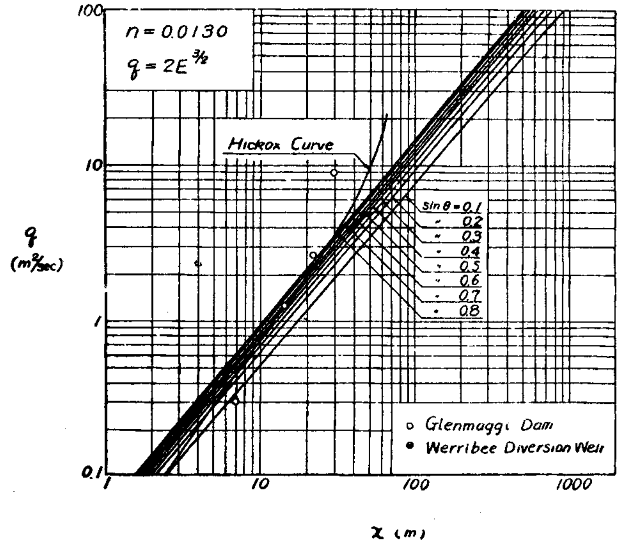

Fig. 16. Location of critical point for a concrete channel.

Conclusion

Although the growth of boundary layer of open channel flows and its associated problem have not yet been established by the present study, the following conclusion seems justified on the basis of the foregoing results.

1. The hydraulic behavior of boundary layer of open channel flows has to be concerned by the combination of three bacic relations of main flow, boundary layer and discharge together and the interrelationship among these equations is the same equation commonly used in classical hydraulics.

2. It seems that for a smooth boundary the shape of velocity profile of turbulent layer is better approximated by the power type of law than the logarithmic law as Bauer concluded. However, it requires the further study concerned the details of velocity profile.

3. The hydraulic behavior of boundary layer growth is closely approximated by Eq. (65) used the 7th type of power law within the limit of experimentation conducted.

4. It is readily understood from observations that the flow pattern and the free surface color change suddenly at the critical point where the boundary layer reaches the free surface. However, it stands far away that the establishment of mechanical principle of air-entrained flow will be obtained.

5. For the practical application to problems of channel design commonly used procedure is also available in the light of knowlegde of boundary layer theory developed in here.

The initial purpose expected for the study of boundary layer growth in open channel flows has not yet reached the final goal contributable to the practical application to engineering problem. Neverthless, it seems that a rational procedure of approach for studying the problem of boundary layer has been formed in the light of possible mathematical form. However, as Craya and Delleur did, there still remains unsolved for the mathematical interest of behavior of the boundary layer equation near the critical regime and the application problem for hydraulics of mild slope. It is hoped that the clear formulation of behaviors of boundary layer of open channel flows will be attained by the useful process of mathematics and systematic experimentation, with the development of further study for a rough bed now in progress. Furthermore, much of the true nature of fluid motion still unsolved in the present hydraulics will be soundly clearified through many physical knowledge obtained by the boundary layer theory.

Acknowledgments

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