On the Electric Field due to Tides

By

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Abstract

It has been pointed out that one of the main causes of faults on submarine cables would be the electrolytic corrosion due to the current which is induced in the cable by tides moving across the earth's magnetic field.

In this paper, some formulas for electric field due to a stream of water which flows in a tide-way of rectangular cross-section are derived, and results of numerical computations are shown. The effects of conductivity of the bottom are also treated as boundary value problems. It is found that the sides of tide-way are the most dangerous places as the intense electrolytic corrosion of cable occurs there.

1. Introduction

It was already predicted by Michael Faraday in 1832 that an e.m.f. would be produced in a volume of water moving across the earth's magnetic field[1]. During the year of 1918, Young, Gerrard and Jevons[2] carried out a series of sea experiments at Dartmouth Harbour, England, and observed electric disturbances which were definitely traced to movements of sea-water cutting the vertical component of the earth's magnetic field.

On the other hand, it has been found by one of the authors of this paper that there exists an electric current in submarine cable sheath, which varies according to the velocity and direction of the tidal flow. He pointed out in a previous publication[3] that this current would cause electrolytic corrosion of the sheath, and he estimated that it would be one of the most important causes of cable faults. His experiments and observations have shown that the electrolytic corrosion of the cable sheath due to the electric current produced by tides in the earth's magnetic field sometimes exceeds the mechanical erosion which has been believed to be the main cause of cable faults for many years.

Although the phenomenon has been known since the 19th century, theoretical studies on the subject have not yet been carried out, as little attention has been payed to its practical meaning. It is important to know the distributions of the electric fields and the currents in the sea-water, since the corrosion of the cable sheath will occur at a place where a large amount of current flows out from it. On the other hand, the knowledge of the field distribution will make it possible to apply this phenomenon on oceanographic problems such as the measurement of the ocean current by an electrical aid[5].

In this paper, the authors will analytically treat the problem of the electric field due to tides, and give some numerical results of the computations, which would be of practical interest.

2. General Consideration

In Fig. 1, let the space $y>0$ be a semi-infinite conductor of conductivity σ, and the domain (A) be flowing in the direction of the z-axis. If we assume that the magnetic field H is vertical and uniform, an e.m.f. is induced in the flowing medium (A) having the direction of the x-axis and its magnitude E_0 per unit distance is given by the relation

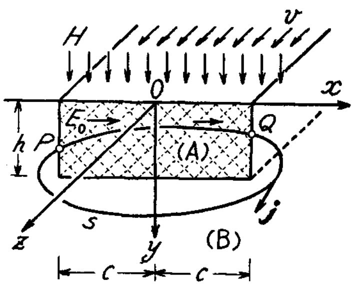

Fig. 1

$$E_0 = \mu_0 H v\,, \tag{1}$$

where μ_0 is the permeability of the medium, and v is the velocity of the domain (A). If the magnetic field H is taken as 32 AT/m, or $\mu_0 H$ as 4×10^{-5} weber/m^2=0.4 gauss, then

$$E_0 \cong 20 v_N \text{ mV/km}\,, \tag{1'}$$

where v_N is the velocity in knot.

By this e.m.f. an electric current circulating in both domains (A) and (B) will be produced. If we assume a closed curve s as shown in Fig. 1, and apply Ohm's law to this circuit, then we have

$$\oint \frac{1}{\sigma} \boldsymbol{j}\cdot d\boldsymbol{s} = \int_P^Q E_0 dx\,, \tag{2}$$

where $\boldsymbol{j}$ is the current density at any point. By means of the relation between the current density $\boldsymbol{j}$ and the electric field $\boldsymbol{E}$:

$$\boldsymbol{j} = \sigma \boldsymbol{E}\,, \tag{3}$$

Eq. (2) can be written as follows:

$$\oint \boldsymbol{E}\cdot d\boldsymbol{s} = \int_P^Q E_0 dx\,. \tag{4}$$

It is evident from this relation that $\boldsymbol{E}$ cannot be derived from a scalar potential, because the integral of the right hand side of Eq. (4) does not vanish.

If we put

$$\boldsymbol{E} = \boldsymbol{E}' + \boldsymbol{E}'', \tag{5}$$

and assume that the partial field $\boldsymbol{E}'$ satisfies the relation:

$$\operatorname{curl} \boldsymbol{E}' = 0, \tag{6}$$

then $\boldsymbol{E}'$ must be derived from a scalar potential V' by the formula:

$$\boldsymbol{E}' = -\operatorname{grad} V', \tag{7}$$

and should satisfy the equation:

$$\oint \boldsymbol{E}' \cdot d\boldsymbol{s} = 0. \tag{8}$$

Substituting Eqs. (5) and (8) into Eq. (4), we get

$$\oint \boldsymbol{E}'' \cdot d\boldsymbol{s} = \int_P^Q E_0 dx. \tag{9}$$

On the other hand, since the stationary medium (B) has not any distributed e.m.f. in it, then

$$\operatorname{curl} \boldsymbol{E} = 0 \qquad \text{in (B)}, \tag{10}$$

and by Eqs. (5) and (6) we get

$$\boldsymbol{E}'' = 0 \qquad \text{in (B)}. \tag{11}$$

Writing

$$\boldsymbol{E} = \boldsymbol{E}_A, \ \boldsymbol{E}' = \boldsymbol{E}'_A, \ \boldsymbol{E}'' = \boldsymbol{E}''_A \qquad \text{in (A)},$$

and

$$\boldsymbol{E} = \boldsymbol{E}_B, \ \boldsymbol{E}' = \boldsymbol{E}'_B, \ \boldsymbol{E}'' = \boldsymbol{E}''_B \qquad \text{in (B)},$$

and substituting Eq. (11) into Eq. (9), we find the relation:

$$\int_P^Q \boldsymbol{E}_A'' \cdot d\boldsymbol{s} = \int_P^Q E_0 dx, \tag{12}$$

which states that the vector $\boldsymbol{E}_A''$ is horizontal and its magnitude E_A'' is equal to the distributed e.m.f. per unit distance:

$$E_A'' = E_0. \tag{13}$$

This is a very simple but important relation in the analysis of the problem.

In general, the vector $\boldsymbol{j}$ must satisfy the relations:

$$\operatorname{div} \boldsymbol{j} = 0, \quad \text{or} \quad \operatorname{div} \sigma \boldsymbol{E} = 0, \tag{14}$$

so that the fields $\boldsymbol{E}_A$ and $\boldsymbol{E}_B$ would be continuous on the both sides of the domain (A) in case the medium is uniform (σ=const):

$$\boldsymbol{E}_A = \boldsymbol{E}_B \quad \text{at} \quad x = \pm c, \ 0 < y < h, \tag{15}$$

On the bottom of (A), however, the horizontal components E_{Ax} and E_{Bx} are not necessarily continuous, even if the medium is homogeneous, because $\boldsymbol{E}_B''=0$ at every point under the bottom of (A), whereas $E_A''=E_0$ and E_0 is not necessarily zero on the bottom of (A). The necessary conditions to be satisfied on the bottom are the continuity of the horizontal components of the partial field $\boldsymbol{E}'$ and that of the vertical components of the current density $\sigma\boldsymbol{E}$:

$$E'_{Ax}=E'_{Bx}\,,\ \sigma_A E_{Ay}=\sigma_B E_{By} \quad \text{at} \quad y=h,\ -c<x<+c, \tag{16}$$

where σ_A and σ_B are the conductivities of the medium on the upper and lower sides of the plane $y=h$, $-c<x<+c$ respectively. If the medium is homogeneous, it can be written as follows:

$$E'_{Ax}=E_{Bx},\ E_{Ay}=E_{By} \quad \text{at} \quad y=h,\ -c<x<+c. \tag{16'}$$

3. Case I: Uniform Velocity.

3.1. Homogeneous Medium.

At first let us consider the electric field in an uniform semi-infinite medium of conductivity σ. Now assuming that the medium in the domain (A) of Fig. 2 is flowing in the z-direction with a constant velocity v_0, then the distributed e.m.f. E_0 per unit distance in the region of (A) will be uniform and its magnitude is given by Eq. (1).

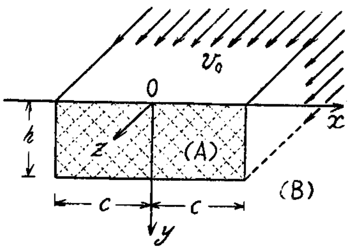

Fig. 2

The scalar potential V' from which the partial electric field $\boldsymbol{E}'$ is derived by Eq. (7) must be a solution of Laplace's equation:

$$\nabla^2 V'=0, \tag{17}$$

and x- and y-components of the total field $\boldsymbol{E}$ must satisfy the conditions:

$$E_{Ax}=E_{Bx}\,,\ E_{Ay}=E_{By} \quad \text{at} \quad x=\pm c,\ 0<y<h, \tag{18}$$

and

$$E_{Ax}-E_0=E_{Bx},\ E_{Ay}=E_{By} \quad \text{at} \quad y=h,\ -c<x<+c, \tag{19}$$

which are given by Eqs. (15) and (16'). These relations can be rewritten as follows:

$$E'_{Ax}+E_0=E'_{Bx},\ E'_{Ay}=E'_{By} \quad \text{at} \quad x=\pm c,\ 0<y<h; \tag{18'}$$

and

$$E'_{Ax}=E'_{Bx},\ E'_{Ay}=E'_{By} \quad \text{at} \quad y=h,\ -c<x<+c, \tag{19'}$$

using the relations:

$$\left.\begin{aligned} E_{Ax}&=E'_{Ax}+E''_{Ax},\ E''_{Ax}=E_0\,, \\ E_{Ay}&=E'_{Ay}+E''_{Ay},\ E''_{Ay}=0\,, \end{aligned}\right\}$$

and

$$\left.\begin{aligned} E_{Bx} &= E'_{Bx}+E''_{Bx},\ E''_{Bx}=0\,, \\ E_{By} &= E'_{By}+E''_{yB},\ E''_{By}=0\,, \end{aligned}\right\}$$

which are given by Eqs. (13) and (11).

If we assume a surface distribution of current source, the density of which is equal to σE_0, on the plane $x=c$ $(0<y<h)$, and another current source of density $-\sigma E_0$ on the other plane $x=-c$ $(0<y<h)$, then the potential due to these plane sources is given by

$$\begin{aligned} V' = -\frac{E_0}{4\pi}[&H(x-c,\ h-y)+H(x-c,\ h+y) \\ &-H(x+c,\ h-y)-H(x+c,\ h+y)]\,, \end{aligned} \tag{20}$$

where

$$H(\xi,\ \eta)=\eta\log(\xi^2+\eta^2)+2\xi\tan^{-1}(\eta/\xi)\,. \tag{21}$$

It can be easily shown that the potential V' given by Eq. (20) satisfies the necessary conditions mentioned above. And the partial fields are derived as follows:

$$\left.\begin{matrix} E'_{Ax} \\ E'_{Bx} \end{matrix}\right\} = -\frac{\partial V'}{\partial x}=\frac{E_0}{2\pi}\left[\tan^{-1}\frac{h-y}{x-c}+\tan^{-1}\frac{h+y}{x-c}-\tan^{-1}\frac{h-y}{x+c}-\tan^{-1}\frac{h+y}{x+c}\right], \tag{22}$$

$$\left.\begin{matrix} E'_{Ay} \\ E'_{By} \end{matrix}\right\} = -\frac{\partial V'}{\partial y}=\frac{E_0}{4\pi}\left[\log\frac{(x-c)^2+(h+y)^2}{(x-c)^2+(h-y)^2}-\log\frac{(x+c)^2+(h+y)^2}{(x+c)^2+(h-y)^2}\right]. \tag{23}$$

The components of the total field are calculated from

$$E_{Ax}=E_0+E'_{Ax},\quad E_{Ay}=E'_{Ay}\quad \text{in (A)}, \tag{24}$$

and

$$E_{Bx}=E'_{Bx},\quad E_{By}=E'_{By}\quad \text{in (B)}. \tag{25}$$

Figs. 3 and 4 show the distributions of the electric field along the horizontal lines at various depths. It is found from

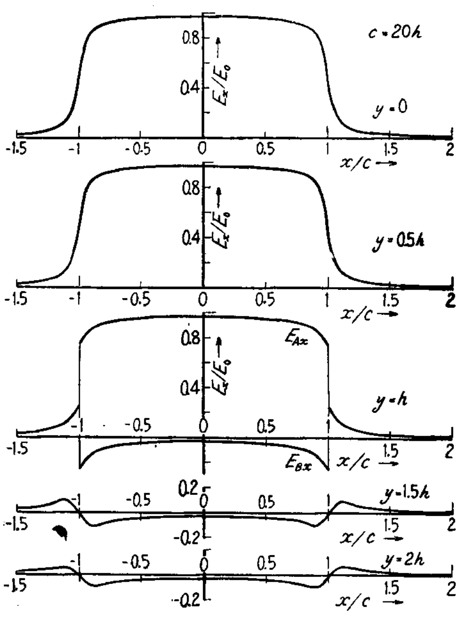

Fig. 3

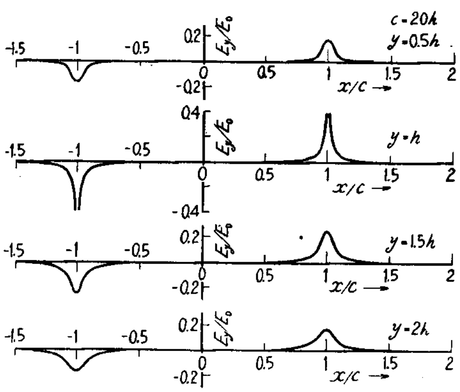

Fig. 4

these figures, that the electric field in the domain (A) is almost equal to the distributed e.m.f. E_0 per unit distance, so that

$$E_x/E_0 \cong 1, \quad E_y \cong 0 \quad \text{in (A)}; \ |x|<c, \ 0<y<h.$$

However, the horizontal component E_x varies discontinuously at the points $y=h$, $x=\pm c$, and the vertical component E_y becomes infinite at the same points. It should be important to note that the horizontal component E_{Bx} $(y \geq h)$ in (B) changes its sign at the points $x \simeq \pm c$.

It will be very interesting and useful to plot a set of current lines in the medium. For this purpose, we use the current flux J which flows across a vertical plane $x=x_1=$const., between $y=0$ and $y=y_1$. This can be calculated as follow:

$$\begin{aligned} J=\sigma\int_0^{y_1} E_{Ax}dy = \frac{\sigma E_0}{4\pi}[&4\pi y_1 - G(c-x_1, \ h-y_1)+G(c-x_1, \ h+y_1) \\ &-G(c+x_1, \ h-y_1)+G(c+x_1, \ h+y_1)], \\ &-c<x_1<+c, \ 0<y_1<h; \end{aligned} \tag{26a}$$

$$\begin{aligned} J=\sigma\int_0^{h} E_{Ax}\,dy+\sigma\int_h^{y_1} E_{Bx}\,dy &= \frac{\sigma E_0}{4\pi}[4\pi h - G(c-x_1, \ y_1-h) \\ +G(c-x_1, \ y_1+h)-G(c+x_1, \ y_1-h)&+G(c+x_1, \ y_1+h)], \\ &-c<x<+c, \ h<y_1; \end{aligned} \tag{26b}$$

$$\begin{aligned} J=\sigma\int_0^{y_1} E_{Bx}\,dy = \frac{\sigma E_0}{4\pi}[&G(x_1-c, \ h-y_1)-G(x_1-c, \ h+y_1) \\ &-G(x_1+c, \ h-y_1)+G(x_1+c, \ h+y_1)], \quad c<|x_1|, \end{aligned} \tag{26c}$$

where

$$G(\xi, \ \eta) = \xi \log(\xi^2+\eta^2) - 2\eta \tan^{-1}(\eta/\xi). \tag{27}$$

The total current J_t which circulates in the medium is given by substituting $x_1=0$ and $y_1=h$ into Eq. (26a):

$$J_t = \frac{c\sigma E_0}{2\pi}\left[\log\frac{4h^2+c^2}{c^2}+\frac{4h}{c}\tan^{-1}\frac{c}{2h}\right]. \tag{28}$$

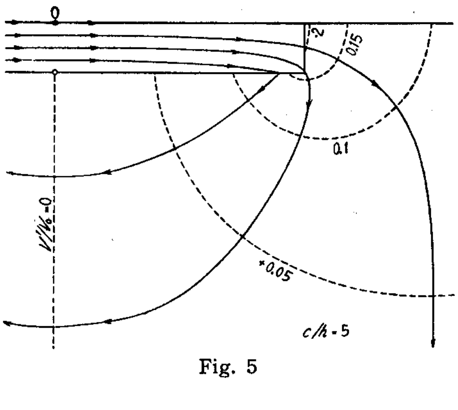

Fig. 5

Fig. 5 shows an example of current distribution in the uniform medium. In this figure, the equipotential lines ($V'=$const.) in the domain (B) are also plotted. It is found that the current flows almost uniformly at the center of (A), but the amount of the current passing through the vertical sides of the region (A) is about 50% of the total current J_t. The current lines are refracted at the boundary plane $y=h$, $-c<x<+c$, and show sharp bends on that plane.

3.2. Two Layer Problem.

Now let us consider the case where the space consists of two media stratified horizontally as shown in Fig. 6. Assuming that the velocity of medium in (A) ($-c<x<+c$, $0<y<h$) is uniform, the electric field $\boldsymbol{E}_1$ in the upper layer can be expressed by

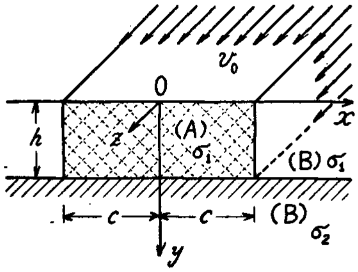

Fig. 6

$$\boldsymbol{E}_1=\boldsymbol{E}_p+\boldsymbol{E}_1^*, \quad 0<y<h, \tag{29}$$

where $\boldsymbol{E}_p$ represents the primary field (i.e. the field in case the conductivity of the lower layer is equal to that of the upper, $\sigma_2=\sigma_1$), and $\boldsymbol{E}_1^*$ represents the secondary field which is produced by the influence of the lower layer.

The conditions to be satisfied on the boundary plane $y=h$ are as follows:

$$E'_{px}+E^*_{1x}=E_{2x}, \quad E'_{py}+E^*_{1y}=\kappa E_{2y}, \quad y=h, \tag{30}$$

where

$$\kappa=\sigma_2/\sigma_1, \tag{31}$$

and

$$\left.\begin{aligned} E'_{px}&=E_{px}-E_0, & -c<x<+c,\\ &=E_{px}, & c<|x|;\\ E'_{py}&=E_{py}. \end{aligned}\right\} \tag{32}$$

It is evident that the secondary field $\boldsymbol{E}_1^*$ in the upper layer and the resultant field $\boldsymbol{E}_2$ in the lower layer can be derived from potentials V_1^* and V_2 respectively, which are solutions of Laplace's equations:

$$\nabla^2 V_1^*=0 \quad \text{and} \quad \nabla^2 V_2=0. \tag{33}$$

The appropriate expressions for these potentials are as follows:

$$\left.\begin{aligned} V_1^*&=\int_0^\infty L_1(\lambda)(e^{\lambda y}+e^{-\lambda y})\sin\lambda x d\lambda, \quad 0<y<h,\\ V_2&=\int_0^\infty L_2(\lambda)e^{-\lambda y}\sin\lambda x d\lambda, \quad h<y, \end{aligned}\right\} \tag{34}$$

from which we get:

$$\left.\begin{aligned} E^*_{1x}&=-\int_0^\infty \lambda L_1(\lambda)(e^{\lambda y}+e^{-\lambda y})\cos\lambda x d\lambda,\\ E^*_{1y}&=-\int_0^\infty \lambda L_1(\lambda)(e^{\lambda y}-e^{-\lambda y})\sin\lambda x d\lambda; \end{aligned}\right\} \tag{35}$$

and

$$\left.\begin{aligned} E_{2x} &= -\int_0^\infty \lambda L_2(\lambda) e^{-\lambda y} \cos \lambda x d\lambda , \\ E_{2y} &= +\int_0^\infty \lambda L_2(\lambda) e^{-\lambda y} \sin \lambda x d\lambda . \end{aligned}\right\} \tag{36}$$

The integral expression of $\boldsymbol{E}_p'$ is obtained from Eqs. (22) and (23):

$$\left.\begin{aligned} E'_{px} &= -\frac{E_0}{\pi}\int_0^\infty [2-e^{-\lambda h}(e^{\lambda y}+e^{-\lambda y})] \sin \lambda c \cos \lambda x \frac{d\lambda}{\lambda} , \\ E'_{py} &= +\frac{E_0}{\pi}\int_0^\infty e^{-\lambda h}(e^{\lambda y}-e^{-\lambda y}) \sin \lambda c \sin \lambda x \frac{d\lambda}{\lambda} . \end{aligned}\right\} \tag{37}$$

Substituting Eqs. (35), (36) and (37) into Eq. (30) we obtain following results:

$$\left.\begin{aligned} E_{1x} &= \frac{k'E_0}{\pi}\int_0^\infty \frac{e^{-\lambda h}}{1-ke^{-2\lambda h}}(e^{\lambda y}+e^{-\lambda y}) \sin \lambda c \cos \lambda x \frac{d\lambda}{\lambda} , \\ E_{1y} &= \frac{k'E_0}{\pi}\int_0^\infty \frac{e^{-\lambda h}}{1-ke^{-2\lambda h}}(e^{\lambda y}-e^{-\lambda y}) \sin \lambda c \sin \lambda x \frac{d\lambda}{\lambda} , \end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned} E_{2x} &= -\frac{k''E_0}{\pi}\int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-2\lambda h}} e^{-\lambda(y-h)} \sin \lambda c \cos \lambda x \frac{d\lambda}{\lambda} , \\ E_{2y} &= +\frac{k''E_0}{\pi}\int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-2\lambda h}} e^{-\lambda(y-h)} \sin \lambda c \sin \lambda x \frac{d\lambda}{\lambda} , \end{aligned}\right\} \tag{39}$$

where

$$k=\frac{1-\kappa}{1+\kappa}, \quad k'=1-k=\frac{2\kappa}{1+\kappa}, \quad k''=1+k=\frac{2}{1+\kappa}. \tag{40}$$

For numerical computations, it is necessary to expand above expressions into series. By means of the relation:

$$1/(1-ke^{-2\lambda h}) = \sum_{n=0}^{\infty} k^n e^{-2n\lambda h} ,$$

and formulas:

$$\int_0^\infty e^{-\lambda p} \sin \lambda a \cos \lambda b \frac{d\lambda}{\lambda} = \frac{1}{2}\left(\tan^{-1}\frac{a+b}{p}+\tan^{-1}\frac{a-b}{p}\right), \quad p>0 , \tag{41}$$

$$\int_0^\infty e^{-\lambda p} \sin \lambda a \sin \lambda b \frac{d\lambda}{\lambda} = \frac{1}{4}\log\frac{(a+b)^2+p^2}{(a-b)^2+p^2}, \quad p>0 , \tag{42}$$

the following infinite series are obtained:

$$\begin{aligned} E_{1x} = \frac{k'E_0}{2\pi}\sum_{n=0}^{\infty} k^n \Big[&\tan^{-1}\frac{c-x}{(2n+1)h-y}+\tan^{-1}\frac{c-x}{(2n+1)h+y} \\ &+\tan^{-1}\frac{c+x}{(2n+1)h-y}+\tan^{-1}\frac{c+x}{(2n+1)h+y}\Big] , \end{aligned} \tag{43}$$

$$\begin{aligned} E_{1y} = \frac{k'E_0}{4\pi}\sum_{n=0}^{\infty} k^n \Big[&\log\frac{(c-x)^2+\{(2n+1)h+y\}^2}{(c-x)^2+\{(2n+1)h-y\}^2} \\ &-\log\frac{(c+x)^2+\{(2n+1)h+y\}^2}{(c+x)^2+\{(2n+1)h-y\}^2}\Big] , \end{aligned} \tag{44}$$

$$E_{2x}=-\frac{k''E_0}{2\pi}\Big[\tan^{-1}\frac{c-x}{y-h}+\tan^{-1}\frac{c+x}{y-h}$$
$$-k'\sum_{n=0}^{\infty}k^n\Big(\tan^{-1}\frac{c-x}{(2n+1)h+y}+\tan^{-1}\frac{c+x}{(2n+1)h+y}\Big)\Big], \tag{45}$$
$$E_{2y}=\frac{k''E_0}{4\pi}\Big[\log\frac{(c+x)^2+(y-h)^2}{(c-x)^2+(y-h)^2}$$
$$-k'\sum_{n=0}^{\infty}k^n\log\frac{(c+x)^2+\{(2n+1)h+y\}^2}{(c-x)^2+\{(2n+1)h+y\}^2}\Big]. \tag{46}$$

At the boundary plane $y=h$ these components become as follows:

$$E_{1x}=\frac{k'k''E_0}{2\pi}\sum_{n=1}^{\infty}k^{n-1}\Big(\tan^{-1}\frac{c-x}{2nh}+\tan^{-1}\frac{c+x}{2nh}\Big)+\begin{cases}\frac{k'}{2}E_0, & -c<x<+c,\\ 0, & c<|x|,\end{cases}$$
$$y=h_-; \tag{47.1}$$
$$E_{2x}=\frac{k'k''E_0}{2\pi}\sum_{n=1}^{\infty}k^{n-1}\Big(\tan^{-1}\frac{c-x}{2nh}+\tan^{-1}\frac{c+x}{2nh}\Big)-\begin{cases}\frac{k''}{2}E_0, & -c<x<+c,\\ 0, & c<|x|,\end{cases}$$
$$y=h_+; \tag{47.2}$$
$$\left.\begin{aligned}E_{1y}&=\frac{k'E_0}{4\pi}\Big[\log\frac{(c+x)^2}{(c-x)^2}-k'\sum_{n=1}^{\infty}k^{n-1}\log\frac{(c+x)^2+(2nh)^2}{(c-x)^2+(2nh)^2}\Big], \quad y=h_-;\\ E_{2y}&=\frac{1}{\kappa}E_{1y}, \quad y=h_+.\end{aligned}\right\} \tag{48}$$

Fig. 7 shows the horizontal components E_{1x} and E_{2x} on the boundary of two layers, for $\kappa=1/10$. There is a discontinuity of amount E_0 between E_{1x} and E_{2x} on both sides of the plane $y=h$, $-c<x<+c$. Fig. 8 shows the vertical components E_{1y} and E_{2y} on the plane $y=h$, and these components become infinite at the points $x=\pm c$, $y=h$. This is due to the assumption that the velocity of flow of the domain (A) vanishes suddenly at the corners of the domain.

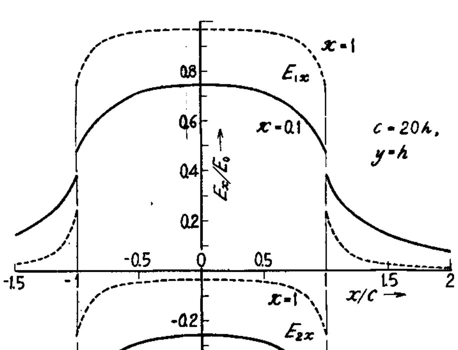

Fig. 7

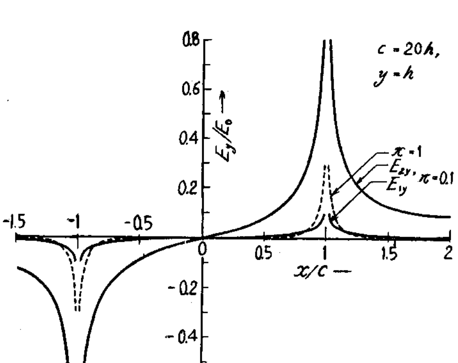

Fig. 8

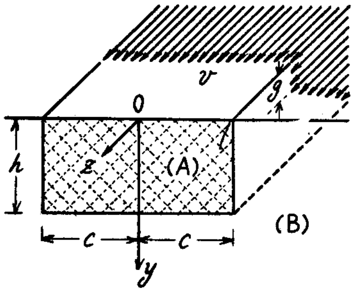

Fig. 9

4. Case II: The velocity diminishes towards the bottom.

4.1. Homogeneous Medium.

If the velocity of the moving medium (A) in Fig. 9 is given by

$$\left.\begin{aligned} v &= v_0 = \text{const.}, \quad 0<y<g, \\ v &= v_0\frac{h-y}{h-g}, \quad g<y<h, \end{aligned}\right\} \tag{49}$$

the expression for the potential V' becomes as follows:

$$\begin{aligned} V' = \frac{E_0}{4\pi(h-g)}[&F(c-x,\ h-y)-F(c-x,\ g-y)+F(c-x,\ h+y) \\ &-F(c-x,\ g+y)-F(c+x,\ h-y)+F(c+x,\ g-y) \\ &-F(c+x,\ h+y)+F(c+x,\ g+y)], \end{aligned} \tag{50}$$

where the function F is defined as

$$F(\xi,\ \eta) = \frac{1}{2}(\xi^2-\eta^2)\log(\xi^2+\eta^2)-2\xi\eta\tan^{-1}(\eta/\xi). \tag{51}$$

By means of Eq. (50), we get following expressions for the electric fields:

$$\begin{aligned} E_x' = \frac{E_0}{4\pi(h-g)}[&G(c-x,\ h-y)-G(c-x,\ g-y)+G(c-x,\ h+y) \\ &-G(c-x,\ g+y)+G(c+x,\ h-y)-G(c+x,\ g-y) \\ &+G(c+x,\ h+y)-G(c+x,\ g+y)], \end{aligned} \tag{52}$$

$$\begin{aligned} E_y' = -\frac{E_0}{4\pi(h-g)}[&H(c-x,\ h-y)-H(c-x,\ g-y)-H(c-x,\ h+y) \\ &+H(c-x,\ g+y)-H(c+x,\ h-y)+H(c+x,\ g-y) \\ &+H(c+x,\ h+y)-H(c+x,\ g+y)], \end{aligned} \tag{53}$$

$$E_x = E_x' + \begin{cases} E_0, & -c<x<+c,\ 0<y<g, \\ E_0\dfrac{h-y}{h-g}, & -c<x<+c,\ g<y<h, \end{cases} \text{ in (A)} \quad \begin{cases} 0, & -c<x<+c,\ h<y, \\ & \text{or}\ \ c<|x|,\ \ 0<y, \end{cases} \text{ in (B)} \tag{54}$$

$$E_y = E_y', \qquad \text{in (A) and (B).} \tag{55}$$

The current flux across the plane $x=x_1$, $0<y<y_1$, is given by

$$J = J' + \begin{cases} \sigma E_0 y_1, & -c<x_1<+c,\ 0<y_1<g, \\ \sigma E_0\dfrac{2hy_1-(g^2+y_1^2)}{2(h-g)}, & -c<x_1<+c,\ g<y_1<h, \\ \sigma E_0\dfrac{h+g}{2}, & -c<x_1<+c,\ h<y_1, \\ 0, & c<|x_1|,\ 0<y_1, \end{cases} \tag{56}$$

where

$$J' = -\frac{\sigma E_0}{4\pi(h-g)}[I(c-x_1,\ h-y_1)-I(c-x_1,\ g-y_1)-I(c-x_1,\ h+y_1)$$
$$+I(c-x_1,\ g+y_1)+I(c+x_1,\ h-y_1)-I(c+x_1,\ g-y_1)$$
$$-I(c+x_1,\ h+y_1)+I(c+x_1,\ g+y_1)], \tag{57}$$

and

$$I(\xi,\ \eta) = \xi\eta \log(\xi^2+\eta^2)+(\xi^2-\eta^2)\tan^{-1}(\eta/\xi)\,. \tag{58}$$

The distribution of the field at various depths are shown in Figs. 10 and 11. In this case, E_x and E_y are finite everywhere, and on the bottom of the domain (A) $E_{A_x}=E_{B_x}$, since the velocity of the medium is zero on the bottom.

The current lines and equipotential lines are plotted in Fig. 12. The center of circulation of the current is not on the bottom of the domain (A), but in the region $g<y_1<h$, $x_1=0$. This point and the amount of total current J_t is calculated numerically from Eq. (56), substituting $x_1=0$ into it. The point $x_1=0$, $y_1=y_m$ at which the current J becomes maximum is the center of circulation, and the maximum value of J is the total current J_t. In Fig. 12, $y_m=0.954\,h$, and $J_t=0.659\,\sigma E_0 h$.

It can be found from this figure, that the greater part of the current flows in the upper region of (A), and more than half of the total current passes the sides of (A). Every current line is a smooth curve, and some of the turning points, at which the hosizontal component of

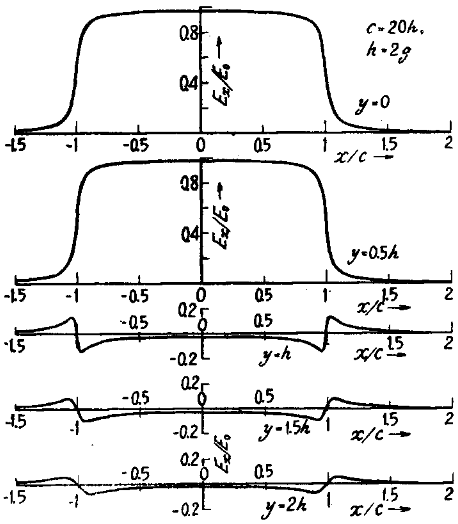

Fig. 10

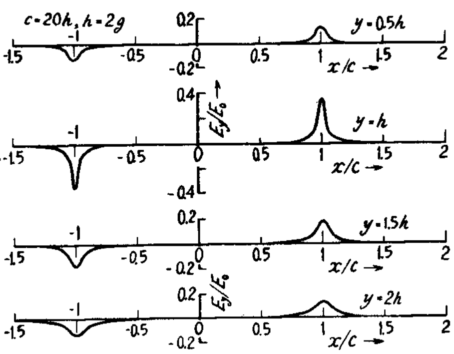

Fig. 11

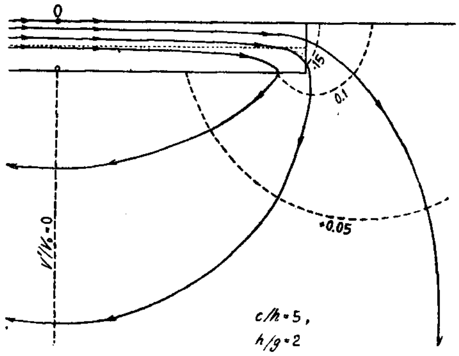

Fig. 12

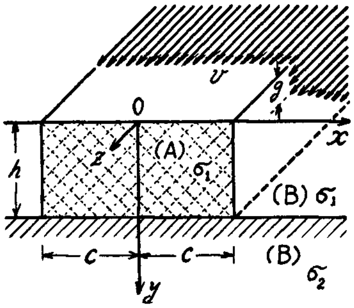

Fig. 13

current density vanishes, are in the domain (A). This means that a small part of the current flows against the distributed e.m.f. in the lower region of (A).

4.2. Two Layer Problem.

We can solve the boundary value problem as shown in Fig. 13, by a similar way described in §3.2. The results are as follows:

$$E_{1x}=E_{px}+E_{1x}^*,\quad E_{1y}=E_{py}+E_{1y}^*, \tag{59}$$

$$E_{px}=-\frac{E_0}{\pi(h-g)}\int_0^\infty[(e^{-\lambda(h-y)}+e^{-\lambda(h+y)})-(e^{-\lambda(g-y)}+e^{-\lambda(g+y)})]\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda^2},$$
$$0<y<g; \tag{60 a}$$

$$E_{px}=-\frac{E_0}{\pi(h-g)}\int_0^\infty[(e^{-\lambda(h-y)}+e^{-\lambda(h+y)})-(e^{-\lambda(y-g)}+e^{-\lambda(y+g)})]\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda^2},$$
$$g<y<h, \tag{60 b}$$

$$E_{py}=-\frac{E_0}{\pi(h-g)}\int_0^\infty(e^{-\lambda h}-e^{-\lambda g})(e^{\lambda y}-e^{-\lambda y})\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda^2},\quad 0<y<g; \tag{61 a}$$

$$E_{py}=+\frac{E_0}{\pi(h-g)}\int_0^\infty[2-(e^{-\lambda(h-y)}-e^{-\lambda(h+y)}+e^{-\lambda(y-g)}+e^{-\lambda(y+g)})]\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda^2},$$
$$g<y<h, \tag{61 b}$$

$$E_{1x}^*=-\frac{kE_0}{\pi(h-g)}\int_0^\infty\frac{(e^{\lambda h}+e^{-\lambda h})-(e^{\lambda g}+e^{-\lambda g})}{1-ke^{-2\lambda h}}e^{-2\lambda h}(e^{\lambda y}+e^{-\lambda y})\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda^2},$$
$$0<y<h, \tag{62}$$

$$E_{1y}^*=-\frac{kE_0}{\pi(h-g)}\int_0^\infty\frac{(e^{\lambda h}+e^{-\lambda h})-(e^{\lambda g}+e^{-\lambda g})}{1-ke^{-2\lambda h}}e^{-2\lambda h}(e^{\lambda y}-e^{-\lambda y})\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda^2},$$
$$0<y<h, \tag{63}$$

$$E_{2x}=-\frac{k''E_0}{\pi(h-g)}\int_0^\infty\frac{(e^{\lambda h}+e^{-\lambda h})-(e^{\lambda g}+e^{-\lambda g})}{1-ke^{-2\lambda h}}e^{-\lambda y}\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda^2},\quad h<y, \tag{64}$$

$$E_{2y}=+\frac{k''E_0}{\pi(h-g)}\int_0^\infty\frac{(e^{\lambda h}+e^{-\lambda h})-(e^{\lambda g}+e^{-\lambda g})}{1-ke^{-2\lambda h}}e^{-\lambda y}\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda^2},\quad h<y. \tag{65}$$

These can be expanded into the following infinite series:

$$\begin{aligned}E_{1x}^*=\frac{E_0}{4\pi(h-g)}\sum_{n=1}^\infty k^n\\
\times[\{&M(c-x,\ 2nh+h-y)+M(c-x,\ 2nh+h+y)\\
&+M(c-x,\ 2nh-h+y)+M(c-x,\ 2nh-h-y)\\
&+M(c+x,\ 2nh+h-y)+M(c+x,\ 2nh+h+y)\\
&+M(c+x,\ 2nh-h+y)+M(c+x,\ 2nh-h-y)\\
-\{&M(c-x,\ 2nh+g-y)+M(c-x,\ 2nh+g+y)\\
&+M(c-x,\ 2nh-g+y)+M(c-x,\ 2nh-g-y)\end{aligned}$$

$$+M(c+x,\ 2nh+g-y)+M(c+x,\ 2nh+g+y)$$
$$+M(c+x,\ 2nh-g+y)+M(c+x,\ 2nh-g-y)\}],\quad 0<y<h, \tag{66}$$

$$E_{px}=\frac{E_0}{4\pi(h-g)}[M(c-x,\ h-y)+M(c-x,\ h+y)+M(c+x,\ h-y)$$
$$+M(c+x,\ h+y)-M(c-x,\ g-y)-M(c-x,\ g+y)$$
$$-M(c+x,\ g-y)-M(c+x,\ g+y)],\quad 0<y<h, \tag{67}$$

$$E_{1y}^{*}=\frac{E_0}{4\pi(h-g)}\sum_{n=1}^{\infty}k^n$$
$$\times[\{H(c-x,\ 2nh+h-y)-H(c-x,\ 2nh+h+y)$$
$$-H(c-x,\ 2nh-h+y)+H(c-x,\ 2nh-h-y)$$
$$-H(c+x,\ 2nh+h-y)+H(c+x,\ 2nh+h+y)$$
$$+H(c+x,\ 2nh-h+y)-H(c+x,\ 2nh-h-y)$$
$$-\{H(c-x,\ 2nh+g-y)-H(c-x,\ 2nh+g+y)$$
$$-H(c-x,\ 2nh-g+y)+H(c-x,\ 2nh-g-y)$$
$$-H(c+x,\ 2nh+g-h)+H(c+x,\ 2nh+g+y)$$
$$+H(c+x,\ 2nh-g+y)-H(c+x,\ 2nh-g-y)\}],\quad 0<y<h, \tag{68}$$

$$E_{py}=-\frac{E_0}{4\pi(h-g)}[H(c-x,\ h-y)-H(c-x,\ h+y)-H(c+x,\ h-y)$$
$$+H(c+x,\ h+y)-H(c-x,\ g-y)+H(c-x,\ g+y)$$
$$+H(c+x,\ g-y)-H(c+x,\ g+y)],\quad 0<y<h, \tag{69}$$

$$E_{2x}=\frac{k''E_0}{4\pi(h-g)}\sum_{n=0}^{\infty}k^n$$
$$\times[M(c-x\ \ 2nh-h+y)+M(c-x,\ 2nh+h+y)$$
$$+M(c+x,\ 2nh+h+y)+M(c+x,\ 2nh+h+y)$$
$$-M(c-x,\ 2nh-g+y)-M(c-x,\ 2nh+g+y)$$
$$-M(c+x,\ 2nh+g+y)-M(c+x,\ 2nh+g+y)],\quad h<y, \tag{70}$$

$$E_{2y}=\frac{k''E_0}{4\pi(h-g)}\sum_{n=0}^{\infty}k^n$$
$$\times[H(c-x,\ 2nh-h+y)+H(c-x,\ 2nh+h+y)$$
$$-H(c+x,\ 2nh-h+y)-H(c+x,\ 2nh+h+y)$$
$$-H(c-x,\ 2nh-g+y)-H(c-x,\ 2nh+g+y)$$
$$+H(c+x,\ 2nh-g+y)+H(c+x,\ 2nh+g+y)],\quad h<y, \tag{71}$$

where

$$M(\xi,\eta)=\xi\log(\xi^2+\eta^2)+2\eta\tan^{-1}(\xi/\eta)$$
$$=H(\eta,\xi), \tag{72}$$

and $H(\xi,\eta)$ has been defined by Eq. (21).

On the bottom $y=h$, the expressions for the field become as follows;

$$
\begin{aligned}
E_{1x} = E_{2x} = \frac{k'' E_0}{4\pi(h-g)} \sum_{n=0}^{\infty} k^n [& M(c-x,\ 2nh) + M(c-x,\ 2nh+2h) \\
& + M(c+x,\ 2nh) + M(c+x,\ 2nh+2h) \\
& - M(c-x,\ 2nh+h-g) - M(c-x,\ 2nh+h+g) \\
& - M(c+x,\ 2nh+h-g) - M(c+x,\ 2nh+h+g)], \quad y = h,
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
E_{1y} = \kappa E_{2y} = \frac{k' E_0}{4\pi(h-g)} \sum_{n=0}^{\infty} k^n [& H(c-x,\ 2nh) + H(c-x,\ 2nh+2h) \\
& - H(c+x,\ 2nh) - H(c+x,\ 2nh+2h) \\
& - H(c-x,\ 2nh+h-g) - H(c-x,\ 2nh+h+g) \\
& + H(c+x,\ 2nh+h-g) + H(c+x,\ 2nh+h+g)], \quad y = h.
\end{aligned} \tag{74}
$$

In Fig. 14, the horizontal component of electric field on the bottom is shown. As mentioned above, the field in this case is continuous everywhere, so that no discontinuity occurs even at the points $x = \pm c$, $y = h$. Fig. 15 shows the vertical component on the same plane $y = h$.

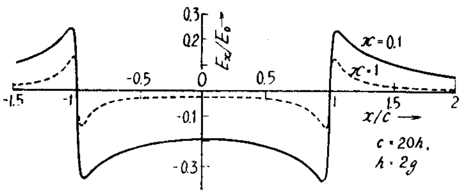

Fig. 14

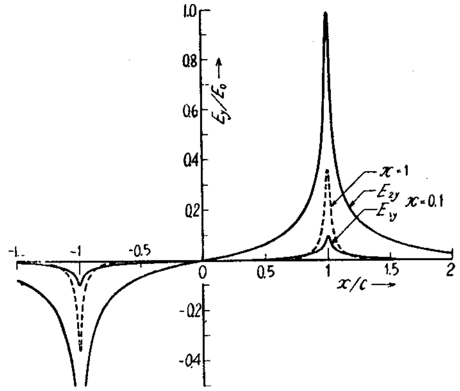

Fig. 15

5. Case III: The velocity diminishes towards the sides.

5.1. Homogeneous Medium.

In case the velocity of the medium in the domain (A) of Fig. 16 is given by

$$
\left.
\begin{aligned}
v &= v_0 = \text{const.}, & -d < x < +d, \\
v &= v_0 \frac{c-x}{c-d}, & d < |x| < c,
\end{aligned}
\right\}
\quad 0 < y < h, \tag{75}
$$

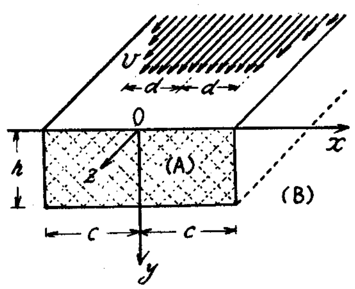

Fig. 16

the potential V' is expressed as follows:

$$
\begin{aligned}
V' = -\frac{E_0}{4\pi(c-d)} [& K(c-x,\ h-y) + K(c-x,\ h+y) - K(c+x,\ h-y) \\
& - K(c+x,\ h+y) - K(d-x,\ h-y) - K(d-x,\ h+y) \\
& + K(d+x,\ h-y) + K(d+x,\ h+y)],
\end{aligned} \tag{76}
$$

where

$$K(\xi, \eta) = \xi\eta \log (\xi^2+\eta^2) + \xi^2 \tan^{-1} (\eta/\xi) + \eta^2 \tan^{-1} (\xi/\eta) \,. \tag{77}$$

The components of electric field are given by

$$\begin{aligned} E_x' = -\frac{E_0}{4\pi(c-d)} [&H(c-x,\ h-y)+H(c-x,\ h+y)+H(c+x,\ h-y) \\ &+H(c+x,\ h+y)-H(d-x,\ h-y)-H(d-x,\ h+y) \\ &-H(d+x,\ h-y)-H(d+x,\ h+y)]\,, \end{aligned} \tag{78}$$

$$E_x = E_x' + \left\{ \begin{array}{lll} E_0\,, & -d<x<+d\,, & 0<y<h\,, \\ E_0 \dfrac{c-x}{c-d}\,, & d<|x|<c\,, & 0<y<h\,, \\ 0\,, & -c<x<+c\,,\ \ h<y\,, & \text{or}\ \ c<|x|\,,\ \ 0<y\,. \end{array} \right\} \tag{79}$$

$$\begin{aligned} E_y = -\frac{E_0}{4\pi(c-d)} [&M(c-x,\ h-y)-M(c-x\ \ h+y)-M(c+x,\ h-y) \\ &+M(c+x,\ h+y)-M(d-x,\ h-y)+M(d-x,\ h+y) \\ &+M(d+x,\ h-y)-M(d+x,\ h+y)]\,. \end{aligned} \tag{80}$$

The current flux J can be calculated by the following relations:

$$J = J' + \left\{ \begin{array}{lll} \sigma E_0 y_1\,, & -d<x_1<+d\,, & 0<y_1<h\,, \\ \sigma E_0 h\,, & -d<x_1<+d\,, & h<y_1\,, \\ \sigma E_0 \dfrac{c-x_1}{c-d} y_1\,, & d<|x_1|<c\,, & 0<y_1<h\,, \\ \sigma E_0 \dfrac{c-x_1}{c-d} h\,, & d<|x_1|<c\,, & h_1<y_1\,, \\ 0\,, & c<|x_1|\,, & 0<y_1\,, \end{array} \right\} \tag{81}$$

where,

$$\begin{aligned} J' = -\frac{\sigma E_0}{4\pi(c-d)} [&F(c-x,\ h-y) \\ &-F(c-x,\ h+y)+F(c+x,\ h-y) \\ &-F(c+x,\ h+y)-F(d-x,\ h-y) \\ &+F(d-x,\ h+y)-F(d+x,\ h-y) \\ &+F(d+x,\ h+y)]\,, \end{aligned} \tag{82}$$

and the function F has been defined by Eq. (51).

The center of the circulation of current is at the point $x=0$, $y=h$, and the total current is given by

$$\begin{aligned} J_t = \sigma E_0 h - \frac{\sigma E_0}{2\pi(c-d)} [&F(c,\ 0)-F(c,\ 2h) \\ &-F(d,\ 0)+F(d,\ 2h)]\,. \end{aligned} \tag{83}$$

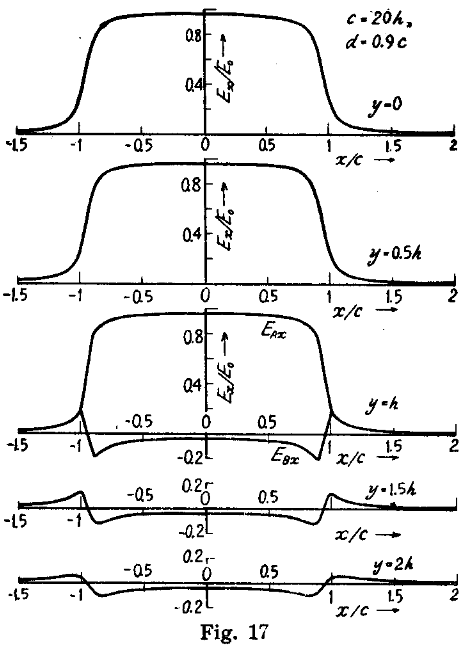

Fig. 17

Figs. 17 and 18 show the distributions of horizontal and vertical components of

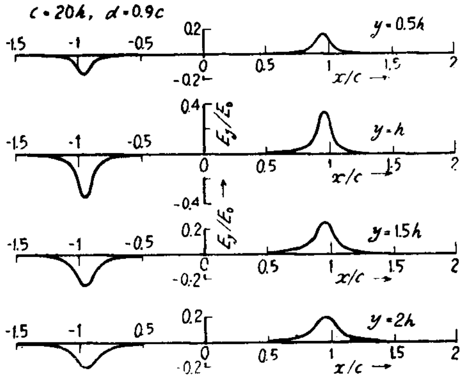

Fig. 18

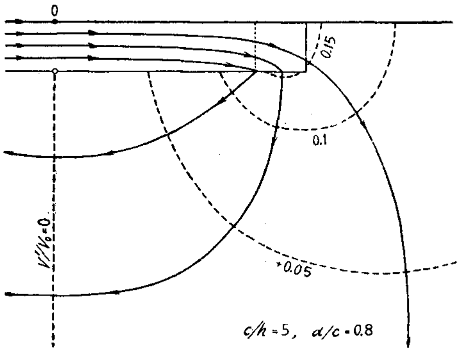

Fig. 19

the electric field at various depths. These curves are similar to that of Figs. 3 and 4, but the horizontal component on the plane $y=h$ varies continuously, and the vertical component does not become infinite even at the points $y=h$, $x=\pm c$. The minima and maxima of E_{2x} for $y=\text{const.} \geqq h$ occur at the points $x \simeq \pm d$ and $x \simeq \pm c$ respectively.

Fig. 19 shows the current lines and equipotential curves. These curves are similar to that of Fig. 5, but the effects of the transition region of velocity ($d<|x|<c$, $0<y<h$) are shown very clearly.

5. 2. Two Layer Problem.

Also the problem of Fig. 20 can be solved by means of the method described in §3. 2. The results are as follows:

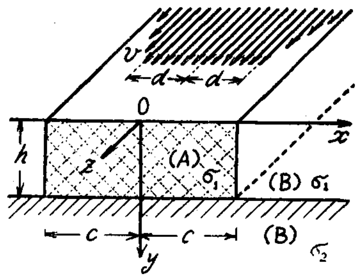

Fig. 20

$$E_{1x} = \frac{k'E_0}{\pi(c-d)} \int_0^\infty \frac{e^{-\lambda h}}{1-ke^{-2\lambda h}} (e^{\lambda y}+e^{-\lambda y})(\cos\lambda d - \cos\lambda c)\cos\lambda x \frac{d\lambda}{\lambda^2}, \quad 0<y<h, \tag{84 a}$$

$$E_{1y} = \frac{k'E_0}{\pi(c-d)} \int_0^\infty \frac{e^{-\lambda h}}{1-ke^{-2\lambda h}} (e^{\lambda y}-e^{-\lambda y})(\cos\lambda d - \cos\lambda c)\sin\lambda x \frac{d\lambda}{\lambda^2}, \quad 0<y<h, \tag{84 b}$$

$$E_{2x} = -\frac{k''E_0}{\pi(c-d)} \int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-2\lambda h}} e^{-\lambda(y-h)}(\cos\lambda d - \cos\lambda c)\cos\lambda x \frac{d\lambda}{\lambda^2}, \quad h<y, \tag{85 a}$$

$$E_{2y} = +\frac{k''E_0}{\pi(c-d)} \int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-2\lambda h}} e^{-\lambda(y-h)}(\cos\lambda d - \cos\lambda c)\sin\lambda x \frac{d\lambda}{\lambda^2}, \quad h<y, \tag{85 b}$$

$$E_{1x} = E'_{1x} + \left\{ \begin{array}{ll} E_0, & -d<x<+d, \\ E_0 \dfrac{c-x}{c-d}, & d<|x|<c, \\ 0, & c<|x|, \end{array} \right\} \tag{86}$$

$$E'_{1x} = -\frac{k'E_0}{4\pi(c-d)} \sum_{n=0}^{\infty} k^n[H(c-x,\ 2nh+h-y)+H(c-x,\ 2nh+h+y)$$
$$+H(c+x,\ 2nh+h-y)+H(c+x,\ 2nh+h+y)$$
$$-H(d-x,\ 2nh+h-y)-H(d-x,\ 2nh+h+y)$$
$$-H(d+x,\ 2nh+h-y)-H(d+x,\ 2nh+h+y)], \tag{86'}$$

$$E_{1y} = -\frac{k'E_0}{4\pi(c-d)} \sum_{n=0}^{\infty} k^n[M(c-x,\ 2nh+h-y)-M(c-x,\ 2nh+h+y)$$
$$-M(c+x,\ 2nh+h-y)+M(c+x,\ 2nh+h+y)$$
$$-M(d-x,\ 2nh+h-y)+M(d-x,\ 2nh+h+y)$$
$$+M(d+x,\ 2nh+h-y)-M(d+x,\ 2nh+h+y)], \tag{87}$$

$$E_{2x} = +\frac{k''E_0}{4\pi(c-d)} \sum_{n=0}^{\infty} k^n[H(c-x,\ 2nh-h+y)-H(c-x,\ 2nh+h+y)$$
$$+H(c+x,\ 2nh-h+y)-H(c+x,\ 2nh+h+y)$$
$$-H(d-x,\ 2nh-h+y)+H(d-x,\ 2nh+h+y)$$
$$-H(d+x,\ 2nh-h+y)+H(d+x,\ 2nh+h+y)], \tag{88}$$

$$E_{2y} = -\frac{k''E_0}{4\pi(c-d)} \sum_{n=0}^{\infty} k^n[M(c-x,\ 2nh-h+y)-M(c-x,\ 2nh+h+y)$$
$$-M(c+x,\ 2nh-h+y)+M(c+x,\ 2nh+h+y)$$
$$-M(d-x,\ 2nh-h+y)+M(d-x,\ 2nh+h+h)$$
$$+M(d+x,\ 2nh-h+y)-M(d+x,\ 2nh+h+y)]. \tag{89}$$

On the bottom $y=h$,

$$E_{1x} = E_{2x} + \left\{ \begin{array}{ll} E_0, & -d<x<+d, \\ E_0 \dfrac{c-x}{c-d}, & d<|x|<c, \\ 0, & c<|x|, \end{array} \right\} \tag{90}$$

$$E_{2x} = -\frac{k'k''E_0}{4\pi(c-d)} \sum_{n=1}^{\infty} k^{n-1}[H(c-x,\ 2nh)+H(c+x,\ 2nh)$$
$$-H(d-x,\ 2nh)-H(d+x,\ 2nh)], \tag{91}$$

$$E_{1y} = \kappa E_{2y} = -\frac{k'E_0}{4\pi(c-d)}[M(c-x,\ 0)$$
$$-M(c+x,\ 0)-M(d-x,\ 0)$$
$$+M(d+x,\ 0)$$
$$+k' \sum_{n=1}^{\infty} k^{n-1}\{M(c-x,\ 2nh)$$
$$-M(c+x,\ 2nh)-M(d-x,\ 2nh)$$
$$+M(d+x,\ 2nh)\}]. \tag{92}$$

Fig. 21 shows an example of the distribution of the horizontal component of electric field on the bottom ($y=h$). This is very similar to Fig. 7, but in Fig. 21,

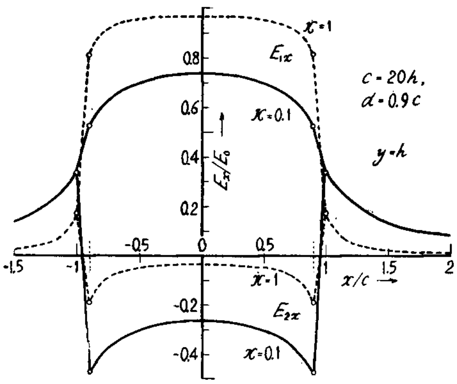

Fig. 21

there is no discontinuity on the curves, and the negative peaks of E_{2x} are somewhat smaller than that of Fig. 7.

6. Potentital Difference Observed Between Two Electrodes.

When we lower two point electrodes into the medium, a potential difference will be observed between them. If both electrodes are in the stationary domain (B), the potential difference will be simply

$$\varDelta V = V'_{(2)} - V'_{(1)}\,, \tag{93}$$

where $V'_{(1)}$ and $V'_{(2)}$ are the values of potential V' at the points (1) and (2) where the two electrodes are placed. The potential V' has been defind in §2.

If one of the electrodes is placed in the domain (A), and the other in the region (A) or (B), the potential difference between them must be calculated from the relation:

$$\varDelta V = \int_{(1)}^{(2)} \left(E_s'' - \frac{j_s}{\sigma}\right) ds\,, \tag{94}$$

where s is an arbitrary curve connecting the points (1) and (2), and E_s'' is the tangential component of the distributed e.m.f. per unit length along the curve. The tangential component j_s of the current density is equal to σE_s, where E_s is the tangential component of the resultant field, so that Eq. (94) becomes as follows:

$$\varDelta V = \int_{(1)}^{(2)} (E_s'' - E_s)\, ds = -\int_{(1)}^{(2)} E_s' ds\,, \tag{95}$$

since E_s is equal to $E_s' + E_s''$.

Since the potential V' at any point P is given by

$$V' = -\int_{\infty}^{P} E_s' ds\,,$$

Eq. (95) can be written as follows:

$$\varDelta V = V'_{(2)} - V'_{(1)}\,, \tag{96}$$

which is the same as Eq. (93).

As mentioned above, V' is continuous everywhere in whole domain, so that $\varDelta V$ is determined uniquely for the given points (1) and (2), wherever they are placed.

7. Current Induced in a Linear Conductor.

In general, a conductor placed in a stationary current field would produce a secondary electric field, so that the resultant field near the conductor would be different from the primary field, i.e. the field before the conductor had been placed. However, a very thin conductor such as a submarine cable does not affect the field materially, and the secondary field due to the concuctor can be neglected. In this case, the resultant field on the conductor's surface is given by the field when the conductor does not exist.

When the primary field is produced by tides, a cable laid in the stationary region (B), which acts always as a part of the return circuit of the current, would obviously have a current density in its sheath given by

$$\boldsymbol{j}_c = \sigma_c \boldsymbol{E} \qquad \text{in (B)}\,, \tag{97}$$

where σ_c is the conductivity of the cable sheath and $\boldsymbol{E}$ is the field intensity in the domain (B).

If the cable is laid in the tide-way (A), the cable acts as a return path of the current, and a discontinuity of tangential component of the field will occur on the surface of the cable. The field outside the cable is $\boldsymbol{E}$, but the interior field is $\boldsymbol{E}'$. The difference $\boldsymbol{E}-\boldsymbol{E}'$ corresponds to the distributed e.m.f. per unit length in the domain (A), which is equal to $\boldsymbol{E}''$. Hence we can write

$$\boldsymbol{j}_c = \sigma_c \boldsymbol{E}' \qquad \text{in (A)}\,. \tag{98}$$

Since in the domain (B) $\boldsymbol{E}=\boldsymbol{E}'$, we can always express the magnitude of axial current density in the cable sheath as follows:

$$j_c = \sigma_c E_s'\,, \tag{99}$$

where E_s' is the tangential component of the field $\boldsymbol{E}'$.

Since the cable is normally laid on the sea bed, the axial current in the cable sheath can be calculated by the relation:

$$J_c = \sigma_c S_c E_x'(y=h)\,, \tag{100}$$

where S_c is the effective cross-sectional area of the cable sheath.

We have found in §§ 3, 4 and 5, that the horizontal component $E_x'(y=h)$ of the partial field $\boldsymbol{E}'$ changes its sign at the points near the both ends of the tide-way. This means that the amount of the sheath current varies rapidly in these regions, and the possibility of the electrolytic corrosion will be also very large. The facts that the cable faults occur very often at the ends of tide-way can be explained by the theoretical results obtained above.

The effect of the conductivity of the bed on the sheath current and other factors such as the existence of highly resistive rocks are also understood by this theory.

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