On the Electric Field due to Tides. II.

By

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In the previous paper, the electric field induced by tidal current was discussed in some detail for the case where the earth's magnetic field is vertical. In this paper we shall consider the effect of the horizontal magnetic field on the electric field and current distribution in the sea water and the sea bed.

A general description will be made in Section 2 concerning the phenomenon in the tidal current of rectangular cross-section, and a method of obtaining the electric potential in general case of velocity distribution will be given Section 3.

In Section 4, a method of solving so-called two-layer problems will be discussed in general, and in Sections 5 and 6, some results of numerical computation will be shown. The method by which some of these results have been obtained will be explained in Appendix I, while Appendix II will be devoted to a list of the formulas for electric potential, electric field and electric current due to horizontal component of the earth's magnetic field in the streams of rectangular cross-section with various velocity distributions.

1. Introduction

The nature of the electric field induced in sea water moving across the earth's magnetic field has been discussed theoretically by several authors, and the results of observations have been also reported. Among these, Dr. Longuet-Higgins (1949)[1] has reported the results of experimental and theoretical research on this phenomenon, and has given a complete list of papers published before 1948 concerning this problem.

A more recently published work by Dr. Longuet-Higgins, Dr. Stern and Dr. Stommel (1945)[2] covers a wide range of theoretical analysis, and results of numerical computation are also given there in some detail.

The author of this paper and Dr. Ezoe have published a theoretical work (Part I, 1957)[3], the purpose of which was mainly to explain a cause of faults on submarine

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cables due to electrolytic corrosion[4]. At that time they could not refer to the important works cited above, and solved the problem by their own method, which differed a little from the method of approach followed by Dr. Longuet-Higgins and others. Although some parts of the previous paper (Part I) duplicate the research by the staffs of W.H.O.I., it seems still to involve somewhat different results, by taking account the conductivity of sea bed and the non-uniformity of velocity of water*.

However the previous paper was not sufficiently general in analytical form, because the effect of horizontal magnetic field had been wholly ignored. It is the purpose of this paper to generalize the basic mathematical description, and to discuss the effect of the horizontal component of the earth's magnetic field, and to propose a useful method of numerical computaion of the electric field in the case of two-layer structure.

2. Basic Relations

We shall consider a stream of rectangular cross-section in which the stream velocity is a function of x and y, and independent of z:

$$\left.\begin{aligned} v &= v(x, y), \quad -c<x<+c;\ 0<y<h, \\ &= 0, \qquad \text{elsewhere,} \end{aligned}\right\} \tag{1.1}$$

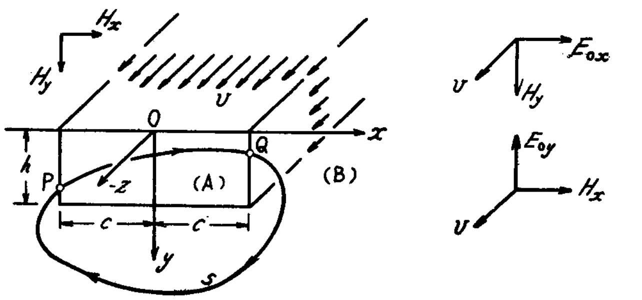

Fig. 1.

as shown in Fig. 1.

In the water moving across the earth's magnetic field $\boldsymbol{H}$, an e.m.f. per unit length

$$\boldsymbol{E}_0 = \mu_0 \boldsymbol{v} \times \boldsymbol{H} \tag{1.2}$$

will be induced. As the velocity $\boldsymbol{v}$ has been assumed to have the z-component v only, the x- and y-components of vector $\boldsymbol{E}_0$ can be expressed as follows:

$$\left.\begin{aligned} E_{0x}(x, y) &= \mu_0 v(x, y) H_y, \\ E_{0y}(x, y) &= -\mu_0 v(x, y) H_x. \end{aligned}\right\} \tag{1.3}$$

Since the medium in the domain A ($-c<x<+c$, $0<y<h$) is the sea water (a good conductor), and, in general, the medium in the stationary domain B is also

* On the other hand, the very interesting problem concerning the streams of elliptic cross-section which has been discussed first by Dr. Longuet-Higgins (1949) has not been included in our research. The author of this paper has tried to generalize the theory of Dr. Longuet-Higgins and others to the case of non-uniform velocity and got some results, which will be published at some later date.

conducting, an electric current through both domains will be produced by the distributed e.m.f. $\boldsymbol{E}_0$. Suppose a closed curve s through both domains A and B (see Fig. 1), and apply Ohm's law to this circuit, we get

$$\oint \frac{1}{\sigma} \boldsymbol{j} \cdot d\boldsymbol{s} = \oint \boldsymbol{E}_0 \cdot d\boldsymbol{s}. \tag{1.4}$$

It can be written as

$$\oint \frac{1}{\sigma} \boldsymbol{j} \cdot d\boldsymbol{s} = \int_P^Q \boldsymbol{E}_0 \cdot d\boldsymbol{s}, \tag{1.4'}$$

since the integral on the right hand side of Eq. (1.4) will vanish except along the part of s in the domain A. As the electric field at any point can be expressed by the relation:

$$\boldsymbol{E} = \frac{1}{\sigma} \boldsymbol{j}, \tag{1.5}$$

Eq. (1.4′) can be modified as follows:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{s} = \int_P^Q \boldsymbol{E}_0 \cdot d\boldsymbol{s}. \tag{1.6}$$

If we assume the field $\boldsymbol{E}$ is given by a sum of two partial fields:

$$\boldsymbol{E} = \boldsymbol{E}' + \boldsymbol{E}'', \tag{1.7}$$

and the partial field $\boldsymbol{E}'$ satisfies the relation:

$$\mathrm{curl}\ \boldsymbol{E}' = 0, \tag{1.8}$$

then we have

$$\oint \boldsymbol{E}' \cdot d\boldsymbol{s} = 0, \tag{1.9}$$

and Eq. (1.6) becomes as follows:

$$\oint \boldsymbol{E}'' \cdot d\boldsymbol{s} = \int_P^Q \boldsymbol{E}_0 \cdot d\boldsymbol{s}. \tag{1.10}$$

Since the electric field in the stationary domain B is irrotational, Eq. (1.10) can be written:

$$\int_P^Q (\boldsymbol{E}'' - \boldsymbol{E}_0) \cdot d\boldsymbol{s} = 0 \qquad \text{in A}, \tag{1.11}$$

from which we obtain the following relation:

$$\boldsymbol{E}'' = \boldsymbol{E}_0 \qquad \text{in A}. \tag{1.12}$$

On the other hand, the partial electric field $\boldsymbol{E}'$ can be derived from a scalar potential V', because of the assumption of Eq. (1.8):

$$\boldsymbol{E}' = -\mathrm{grad}\ V'. \tag{1.13}$$

Hence, we can derive the electric field at any point from the following relations:

$$\left.\begin{aligned} \boldsymbol{E} &= -\text{grad } V'+\boldsymbol{E}_0, && \text{in A;} \\ &= -\text{grad } V', && \text{in B.} \end{aligned}\right\} \tag{1.14}$$

The potential difference which should be observed between two point electrodes lowered into the domain A and (or) B, is the difference of the values of potential V' at these point electrodes.

Even if the conductivity or velocity varies discontinuously at the boundary planes between A and B, or at any other plane in A or B, the potential V' and the normal component j_n of current density $\boldsymbol{j}$ must be continuous at these planes.

Since the distributed e.m.f. $\boldsymbol{E}_0$ in Eq. (1.14) can be easily obtained by Eq. (1.3) if the velocity distribution $v(x, y)$ and the earth's magnetic field (H_x, H_y) are given, the behaviours of electric field and current can be described completely if we could find the potential V'.

3. A Method of Obtaining the Potential in a Uniform Medium

Although we shall consider in Section 4 the more general case where the conductivity of the sea bed σ_2 is different from that of the sea water σ_1, it will be helpful to have a potential V' for the simple case where the conductivity is constant over the semi-infinite domain including both domains A and B. For this purpose it seems to be very convenient to use a method of the "fictitious current sources" which will be explained below.

For simplicity, we shall consider at first an elementary case where the stream velocity v_0 is constant over the rectangular domain A, and the magnetic field is vertical. In this case, the distributed e.m.f. is horizontal and its magnitude E_0 is constant over the domain A.

Assume that the potential V' is derived from fictitious current sources, placed on both sides of the stream $(x=\pm c,\ 0<y<h)$, and that the surface intensity of the right hand side source $(x=+c)$ is $+j_0$, and the intensity of the left hand side source $(x=-c)$ is $-j_0$. Furthermore we put

$$V' = V_+ + V_-, \tag{3.1}$$

where V_+ denotes the potential due to the right hand side source $(+j_0)$ and V_- that of the left hand side source $(-j_0)$.

Then the partial electric field $\boldsymbol{E}'$ will be given by

$$\boldsymbol{E}' = -(\text{grad } V_+ + \text{grad } V_-), \tag{3.2}$$

and the x-component of this field will vary discontinuously across both sides of the domain A $(x=\pm c)$. For instance, at the right side $x=c$:

$$\left.\begin{aligned} E_x' &= -\frac{\partial V_-}{\partial x}-\frac{j_0}{2\sigma_1}, &\quad \text{at } x = c_- \text{ (in A)},\\ &= -\frac{\partial V_-}{\partial x}+\frac{j_0}{2\sigma_1}, &\quad \text{at } x = c_+ \text{ (in B)}, \end{aligned}\right\} 0<y<h. \tag{3.3}$$

Therefore, the resultant electric field at this side becomes as follows:

$$\left.\begin{aligned} E_x &= E_x'+E_x'' = -\frac{\partial V_-}{\partial x}-\frac{j_0}{2\sigma_1}+E_0, \quad x = c_-;\\ E_x &= E_x' = -\frac{\partial V_-}{\partial x}+\frac{j_0}{2\sigma_1}, \quad x = c_+, \end{aligned}\right\} 0<y<h. \tag{3.4}$$

Since the component E_x must be continuous across this plane, the value of j_0 should be chosen as

$$j_0 = \sigma_1 E_0, \tag{3.5}$$

by which the similar condition at the other side of the stream $(x=-c)$ will be also satisfied.

This result can be interpreted as follows. A system of fictitious current sources shown in Fig. 2 (a) will produce a current diverging from the right hand side source $(+\sigma_1 E_0)$ which is absorbed by the left hand side source $(-\sigma_1 E_0)$. The potential V' due to these sources can be obtained by simple integrals. The electric field and current density at any point will be $\boldsymbol{E}'=-\text{grad } V'$ and $\boldsymbol{j}'=\sigma_1\boldsymbol{E}'$, respectively.

The second system of ficttious current sources shown in Fig. 2 (b), which is equal to that of the first system in its intensity and configuration but opposite to the first in its signs, combined with the distributed e.m.f. $\boldsymbol{E}_0$ will produce the partial electric field $\boldsymbol{E}''$ in the domain A.

Superposing these two systems of fictitious sources and distributed e.m.f., we obtain the electric field $\boldsymbol{E}'+\boldsymbol{E}''$ in the domain A, and $\boldsymbol{E}'$ (or the potential V') in the domain

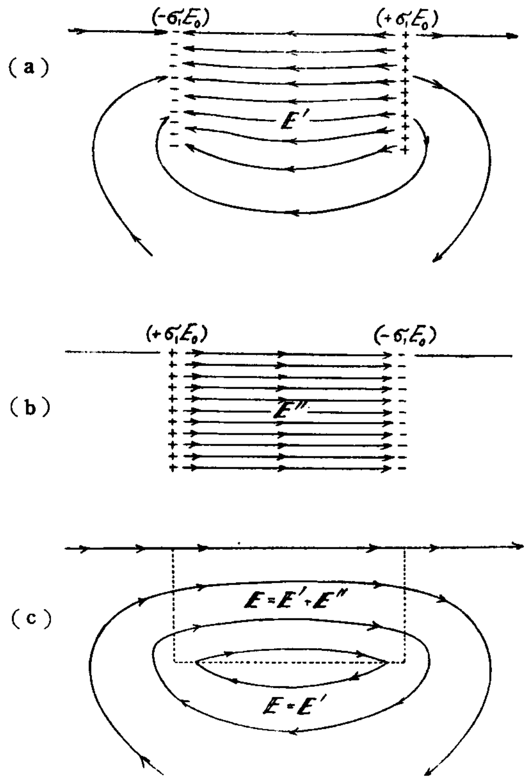

Fig. 2.

B; and the boundary conditions (continuities of V' and j_n) are automatically fulfilled (see Fig. 2 (c)).

This statement can easily be extended to more general cases where the magnetic field is not necessarily vertical and the stream is not constant.

As we have seen above, the fictitious current sources from which the potential V' will be derived, must be placed on the planes of discontinuity of the distributed e.m.f., and the intensity of the surface distrbution of current source should be

$$j_0 = -\sigma_1 \Delta E_{0n}, \tag{3.6}$$

where ΔE_{0n} is the discontinuous increase of the normal component of $\boldsymbol{E}_0$ across the plane of discontinuity. Usually the planes of discontinuity will be the sides, the free surface or the bottom of streams.

If the distributed e.m.f. $\boldsymbol{E}_0$ varies continuously in the domain A (due to the continuous variation of velocity v (x,y)), the fictitious current source should take a form of volume current source, the volume density of which is given by

$$q_0 = -\mathrm{div}\ \sigma_1 \boldsymbol{E}_0, \tag{3.7}$$

or,

$$q_0 = -\sigma_1\left(\frac{\partial E_{0x}}{\partial x}+\frac{\partial E_{0y}}{\partial y}\right). \tag{3.8}$$

It will be evident that the total quantity of fictitious current over the domain A must be zero:

$$\int_S j_0 dS+\int_\tau q_0 d\tau = 0, \tag{3.9}$$

where S denotes the surface of discontinuity and τ the volume in which the velocity varies continuously.

After finding the distribution of fictitious current sources, the potential V' can be obtained by means of integrals of the form:

$$V' = \frac{1}{2\pi\sigma_1}\int_0^\infty dz'\left[\int_{S+S'}\frac{j_0 dS}{r}+\int_{\tau+\tau'}\frac{q_0 d\tau}{r}\right], \tag{3.10}$$

where r is the distance between a current source point (x', y', z'), and a point $(x, y, 0)$ on the xy-plane; and S' and τ' are the electric images of the sources $S(j_0)$ and $\tau(q_0)$ in the free surface $(y=0)$, respectively.

If we assume that the discontinuities of velocity occur only on the boundary planes of the domain A, the potential V' can be given by the formula:

$$V' = -\frac{1}{2\pi\sigma_1}\left[\int_{-h}^{+h} j_0(c,\ y')\log\sqrt{(x-c)^2+(y-y')^2}dy' \right.$$
$$\left. +\int_{-h}^{+h} j_0(-c,\ y')\log\sqrt{(x+c)^2+(y-y')^2}dy' \right.$$

$$+2\int_{-c}^{+c} j_0(x', 0) \log \sqrt{(x-x')^2+y^2}dx'$$
$$+\int_{-c}^{+c} j_0(x', h) \log \sqrt{(x-x')^2+(y-h)^2}dx'$$
$$+\int_{-c}^{+c} j_0(x', -h) \log \sqrt{(x-x')^2+(y+h)^2}dx'$$
$$+\int_{-h}^{+h}\int_{-c}^{+c} q_0(x', y') \log \sqrt{(x-x')^2+(y-y')^2}dx'dy'\Big]. \quad (3.11)$$

Table I shows examples of fictitious current sources for some types of velocity distribution.

Table I.

	Case I	Case II	Case III
Velocity distribution	$v=v_0$, $0<y<h$, $\|x\|<c$.	$v=\begin{cases} v_0, & 0<y<g, \\ v_0\frac{h-y}{h-g}, & g<y<h, \end{cases}$ $\|x\|<c$.	$v=\begin{cases} v_0, & \|x\|<c, \\ v_0\frac{c-\|x\|}{c-d}, & d<\|x\|<c, \end{cases}$ $0<y<h$.
Current sources due to vertical magnetic field (H_y)	$j_0=+\sigma_1 E_{oxm}$, $x=+c$, $0<y<h$.	$j_0=\begin{cases} +\sigma_1 E_{oxm}, & 0<y<g, \\ +\sigma_1 E_{oxm}\frac{h-y}{h-g}, & g<y<h, \end{cases}$ $x=+c$.	$q_0=+\frac{\sigma_1 E_{oxm}}{c-d}$, $d<x<c$, $0<y<h$.
	$j_0=-\sigma_1 E_{oxm}$, $x=-c$, $0<y<h$.	$j_0=\begin{cases} -\sigma_1 E_{oxm}, & 0<y<g, \\ -\sigma_1 E_{oxm}\frac{h-y}{h-g}, & g<y<h, \end{cases}$ $x=-c$.	$q_0=-\frac{\sigma_1 E_{oxm}}{c-d}$, $-c<x<-d$, $0<y<h$.
Current sources due to horizontal magnetic field (H_x)	$j_0=+\sigma_1 E_{oym}$, $y=0$, $-c<x<+c$.	$j_0=+\sigma_1 E_{oym}$, $y=0$, $-c<x<+c$.	$j_0=\begin{cases} +\sigma_1 E_{oym}, & \|x\|<d, \\ +\sigma_1 E_{oym}\frac{c-\|x\|}{c-d}, & d<\|x\|<c, \end{cases}$ $y=0$.
	$j_0=-\sigma_1 E_{oym}$, $y=h$, $-c<x<+c$.	$q_0=-\frac{\sigma_1 E_{oym}}{h-g}$, $g<y<h$, $-c<x<+c$.	$j_0=\begin{cases} -\sigma_1 E_{oym}, & \|x\|<d, \\ -\sigma_1 E_{oym}\frac{c-\|x\|}{c-d}, & d<\|x\|<c, \end{cases}$ $y=h$.

$E_{oxm}=\mu_0 v_0 H_y$, $E_{oym}=\mu_0 v_0 H_x$;
j_0: surface intensity of fictitious current sources,
q_0: volume intensity of fictitious current sources.

4. Two-Layer and Three-Layer Problems

Af first we shall consider the two-layer case where the conductivity of the medium in the domain $0<y<h$ is σ_1 (const.), and the medium in the domain $h<y$ has a different conductivity σ_2 (const.). We assume that a part (A) of the upper layer $(-c<x<+c, 0<y<h)$ is moving parallel to the z-axis with a velocity $v(x, y)$ independent of z (see Fig. 3 (a)).

This problem can be solved, like other boundary value problems, by combining the primary electric field for the case where the conductivity is uniform ($\sigma_2=\sigma_1$), and the secondary electric field due to the lower layer of different conductivity.

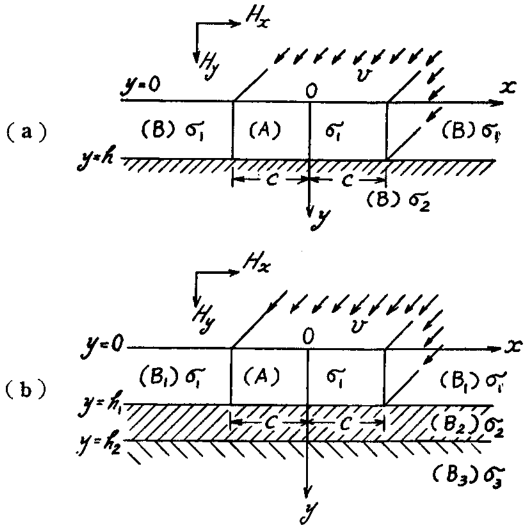

Fig. 3.

Let V_p' be the primary potential when $\sigma_2=\sigma_1$, $V_1{}^*$ the secondary potential in the upper layer ($0<y<h$), and V_2 the resultant potential in the lower layer ($h<y$). The potentials $V_1{}^*$ and V_2 are solutions of Laplace's equation, and generally can be expressed in the forms:

$$\left.\begin{aligned} V^*_{1(x)}&=\int_0^\infty L_x(\lambda)(e^{\lambda y}+e^{-\lambda y})\sin\lambda x d\lambda\,, \\ V^*_{1(y)}&=\int_0^\infty L_y(\lambda)(e^{\lambda y}+e^{-\lambda y})\sin\lambda x d\lambda\,, \end{aligned}\right\}\quad 0<y<h; \tag{4.1}$$

and

$$\left.\begin{aligned} V_{2(x)}&=\int_0^\infty M_x(\lambda)e^{-\lambda y}\sin\lambda x d\lambda\,, \\ V_{2(y)}&=\int_0^\infty M_y(\lambda)e^{-\lambda y}\sin\lambda x d\lambda\,, \end{aligned}\right\}\quad h<y, \tag{4.2}$$

where $V^*_{1(x)}$ and $V_{2(x)}$ are the potentials corresponding to the horizontal component E_{0x} of the distributed e.m.f. (due to the vertical component H_y of the earth's magnetic field), and $V^*_{1(y)}$ and $V_{2(y)}$ represent the potentials corresponding to E_{0y} (due to H_x). When the magnetic field has both vertical and horizontal components, the secondary potential in the upper layer and the resultant potential in the lower layer can be given by

and
$$\left.\begin{aligned} V_1^* &= V^*_{1(x)}+V^*_{1(y)}\,, \\ V_2 &= V_{2(x)}+V_{2(y)}\,, \end{aligned}\right\} \tag{4.3}$$

respectively.

These potentials must satisfy the boundary conditions:

$$\left.\begin{aligned} V_1 &= V_2\,, \\ \sigma_1\frac{\partial V_1}{\partial y} &= \sigma_2\frac{\partial V_2}{\partial y}\,, \end{aligned}\right\}\quad y=h, \tag{4.4}$$

where

$$V_1 = V_p' + V_1^*, \tag{4.5}$$

and V_p' can be assumed to have been obtained by the method described in the preceding section*.

The unknown functions $L(\lambda)$ and $M(\lambda)$ in Eqs. (4.1) and (4.2) can be determined by the conditions of Eq. (4.4); and the electric fields and current densities can be deduced as follows:

$$\left.\begin{aligned} \boldsymbol{E}_1' &= -\text{grad } V_1; \\ \boldsymbol{E}_1 &= \boldsymbol{E}_0 + \boldsymbol{E}_1', \ -c < x < +c, \\ &= \boldsymbol{E}_1', \ c < |x|; \\ \boldsymbol{j} &= \sigma_1 \boldsymbol{E}_1, \end{aligned}\right\} \ 0 < y < h, \tag{4.6}$$

and

$$\left.\begin{aligned} \boldsymbol{E}_2 &= -\text{grad } V_2, \\ \boldsymbol{j}_2 &= \sigma_2 \boldsymbol{E}_2, \end{aligned}\right\} \ h < y. \tag{4.7}$$

If the current lines are to be drawn, we must know the flux of current flowing through a vertical (or horizontal) plane, say $x = x_1$, $0 < y < y_1$. This is given by

$$\left.\begin{aligned} J &= \sigma_1 \int_0^{y_1} E_{1x} dy, \quad 0 < y_1 < h, \\ \text{or} \quad J &= \sigma_1 \int_0^h E_{1x} dy + \sigma_2 \int_h^{y_1} E_{2x} dy, \quad h < y_1. \end{aligned}\right\} \tag{4.8}$$

The locus $J(x_1, y_1) = \text{const.}$ will give a stream line. Usually, it is convenient to take the constant as

$$J(x_1, y_1) = \frac{n}{N} J_t, \quad (n = 0, 1, 2, \cdots, N.), \tag{4.9}$$

where N is an integer and J_t—obtained as the maximum (or minimum) value of the flux J—is the total current circulating through the semi-infinite space. The coordinates (x_m, y_m) at which J becomes maximum (or minimum), will give the center of circulation of the current.

The method described above can be extended to the problems of three or more layers. Suppose a three-layer structure in which the upper layer $(0 < y < h_1)$, the intermediate layer $(h_1 < y < h_2)$ and the lower layer of infinite thickness $(h_2 < y)$ have different conductivities σ_1, σ_2 and σ_3, respectively (see Fig. 3 (b)). The potential due to the vertical magnetic field in each layer can be expressed as follows:

* The primary potential V_p' in Eq. (4.5) must be expressed in the integral form similar to Eq. (4.1). Usually, the integral of Eq. (3.11) does not yield directly the expression of such a form. However, the appropriate integral expression for V_p' (or $\boldsymbol{E}_p'$) can be obtained easily, if the velocity distribution $v(x, y)$ is one of those in Table I.

Three examples for the case of the vertical magnetic field have been given in Part I, and those for the horizontal magnetic field will be shown in Appendix II of this paper.

$$\left.\begin{aligned} V_1 &= V_p'+V_1^*, \\ V_1^* &= \int_0^\infty L(\lambda)(e^{\lambda y}+e^{-\lambda y}) \sin \lambda x d\lambda , \end{aligned}\right\} \quad 0<y<h_1 ; \tag{4.10}$$

$$V_2 = \int_0^\infty [M(\lambda)e^{\lambda y}+N(\lambda)e^{-\lambda y}] \sin \lambda x d\lambda , \quad h_1<y<h_2 ; \tag{4.11}$$

$$V_3 = \int_0^\infty P(\lambda)e^{-\lambda y} \sin \lambda x d\lambda , \quad h_2<y . \tag{4.12}$$

The potentials due to the horizontal magnetic field are given in similar forms except that cos λx enters instead of sin λx.

The unknown functions L, M, N and P can be determined by the following boundary conditions:

$$\left.\begin{aligned} V_1 = V_2, \ \sigma_1\frac{\partial V_1}{\partial y} = \sigma_2\frac{\partial V_2}{\partial y}, \quad \text{when} \quad y = h_1, \\ V_2 = V_3, \ \sigma_2\frac{\partial V_2}{\partial y} = \sigma_3\frac{\partial V_3}{\partial y}, \quad \text{when} \quad y = h_2. \end{aligned}\right\} \tag{4.13}$$

If we put $\sigma_1=\sigma_2(h_1<h_2)$, the problem of shallow streams[2),5)] can be treated.

5. Current Distribution

Some examples of current lines due to the vertical magnetic field have been shown in the previous paper. We shall consider here the current lines for the case of a uniform medium ($\sigma_2=\sigma_1$) and constant stream ($v=v_0$) flowing across the magnetic field which has a horizontal component as well as a vertical component.

As we have already displayed in Part I the formulas for the potential, electric field and current due to the vertical magnetic field, only the formulas for the case of a horizontal magnetic field will be shown below:

$$\begin{aligned} V' = -\frac{E_{0ym}}{4\pi}[&2M(c-x,\ y)+2M(c+x,\ y)-M(c-x,\ h-y) \\ &-M(c+x,\ h-y)-M(c-x,\ h+y)-M(c+x,\ h+y)], \end{aligned} \tag{5.1}$$

$$\begin{aligned} E_x = \frac{E_{0ym}}{4\pi}\Bigg[&2\log\frac{(c+x)^2+y^2}{(c-x)^2+y^2} - \log\frac{(c+x)^2+(h-y)^2}{(c-x)^2+(h-v)^2} \\ &- \log\frac{(c+x)^2+(h+y)^2}{(c-x)^2+(h+y)^2}\Bigg], \end{aligned} \tag{5.2}$$

$$\begin{aligned} E_y' = \frac{E_{0ym}}{2\pi}\Bigg[&2\left(\tan^{-1}\frac{c-x}{y} + \tan^{-1}\frac{c+x}{y}\right)+\left(\tan^{-1}\frac{c-x}{h-y} +\tan^{-1}\frac{c+x}{h-y}\right) \\ &-\left(\tan^{-1}\frac{c-x}{h+y} +\tan^{-1}\frac{c+x}{h+y}\right)\Bigg], \end{aligned} \tag{5.3}$$

$$E_y = \begin{cases} -E_{0ym}+E_y' ; & -c<x< +c,\ 0<y<h , \\ E_y' ; & \text{elsewhere}, \end{cases} \tag{5.4}$$

$$\begin{aligned} J = \frac{\sigma_1 E_{0ym}}{4\pi}[&2H(c-x,\ y)-2H(c+x,\ y)+H(c-x,\ h-y) \\ &-H(c+x,\ h-y)-H(c-x,\ h+y)+H(c+x,\ h+y)], \end{aligned} \tag{5.5}$$

where

$$\left.\begin{aligned} M(\xi,\,\eta) &= \xi \log (\xi^2+\eta^2)+2\eta \tan^{-1}(\xi/\eta), \\ H(\xi,\,\eta) &= \eta \log (\xi^2+\eta^2)+2\xi \tan^{-1}(\eta/\xi). \end{aligned}\right\} \tag{5.6}$$

Two sets of current lines given by J=const. are shown in Figs. 4 (a)* and (b), for the cases where (a) the magnetic field is vertical ($H_y=H_0$, $H_x=0$) and (b) horizontal ($H_x=H_0$, $H_y=0$). In Fig. 4 (b), the e.m.f. due to electromagnetic induction is directed vertically in the domain A, and a pair of eddies is produced symmetrically on both sides of the stream. The centers of circulation of the current are on the lines $x=\pm c$ $(0<y<h)$.

Combining these two results we can draw a set of current lines in case both components of the magnetic field exist. As an example Fig. 5 shows the case where $H_x=H_y=H_0$ (i.e. the dip angle of the earth's magnetic field is equal to 45°, and the

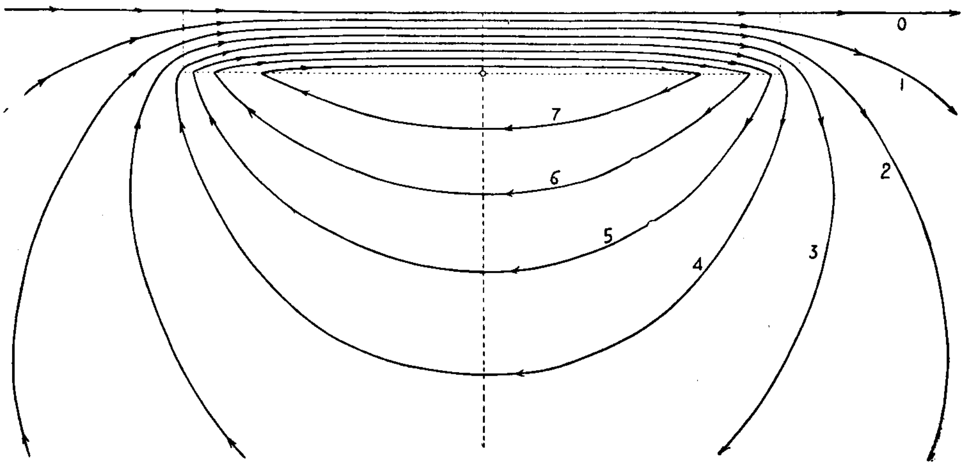

Fig. 4 (a)

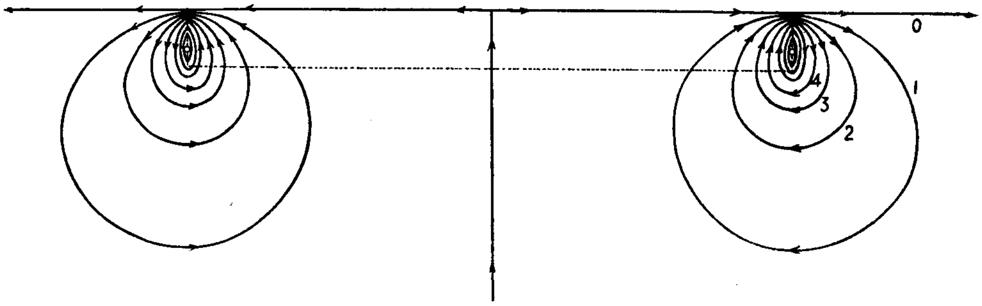

Fig. 4 (b)

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* Fig. 4(a) shows the same distribution of current as in Fig. 5 of Part I, but in a more detailed form.

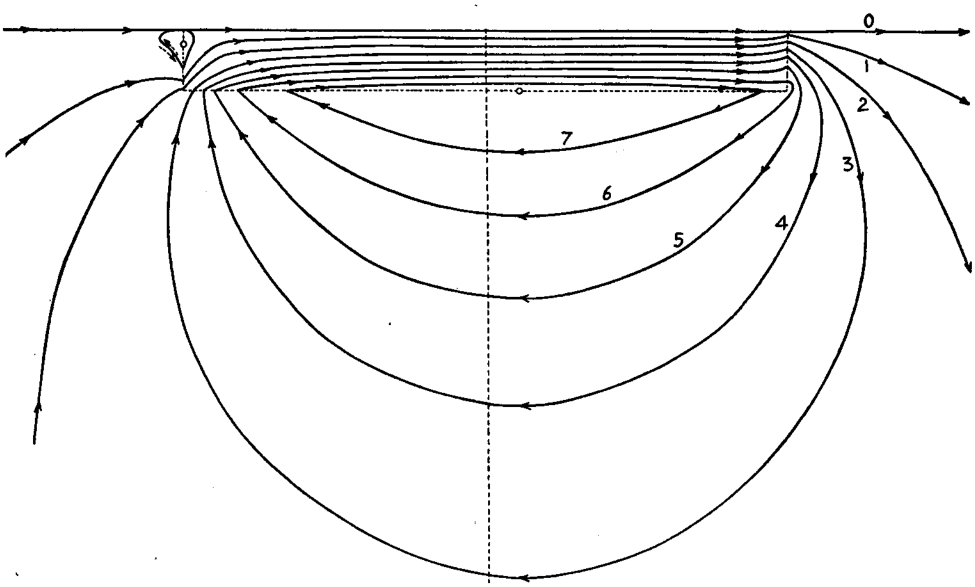

Fig. 5.

stream of water is directed from west to east). In this case, the total ourrent J_t is slightly larger than the value when the magnetic field is vertical ($H_x=0$), and the center of circulation of the current is shifted toward the right from the point $x=0$, $y=h$. It should be noted that a weak eddy current exists near the left edge of the stream, in which the electric current circulates in the direction opposite to the main circulation.

6. Distribution of Electric Field

As shown in Part I, the intensity of the electric field may in some cases become infinite at the corners of the cross-section of the stream. The singularities which occur in some typical distributions of velocity are listed below:

Table II.

		Vertical magnetic field (H_y)	Horizontal magnetic field (H_x)
Case I	$v=v_0$ const. stream	$E_y=\pm\infty$ at $x=\pm c$, $y=h$.	$E_x=\pm\infty$ at $x=\pm c$ $y=0$ and h.
Case II	$v=v_0$, $0<y<g$; $v=v_0\frac{h-y}{h-g}$, $g<y<h$.	E_x and E_y are finite everywhere.	$E_x=\pm\infty$ at $x=\pm c$, $y=0$.
Case III	$v=v_0$, $\|x\|<d$; $v=v_0\frac{c-\|x\|}{c-d}$, $d<\|x\|<c$.	E_x and E_y are finite everywhere.	E_x and E_y are finite everywhere.

It will be clear that the case III is a very suitable model to discuss the effect of vertical and horizontal components of a magnetic field on the electric field and current density at the surface or on the bottom of the sea*. In this case, the electric potential and magnetic fields due to a horizontal magnetic field in a uniform medium ($\sigma_2=\sigma_1$) are expressed as follows:

$$V' = -\frac{E_{0ym}}{4\pi(c-d)}[2D(c-x,\ y)+2D(c+x,\ y)-2D(d-x,\ y)-2D(d+x,\ y)$$
$$-D(c-x,\ h-y)-D(c+x,\ h-y)-D(c-x,\ h+y)-D(c+x,\ h+y)$$
$$+D(d-x,\ h-y)+D(d+x,\ h-y)+D(d-x,\ h+y)+D(d+x,\ h+y)], \quad (6.1)$$

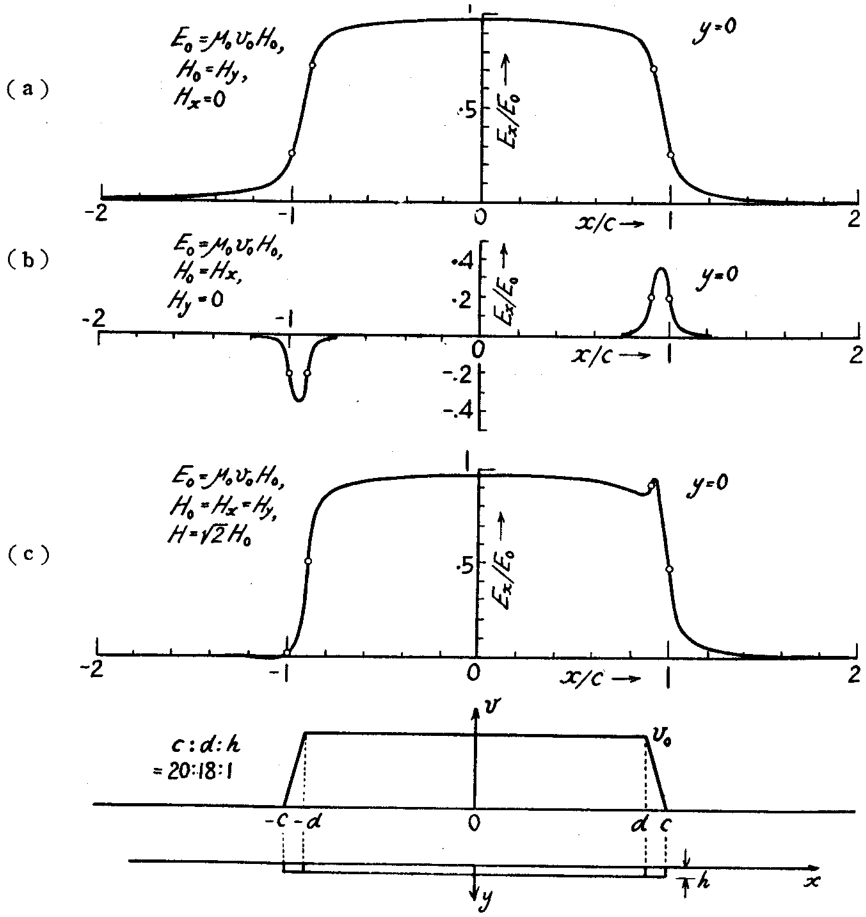

Fig. 6.

* We can also analyse the case where the velocity diminishes toward the bottom and both sides of stream. This model will be more suitable to fit practical cases. However, we will not discuss it here, since the expressions for potential and field are rather lengthy, and numerical computations will be very tedious.

$$E_x = -\frac{E_{0ym}}{4\pi(c-d)}[2M(c-x,\ y)-2M(c+x,\ y)-2M(d-x,\ y)+2M(d+x,\ y)$$
$$-M(c-x,\ h-y)+M(c+x,\ h-y)-M(c-x,\ h+y)+M(c+x,\ h+y)$$
$$+M(d-x,\ h-y)-M(d+x,\ h-y)+M(d-x,\ h+y)-M(d+y,\ h+y)], \quad (6.2)$$

$$E_y = -\frac{E_{0ym}}{4\pi(c-d)}[2H(c-x,\ y)+2H(c+x,\ y)-2H(d-x,\ y)-2H(d+x,\ y)$$
$$+H(c-x,\ h-y)+H(c+x,\ h-y)-H(c-x,\ h+y)-H(c+x,\ h+y)$$
$$-H(d-x,\ h-y)-H(d+x,\ h-y)+H(d-x,\ h+y)+H(d+x,\ h+y)], \quad (6.3)$$

where

$$D(\xi,\ \eta) = \frac{1}{2}(\xi^2-\eta^2)\log(\xi^2+\eta^2)+2\xi\eta\tan^{-1}(\xi/\eta). \quad (6.4)$$

Figs. 6 (a) and (b) show the horizontal components of the electric field at the surface $y=0$ due to the vertical and horizontal magmetic fields (H_y and H_x) respectively; and Fig. 6 (c) shows the horizontal electric fields due to the resultant magnetic

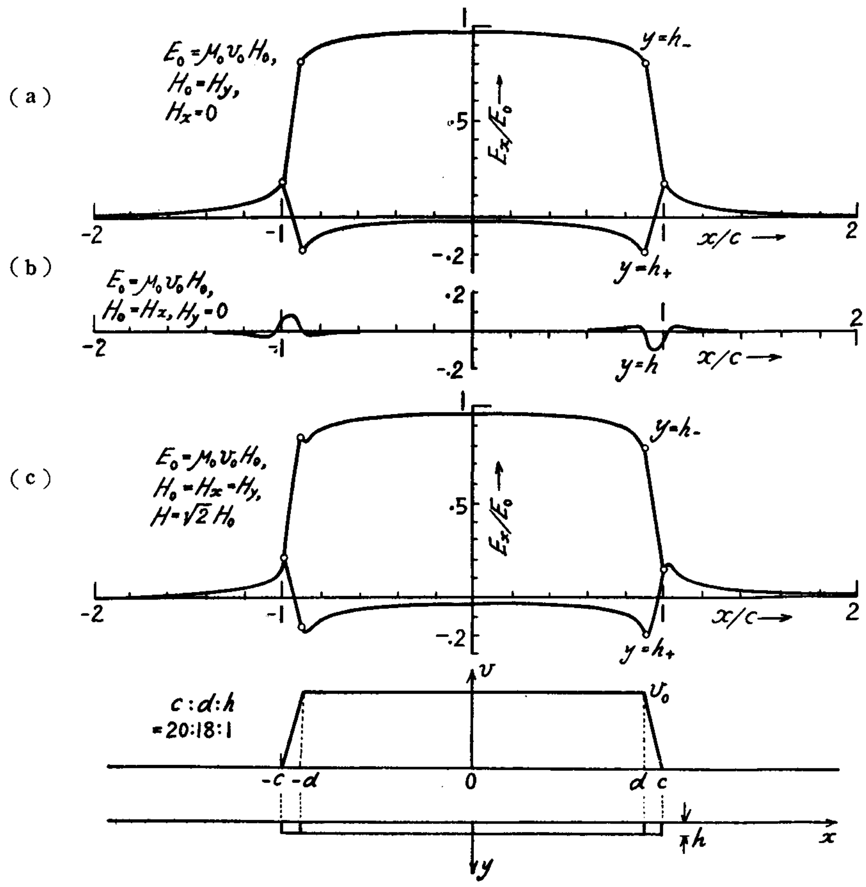

Fig. 7.

field, where $H_x=H_y=H_0$. In Figs. 7 (a), (b) and (c) the electric field distributions on the bottom $y=h$ are shown for the same conditions as in Fig. 6. From these curves it can be seen that the horizontal component of the magnetic field will produce some assymmetries in the electric field, but its effect on the resultant intensity of electric field is not material.

For two-layer structures, the horizontal electric fields in both layers induced by a vertical magnetic field are given by[3]:

$$\left.\begin{aligned} E_{1x} &= \frac{k'E_{0xm}}{\pi(c-d)}\int_0^\infty \frac{e^{-\lambda h}}{1-ke^{-2\lambda h}}(e^{\lambda y}+e^{-\lambda y})(\cos\lambda d-\cos\lambda c)\cos\lambda x\frac{d\lambda}{\lambda^2}, \\ E_{2x} &= -\frac{k''E_{0xm}}{\pi(c-d)}\int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-\lambda h}}e^{-\lambda(y-h)}(\cos\lambda d-\cos\lambda c)\cos\lambda x\frac{d\lambda}{\lambda^2}, \end{aligned}\right\} \quad \text{(due to } H_y\text{)}. \quad (6.5)$$

On the other hand, the horizontal components of the electric field due to a horizontal magnetic field are expressed as follows:

$$\left.\begin{aligned} E_{1x} &= E_{px}+E_{1x}^*, \\ E_{px} &= \frac{E_{0ym}}{\pi(c-d)}\int_0^\infty (2e^{-\lambda y}-e^{-\lambda(h-y)}-e^{-\lambda(h+y)})(\cos\lambda d-\cos\lambda c)\sin\lambda x\frac{d\lambda}{\lambda^2}, \\ E_{1x}^* &= -\frac{kE_{0ym}}{\pi(c-d)}\int_0^\infty \frac{e^{-\lambda h}(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}}(e^{\lambda y}+e^{-\lambda y})(\cos\lambda d-\cos\lambda c)\sin\lambda x\frac{d\lambda}{\lambda^2}, \\ E_{2x} &= -\frac{k'E_{0ym}}{\pi(c-d)}\int_0^\infty \frac{e^{-\lambda(y-h)}(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}}(\cos\lambda d-\cos\lambda c)\sin\lambda x\frac{d\lambda}{\lambda^2}, \end{aligned}\right\} \quad \text{(due to } H_x\text{)}, \quad (6.6)$$

where

$$k=\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},\quad k'=1-k,\quad k''=1+k. \quad (6.7)$$

For our problem of faults on submarine cables[4], it seems to be most important to know the horizontal components of the electric field on the bottom, especially E_{2x} $(y=h_+)$. The expressions for these components are as follows:

$$\left.\begin{aligned} E_{2x} &= -\frac{k''E_{0xm}}{\pi(c-d)}\int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-2\lambda h}}(\cos\lambda d-\cos\lambda c)\cos\lambda x\frac{d\lambda}{\lambda^2}, \\ E_{1x} &= E_{0x}+E_{2x}, \qquad |x|<c, \\ &= E_{2x}, \qquad \text{elsewhere}; \end{aligned}\right\} \quad y=h, \quad \text{(due to } H_y\text{)}, \quad (6.8)$$

$$E_{2x}=E_{1x}=-\frac{k''E_{0ym}}{\pi(c-d)}\int_0^\infty \frac{(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}}(\cos\lambda d-\cos\lambda c)\sin\lambda x\frac{d\lambda}{\lambda^2},\quad y=h, \quad \text{(due to } H_x\text{)}. \quad (6.9)$$

These integrals could be computed by expanding the factor $1/(1-ke^{-2\lambda h})$ into a series, as has been done in the previous work[3),5)]. However, expansions of the type:

$$(1-ke^{-2\lambda h})^{-1} = \sum_{n=0}^{\infty} k^n e^{-2n\lambda h}$$

will yield an infinite series of very slow convergence, especially when the parameter k is nearly equal to unity. This is unfortunately the case in practical situations. To avoid this difficulty, we expand the factors:

$$(1-e^{-2\lambda h})/(1-ke^{-2\lambda h}) \quad \text{and} \quad (1-e^{-\lambda h})^2/(1-ke^{-2\lambda h})$$

in Eqs. (6.8) and (6.9) by means of a system of orthogonal functions. The details of this method will be discussed in Appendix I.

Fig. 8 shows the results of numerical computations for the case of two layer

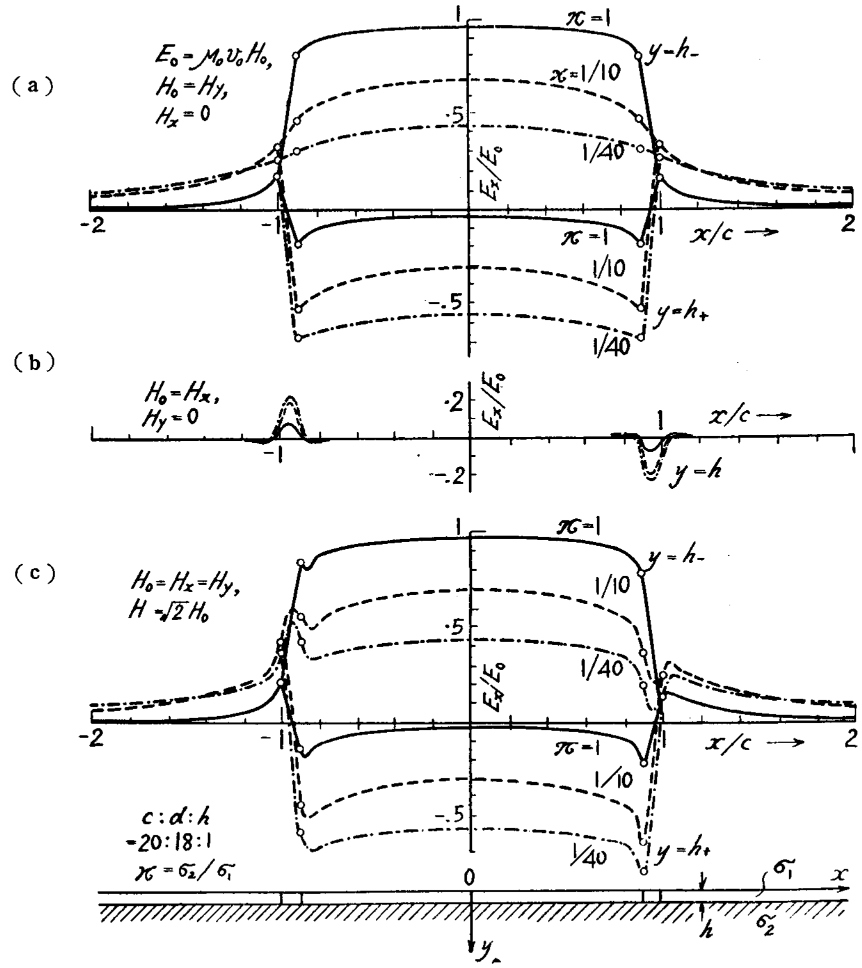

Fig. 8.

structures, in which the ratio of the conductivities are chosen as 1 : 1, 1 : 0.1 and 1 : 0.025. Also, in this case the effect of the horizontal magnetic field is not very large. However, the electric field varies in a rather complicated manner, and a sharp negative peak is seen on each curve of Fig. 8 (c). Although the values of $\kappa = \sigma_2/\sigma_1$ chosen here are not necessarily practical ones, the dependence of the electric field on the conductivity of sea beds will be understood to some extent.

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Appendix I. Computation of Some Integrals

To compute the value of the integrals in Eqs. (6.8) and (6.9):

$$I_1 = \int_0^\infty \frac{1-e^{-2\lambda h}}{1-ke^{-2\lambda h}} (\cos\lambda d - \cos\lambda c)\cos\lambda x \frac{d\lambda}{\lambda^2}, \tag{1}$$

and

$$I_2 = \int_0^\infty \frac{(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}} (\cos\lambda d - \cos\lambda c)\sin\lambda x \frac{d\lambda}{\lambda^2}, \tag{2}$$

we shall consider a method of expanding the functinns:

$$H(\zeta) = \frac{1-\zeta}{1-k\zeta}, \quad \zeta = e^{-2\lambda h}, \quad 0<\zeta<1, \tag{3}$$

and

$$G(\zeta) = \frac{(1-\eta)^2}{1-k\eta^2}, \quad \eta = e^{-\lambda h}, \quad 0<\eta<1, \tag{4}$$

by means of an appropriate system of orthogonal functions*. For this purpose it is required to obtain a system of polynomials $f_r(x)$ of order r, each of which is orthogonal over (0,1) to all polynomials $f_s(x)$, $s<r$, and normalized over (0,1):

$$\left.\begin{aligned} \int_0^1 f_r(x)f_s(x)dx &= 0, \quad s<r; \\ &= 1, \quad s=r. \end{aligned}\right\} \tag{5}$$

From these conditions we get the following functions:

$$\left.\begin{aligned} f_0(x) &= 1, \\ f_1(x) &= \sqrt{3}\,(1-2x), \\ f_2(x) &= \sqrt{5}\,(1-6x+6x^2), \\ f_3(x) &= \sqrt{7}\,(1-12x+30x^2-20x^3), \\ f_4(x) &= \sqrt{9}\,(1-20x+90x^2-140x^3+70x^4), \\ f_5(x) &= \sqrt{11}(1-30x+210x^2-560x^3+630x^4-252x^5), \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{6}$$

It can be easily shown that the following relation exists between successive three polynomials:

$$g_n(x) = g_{n-2}(x) - 2(2n-1)\int_0^x g_{n-1}(x)dx, \tag{7}$$

where

* F. B. Hildebrand: Introduction to Numerical Analysis (1956), 269-272.

$$g_n(x) = \frac{1}{\sqrt{2n+1}} f_n(x). \tag{8}^*$$

Now, by means of the system of orthogonal functions $f_n(\zeta)$, we can expand the function $H(\zeta)$ as follows:

$$H(\zeta) = \sum_{n=0}^{\infty} a_n f_n(\zeta), \tag{9}$$

where

$$a_n = \int_0^1 H(\zeta) f_n(\zeta) d\zeta, \tag{10}$$

or

$$a_n = \sqrt{2n+1} \sum_{m=0}^{n} c_{nm} b_m, \tag{11}$$

$$b_m = \int_0^1 H(\zeta) \zeta^m d\zeta, \tag{12}$$

and c_{nm} is the coefficient of ζ^m in $g_n(\zeta)$ defined by Eq. (8), i.e.

$$f_n(\zeta) = \sqrt{2n+1} \sum_{m=0}^{n} c_{nm} \zeta^m, \quad c_{no} = 1. \tag{13}$$

The results of integration of Eq. (12) are as follows:

$$\left.\begin{aligned}
b_0 &= \frac{1}{k}\left(1 + \frac{k'}{k}\log k'\right), \\
b_1 &= \frac{1}{k}\left(\frac{1}{2} + \frac{k'}{k} + \frac{k'}{k^2}\log k'\right), \\
b_2 &= \frac{1}{k}\left(\frac{1}{3} + \frac{1}{2}\frac{k'}{k} + \frac{k'}{k^2} + \frac{k'}{k^3}\log k'\right), \\
b_3 &= \frac{1}{k}\left(\frac{1}{4} + \frac{1}{3}\frac{k'}{k} + \frac{1}{2}\frac{k'}{k^2} + \frac{k'}{k^3} + \frac{k'}{k^4}\log k'\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{14}$$

Rearranging the terms in Eq. (9), we obtain:

$$H(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^n, \tag{15}$$

---

* Dr. Maeda used another system of orthogonal functions:

$$f_1(x) = \sqrt{3}\,x, \qquad f_2(x) = \sqrt{5}(-3x+4x^2),$$
$$f_3(x) = \sqrt{7}(6x-20x^2+15x^3), \quad \dots \dots \dots$$

in order to calculate the integrals:

$$\int_0^\infty \frac{ke^{-2\lambda d}}{1-ke^{-2\lambda d}} J_0(\lambda r) d\lambda$$

and

$$\int_0^\infty \frac{k_1 e^{-2\lambda d_1} + k_2 e^{-2\lambda(d_1+d_2)}}{1 - k_1 e^{-2\lambda d_1} - k_2 e^{-2\lambda(d_1+d_2)} + k_1 k_2 e^{-2\lambda d_2}} J_0(\lambda r) d\lambda,$$

which appear in the two-layer and threr-layer problems of electrical prospecting. (see K. Maeda: Method of Calculation for Electric Resistance Prospecting, Bulletin of the Railway Technical Laboratory, **6**, No. 6/7 (Nov./Dec., 1949), 11 and 66–68.)

where

$$
\left.\begin{aligned}
A_0 &= a_0c_{00}+a_1c_{10}+a_2c_{20}+\cdots\cdots ,\\
A_1 &= a_1c_{11}+a_2c_{21}+a_3c_{31}+\cdots\cdots ,\\
A_2 &= a_2c_{22}+a_3c_{32}+a_4c_{42}+\cdots\cdots ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{16}
$$

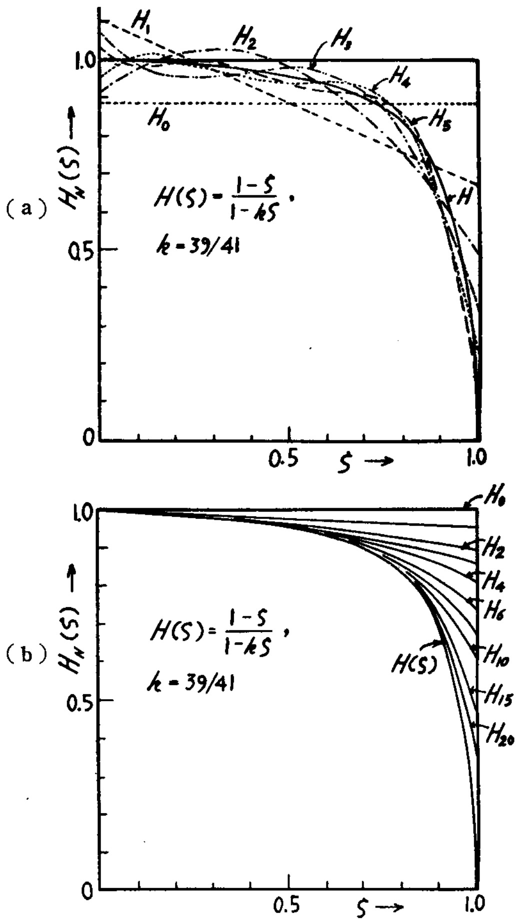

Fig. A-1.

To illustrate the accuracy of expansion of Eq. (15), we shall chose a value $k=39/41$, corresponding to $\sigma_2/\sigma_1=1/40$, and compare the approximate values

$$H_N(\zeta)=\sum_{n=0}^{N}A_n\zeta^n \tag{17}$$

with the true value of Eq. (3). As shown in Fig. A-1 (a), $H_4(\zeta)$ is already fairly close to the accurate curve of $H(\zeta)$, whereas the binomial expansion

$$H_N(\zeta)=\sum_{n=0}^{N}k^n\zeta^n \tag{18}$$

does not give a good approximation even if we allow N to reach 10, except for the region of ζ small, as can be seen from Fig. A-1 (b).

After obtaining the expansion of Eq. (15), we can write down Eq. (1) in a computable form:

$$
\begin{aligned}
I_1 = \frac{1}{4}\sum_{n=0}^{\infty}A_n[&-J(c-x,\ 2nh)\\
&-J(c+x,\ 2nh)+J(d-x,\ 2nh)\\
&+J(d+x,\ 2nh)],
\end{aligned} \tag{19}
$$

where

$$J(\xi,\eta)=\eta\log(\xi^2+\eta^2)-2\xi\tan^{-1}(\xi/\eta). \tag{20}$$

We can obtain another formula for the integral I_1 by means of Eq. (18):

$$
\begin{aligned}
I_1 = \frac{1}{4}k'\sum_{n=1}^{\infty}k^{n-1}[&H(c-x,\ 2nh)+H(c+x,\ 2nh)\\
&-H(d-x,\ 2nh)-H(d+x,\ 2nh)],
\end{aligned} \tag{21}
$$

or

$$I_1 = \frac{1}{4} k' \sum_{n=1}^{\infty} k^{n-1} [J(c-x,\ 2nh) + J(c+x,\ 2nh) - J(d-x,\ 2nh) - J(d+x,\ 2nh)] + \frac{\pi}{2} \begin{cases} (c-d), & |x|<d, \\ (c-x), & d<|x|<c, \\ 0, & c<|x|. \end{cases} \tag{22}$$

Although Eq. (19) looks similar to Eq. (22), the convergence of the former is much faster than that of the latter. For example, the accurate value E_{2x} and approximate values $E_{2x}^{(N)}$ at $x=0$, $y=h$, for $k=39/41$, will become as follows, according to the number of terms N of Eq. (19):

$$E_{2x}/E_0 = -0.559; \quad N=1: \ E_{2x}^{(1)}/E_0 = -0.685,$$
$$2: \ E_{2x}^{(2)}/E_0 = -0.575,$$
$$3: \ E_{2x}^{(3)}/E_0 = -0.538,$$
$$4: \ E_{2x}^{(4)}/E_0 = -0.542,$$
$$5: \ E_{2x}^{(5)}/E_0 = -0.559.$$

whereas, Eq. (22) will not give the same accuracy even if as many as 50 terms are taken into accunt. However, it is not easy to take many terms when Eq. (19) is used, because the determination of the coefficients A_n for larger n will become more and more tedious as n increases. On the other hand, the terms for very large n in Eq. (21) or (22) are obtained so simply, that the sum of terms beyond a certain values of n, say $n>50$, is not difficult to compute. Therefore, if a very accurate value of the integral is wanted, it seems better to use Eq. (21) or (22). The value of E_{2x}/E_0 shown above has been obtained in such a way (see Fig. A-2).

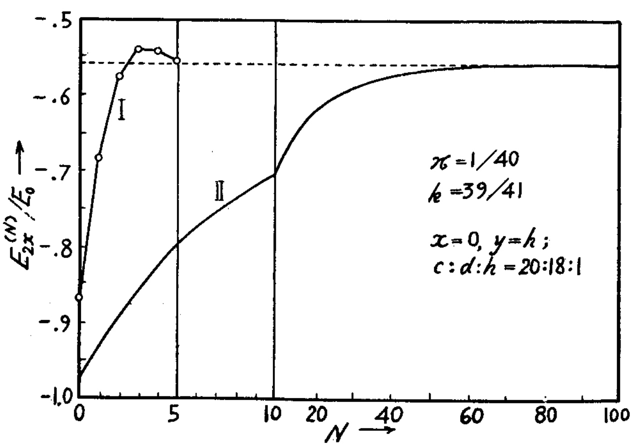

Fig. A-2.

As for Eq. (4), the same type of expansion can be obtained, by means of the orthogonal system given by Eqs. (6):

$$G(\eta) = \sum_{n=0}^{\infty} a_n f_n(\eta), \tag{23}$$

where

$$\left.\begin{aligned} a_n &= \int_0^1 G(\eta) f_n(\eta) d\eta\,, \\ &= \sqrt{2n+1} \sum_{m=0}^{n} c_{nm} b_m\,, \\ b_n &= \int_0^1 G(\eta) \eta^m d\eta\,, \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} b_0 &= \frac{1}{k}(p-1)\,, \\ b_1 &= \frac{1}{k}\left(-q+\frac{3}{1.2}\right), \\ b_2 &= \frac{1}{k}\left(\frac{p}{k}+\frac{4}{2.3}-\frac{k''}{k}\right), \\ b_3 &= \frac{1}{k}\left(-\frac{q}{k}+\frac{5}{3.4}-\frac{1}{2}\frac{k''}{k}+\frac{2}{k}\right), \\ b_4 &= \frac{1}{k}\left(\frac{p}{k^2}+\frac{6}{4.5}-\frac{1}{3}\frac{k''}{k}+\frac{1}{2}\frac{2}{k}-\frac{k''}{k^2}\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{25}$$

where

$$\left.\begin{aligned} p &= \log k' + \frac{k''}{2\sqrt{k}} \log\frac{1+\sqrt{k}}{1-\sqrt{k}}\,, \\ q &= \frac{k''}{2k}\log k' + \frac{1}{\sqrt{k}}\log\frac{1+\sqrt{k}}{1-\sqrt{k}}\,, \end{aligned}\right\} \quad k>0 \quad (\sigma_2<\sigma_1)\,, \tag{26}$$

$$\left.\begin{aligned} p &= \log k' + \frac{k''}{\sqrt{-k}} \tan^{-1}\sqrt{-k}\,, \\ q &= \frac{k''}{2k}\log k' + \frac{2}{\sqrt{-k}}\tan^{-1}\sqrt{-k}\,, \end{aligned}\right\} \quad k<0 \quad (\sigma_2>\sigma_1)\,. \tag{27}$$

The coefficients in the expansion

$$G(\eta) = \sum_{n=0}^{\infty} A_n \eta^n \tag{28}$$

are given by the same form as in Eq. (16). Since the approximation in this case is closer than in the former case, the value of N can be reduced to 3 or 2, for $k \simeq 0.95$.

Appendix II. Listing of Formulas for Electric Field due to Horizontal Magnetic Field.

Case I.

$$\left.\begin{aligned} \text{Velocity distribution:}\quad v &= v_0\,, \quad |x|<c\,,\ 0<y<h\,; \\ &= 0\,, \quad \text{elsewhere.} \end{aligned}\right\} \tag{I.1}$$

Induced e.m.f. per unit distance:

$$\left.\begin{aligned} E_{0y} &= -E_0 = -\mu_0 v_0 H_x\,, \quad |x|<c\,,\ 0<y<h\,; \\ &= 0\,, \quad \text{elsewhere.} \end{aligned}\right\} \tag{I.2}$$

1.1. Uniform Medium

The formulas for potential, field and current have been given in Section 5 of this paper; see Eqs. (5.1)-(5.6).

1.2. Two-Layer Structure

$$\left.\begin{aligned} E_{1x} &= E_{px}+E_{1x}^{*}, \\ E_{px} &= -\frac{E_0}{\pi}\int_0^\infty (e^{-\lambda(h-y)}+e^{-\lambda(h+y)}-2e^{-\lambda y})\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda}, \quad y<h, \\ E_{1x}^{*} &= -k\frac{E_0}{\pi}\int_0^\infty \frac{(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}}e^{-\lambda h}(e^{\lambda y}+e^{-\lambda y})\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda}. \end{aligned}\right\} \quad (\text{I. }3)$$

$$\left.\begin{aligned} E_{1y} &= E_{py}+E_{1y}^{*}, \\ E_{py} &= -\frac{E_0}{\pi}\int_0^\infty [2(1-e^{-\lambda y})-(e^{-\lambda(h-y)}-e^{-\lambda(h+y)})]\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda}, \quad y<h, \\ E_{1y}^{*} &= k\frac{E_0}{\pi}\int_0^\infty \frac{(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}}e^{-\lambda h}(e^{\lambda y}-e^{-\lambda y})\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda}. \end{aligned}\right\} \quad (\text{I. }4)$$

$$\begin{pmatrix} E_{2x} \\ E_{2y} \end{pmatrix} = -k''\frac{E_0}{\pi}\int_0^\infty \frac{(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}}e^{-\lambda(y-h)}\sin\lambda c\begin{pmatrix}\sin\lambda x \\ \cos\lambda x\end{pmatrix}\frac{d\lambda}{\lambda}. \quad (\text{I. }5)$$

Case II.

Velocity distribution:

$$\left.\begin{aligned} v &= v_0, \quad 0<y<g, \\ &= v_0\frac{h-y}{h-g}, \quad g<y<h, \end{aligned}\right\} |x|<c; \quad \\ \quad = 0, \quad \text{elsewhere}, \qquad (\text{II. }1)$$

Induced e.m.f. pre unit distance:

$$\left.\begin{aligned} E_{0y} &= -E_0 = -\mu_0 v_0 H_x, \quad 0<y<g, \\ &= -E_0\frac{h-y}{h-g}, \quad g<y<h, \end{aligned}\right\} |x|<c; \quad \\ \quad = 0, \quad \text{elsewhere}. \qquad (\text{II. }2)$$

2.1. Uniform Medium

$$\begin{aligned} V' = &-\frac{E_0}{2\pi}[M(c-x,\ y)+M(c+x,\ y)] \\ &+\frac{E_0}{4\pi(h-g)}[K(c-x,\ h-y)+K(c+x,\ h-y) \\ &\qquad +K(c-x,\ h+y)+K(c+x,\ h+y) \\ &\qquad -K(c-x,\ g-y)-K(c+x,\ g-y) \\ &\qquad -K(c-x,\ g+y)-K(c+x,\ g+y)]. \end{aligned} \quad (\text{II. }3)$$

$$\begin{aligned} E_x = &\frac{E_0}{2\pi}\log\frac{(c+x)^2+y^2}{(c-x)^2+y^2} \\ &+\frac{E_0}{4\pi(h-g)}[H(c-x,\ h-y)-H(c+x,\ h-y) \\ &\qquad +H(c-x,\ h+y)-H(c+x,\ h+y) \\ &\qquad -H(c-x,\ g-y)+H(c+x,\ g-y) \\ &\qquad -H(c-x,\ g+y)+H(c+x,\ g+y)], \end{aligned} \quad (\text{II. }4)$$

$$E_y = \frac{E_0}{\pi}\left(\tan^{-1}\frac{c-x}{y} + \tan^{-1}\frac{c+x}{y}\right)$$
$$+ \frac{E_0}{4\pi(h-g)}[M(c-x,\ h-y)+M(c+x,\ h-y)$$
$$-M(c-x,\ h+y)-M(c+x,\ h+y)$$
$$-M(c-x,\ g-y)-M(c+x,\ g-y)$$
$$+M(c-x,\ g+y)+M(c+x,\ g+y)]$$
$$-\begin{cases} E_0, & 0<y<g, \\ E_0\dfrac{h-y}{h-g}, & g<y<h, \\ 0, & \text{elsewhere.} \end{cases} \quad |x|<c, \qquad \text{(II. 5)}$$

$$J = -\frac{\sigma_1 E_0}{2\pi}[H(c-x,\ y)-H(c+x,\ y)]$$
$$-\frac{\sigma_1 E_0}{4\pi(h-g)}[F(c-x,\ h-y)-F(c+x,\ h-y)$$
$$-F(c-x,\ h+y)+F(c+x,\ h+y)$$
$$-F(c-x,\ g-y)+F(c+x,\ g-y)$$
$$+F(c-x,\ g+y)-F(c+x,\ g+y)]. \qquad \text{(II. 6)}$$

2.2. Two-Layer Structure

$$\left.\begin{aligned}
E_{1x} &= E_{px}+E^*_{1x}, \\
E_{px} &= \frac{E_0}{\pi(h-g)}\int_0^\infty [2(h-g)\lambda e^{-\lambda y}-2+e^{-\lambda(h-y)}+e^{-\lambda(h+y)} \\
&\qquad +e^{-\lambda(y-g)}-e^{-\lambda(y+g)}]\sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda^2}, \\
&\qquad\qquad g<y<h; \\
E^*_{1x} &= \frac{kE_0}{\pi(h-g)}\int_0^\infty [2(h-g)\lambda-(e^{\lambda h}-e^{-\lambda h})+(e^{\lambda g}-e^{-\lambda g})] \\
&\qquad \times\frac{e^{-2\lambda h}}{1-ke^{-2\lambda h}}(^{\lambda y}+e^{-\lambda y})\ \sin\lambda c\sin\lambda x\frac{d\lambda}{\lambda^2}.
\end{aligned}\right\} \qquad \text{(II. 7)}$$

$$\left.\begin{aligned}
E_{1y} &= E_{py}+E^*_{1y}, \\
E_{py} &= \frac{E_0}{\pi(h-g)}\int_0^\infty [2(h-g)\lambda e^{-\lambda y}-(e^{-\lambda(h-y)}-e^{-\lambda(h+y)}) \\
&\qquad +(e^{-\lambda(y-g)}-e^{-\lambda(y+g)})]\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda^2} \\
&\quad -\begin{cases} E_0\dfrac{h-y}{h-g}, & |x|<c, \\ 0, & \text{elsewhere,} \end{cases} \quad g<y<h; \\
E^*_{1y} &= -\frac{kE_0}{\pi(h-g)}\int_0^\infty [2(h-g)\lambda-(e^{\lambda h}-e^{-\lambda h})+(e^{\lambda g}-e^{-\lambda g})] \\
&\qquad \times\frac{e^{-2\lambda h}}{1-ke^{-2\lambda h}}(e^{\lambda y}-e^{-\lambda y})\sin\lambda c\cos\lambda x\frac{d\lambda}{\lambda^2}.
\end{aligned}\right\} \qquad \text{(II. 8)}$$

$$\begin{pmatrix} E_{2x} \\ E_{2y} \end{pmatrix} = \frac{k''E_0}{\pi(h-g)} \int_0^\infty [2(h-g)\lambda - (e^{\lambda h} - e^{-\lambda h}) + (e^{\lambda g} - e^{-\lambda g})] \times \frac{e^{-\lambda y}}{1-ke^{-2\lambda h}} \sin \lambda c \begin{pmatrix} \sin \lambda x \\ \cos \lambda x \end{pmatrix} \frac{d\lambda}{\lambda^2}. \quad \text{(II. 9)}$$

Case III.

Velocity distribution:

$$\left.\begin{aligned} v &= v_0, \quad |x| < d; \\ &= v_0 \frac{c-|x|}{c-d}, \quad d < |x| < c; \end{aligned}\right\} 0 < y < h, \\ = 0, \quad \text{elsewhere.} \qquad \text{(III. 1)}$$

Induced e.m.f. pre unit distance:

$$\left.\begin{aligned} E_{0y} &= -E_0 = -\mu_0 v_0 H_x, \quad |x| < d, \\ &= -E_0 \frac{c-|x|}{c-d}, \quad d < |x| < c, \end{aligned}\right\} 0 < y < h, \\ = 0, \quad \text{elsewhere.} \qquad \text{(III. 2)}$$

3.1. Uniform Medium

The formulas for V', E_x and E_y have been given in Section 6 of this paper; see Eqs. (6.1)-(6.4).

$$\begin{aligned} J = -\frac{\sigma_1 E_0}{4\pi(c-d)} [&2K(c-x, y) - 2K(c+x, y) \\ &-2K(d-x, y) + 2K(d+x, y) \\ &+K(c-x, h-y) - K(c+x, h-y) \\ &-K(c-x, h+y) + K(c+x, h+y) \\ &-K(d-x, h-y) + K(d+x, h-y) \\ &+K(d-x, h+y) - K(d+x, h+y)]. \end{aligned} \quad \text{(III. 3)}$$

3.2. Two-Layer Structure

The formulas for E_{1x} and E_{2x} have been given in Section 6 of this paper; see Eq. (6.6).

$$\left.\begin{aligned} E_{1y} &= E_{py} + E_{1y}^*, \\ E_{py} &= -\frac{E_0}{\pi(c-d)} \int_0^\infty [2(1-e^{-\lambda y}) - e^{-\lambda(h-y)} + e^{-\lambda(h+y)}](\cos \lambda d - \cos \lambda c) \cos \lambda x \frac{d\lambda}{\lambda^2}, \\ E_{1y}^* &= \frac{kE_0}{\pi(c-d)} \int_0^\infty \frac{e^{-\lambda h}(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}} (e^{\lambda y} - e^{-\lambda y})(\cos \lambda d - \cos \lambda c) \cos \lambda x \frac{d\lambda}{\lambda^2}. \end{aligned}\right\} \text{(III. 4)}$$

$$E_{2y} = -\frac{k''E_0}{\pi(c-d)} \int_0^\infty \frac{e^{-\lambda(y-h)}(1-e^{-\lambda h})^2}{1-ke^{-2\lambda h}} (\cos \lambda d - \cos \lambda c) \cos \lambda x \frac{d\lambda}{\lambda^2}. \quad \text{(III, 5)}$$