Vertical Sounding by Central Induction Method

By

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The apparent resistivity methods are widely used to determine the structure of horizontally stratified earth. But with these methods, some material difficulties are unavoided in cases where there are four or more layers.

On the other hand, the central induction methods which have been studied by Koenigsberger, Nunier and Stefanescu seem to have a distinct advantage over the resistivity methods in resolving structures involving four or more layers.

In this paper the authors discuss the theory of C. I. M. The numerical tables for interpreting the C. I. M. data are presented, and a new procedure in which the phase angle of the resultant magnetic field is measured is proposed.

1. Central Induction Method

When an alternating current is supplied to an insulated circular loop placed near the surface of the earth, the phase of the secondary magnetic field due to the induced currents in the subsurface will generally not be in the same phase as the phase of the primary magnetic field created by the loop. This phase difference at the center of the loop depends upon the earth's conductivity. J. G. Koenigsberger[1),2)] and W. Nunier[3)] have studied this phenomena and proposed a new electromagnetic induction method, called the central induction method (C.I.M). S. S. Stefanescu[4),5)] has made a theoretical study of this method.

The apparent resistivity methods are widely used to determine the structure of horizontally stratified earth. But with these methods, some material difficulties are unavoided in cases involving four or more layers. On the other hand, the C.I.M. seems to have a distinct advantage over the resistivity methods in resolving structures involving four or more layers. In this paper, the authors discuss this method theoretically, and present the numerical tables for interpreting C.I.M. data and propose a new procedure in which the phase angle of the resultant magnetic field is measured.

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1.1. Magnetic Field by a Circular Loop

If we assume that an alternating current $I\cos\omega t$ flowing in a circular loop produces a magnetic field $H_e \cos\omega t$, ω being the angular frequency, and the medium surrounding the loop is the air, then the three components $H_{e\rho}$, $H_{e\phi}$ and H_{ez} of the magnetic field at an arbitrary point (ρ, ϕ, z) are as follows

$$\left.\begin{aligned} H_{e\rho} &= \frac{aI}{2}\int_0^\infty \varepsilon^{-z\lambda} J_1(\rho\lambda)J_1(a\lambda)\lambda d\lambda\,, \\ H_{e\phi} &= 0\,, \\ H_{ez} &= \frac{aI}{2}\int_0^\infty \varepsilon^{-z\lambda} J_0(\rho\lambda)J_1(a\lambda)\lambda d\lambda\,, \end{aligned}\right\} \tag{1.1}$$

where z is assumed to be positive (see Fig. 1).

The magnetic field is in the same phase with the current i and H_e is therefore a real number. When the circular loop is set up at a distance h above the earth's surface as shown in Fig. 2, we denote the magnetic and electric fields in the air $(z<h)$ as $H_e\cos\omega t + H_0^* \cos\omega t$ and $E_0 \cos\omega t$ respectively, and in the earth $(z>h)$ as $H_1\cos\omega t$ and $E_1 \sin\omega t$ respectively. Ignoring the displacement current, Maxwell's equations in cylindrical coordinates (Fig. 2) are as follows

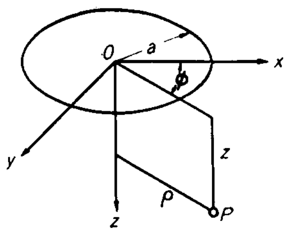

Fig. 1

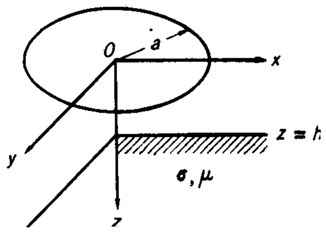

Fig. 2

$$\left.\begin{aligned} &\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = \sigma E_\phi\,, \\ &\frac{\partial E_\phi}{\partial z} = \mu_0\mu_r \frac{\partial H_\rho}{\partial t}\,, \\ &\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho E_\phi)\right] = -\mu_0\mu_r\frac{\partial H_z}{\partial t}\,. \end{aligned}\right\} \tag{1.2}$$

From Eq. 1.2, we obtain

$$\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi)\right] + \frac{\partial^2 E_\phi}{\partial z^2} - \mu_0\mu_r\sigma\frac{\partial E_\phi}{\partial t} = 0\,. \tag{1.3}$$

Considering that $E_\phi = E_{0\phi}\cos\omega t$, $\sigma=0$ and $\mu_r=1$ in the air, and that $E_\phi = E_{1\phi}\cos\omega t$ and σ and μ_r are finite in the earth, Eq. 1.3 can be written as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_{0\phi})\right] + \frac{\partial^2 E_{0\phi}}{\partial z^2} = 0\,, \quad z<h\,, \\ &\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_{1\phi})\right] + \frac{\partial^2 E_{1\phi}}{\partial z^2} + k^2 E_{1\phi} = 0\,, \quad z>h\,, \end{aligned}\right\} \tag{1.4}$$

where

$$k^2 = j\omega\mu_0\mu_r\sigma\,. \tag{1.5}$$

From Eq. 1.4, we get

$$\left.\begin{aligned} E_{0\phi} &= \int_0^\infty L_0(\lambda)\varepsilon^{z\lambda}J_1(\rho\lambda)d\lambda\,, \quad z<h\,, \\ E_{1\phi} &= \int_0^\infty L_1(\lambda)\varepsilon^{-z\sqrt{\lambda^2-k^2}}J_1(\rho\lambda)d\lambda\,, \quad z>h\,, \end{aligned}\right\} \tag{1.6}$$

where L_0 and L_1 are functions of λ only.

On the other hand Eq. 1.2 yields

$$\left.\begin{aligned} H_{0z}^* &= \frac{1}{j\omega\mu_0}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_{0\phi})\,, \\ H_{0\rho}^* &= -\frac{1}{j\omega\mu_0}\frac{\partial E_{0\phi}}{\partial z}\,, \end{aligned}\right\} \tag{1.7}$$

and

$$\left.\begin{aligned} H_{1z} &= \frac{1}{j\omega\mu_0\mu_r}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_{1\phi})\,, \\ H_{1\rho} &= -\frac{1}{j\omega\mu_0\mu_r}\frac{\partial E_{1\phi}}{\partial z}\,. \end{aligned}\right\} \tag{1.8}$$

Substituting the values of $E_{0\phi}$ and $E_{1\phi}$ of Eq. 1.6 into Eq. 1.7 and Eq. 1.8, we obtain

$$\left.\begin{aligned} H_{0z}^* &= \frac{1}{j\omega\mu_0}\int_0^\infty L_0(\lambda)\varepsilon^{z\lambda}J_0(\rho\lambda)\lambda d\lambda\,, \\ H_{0\rho}^* &= -\frac{1}{j\omega\mu_0}\int_0^\infty L_0(\lambda)e^{z\lambda}J_1(\rho\lambda)\lambda d\lambda\,, \end{aligned}\right\} \tag{1.9}$$

$$\left.\begin{aligned} H_{1z} &= \frac{1}{j\omega\mu_0\mu_r}\int_0^\infty L_1(\lambda)\varepsilon^{-z\sqrt{\lambda^2-k^2}}J_0(\rho\lambda)\lambda d\lambda\,, \\ H_{1\rho} &= \frac{1}{j\omega\mu_0\mu_r}\int_0^\infty L_1(\lambda)\varepsilon^{-z\sqrt{\lambda^2-k^2}}J_0(\rho\lambda)\sqrt{\lambda^2-k^2}d\lambda\,. \end{aligned}\right\} \tag{1.10}$$

The conditions to be satisfied at the boundary surface of the air and the earth are

$$\left.\begin{aligned} H_{e\rho}+H_{0\rho}^* &= H_{1\rho}\,, \quad z=h\,, \\ H_{ez}+H_{0z}^* &= \mu_r H_{1z}\,, \quad z=h\,, \end{aligned}\right\} \tag{1.11}$$

from which we can determine the functions L_0 and L_1, combining with Eqs. 1.1, 1.9 and 1.10.

The vertical component of the magnetic field in the air becomes as follows

$$\begin{aligned} H_{0z} &= H_{ez}+H_{0z}^* \\ &= \frac{aI}{2}\int_0^\infty\left[\varepsilon^{-z\lambda}+\frac{\mu_r\lambda-\sqrt{\lambda^2-k^2}}{\mu_r\lambda+\sqrt{\lambda^2-k^2}}\varepsilon^{-(2h-z)\lambda}\right]J_0(\rho\lambda)J_1(a\lambda)\lambda d\lambda\,, \quad 0<z<h\,. \end{aligned} \tag{1.12}$$

When the circluar loop is placed at the surface level, we obtain,

$$H_{0z} = \frac{aI}{2}\int_0^\infty \frac{2\mu_r\lambda^2}{\mu_r\lambda+\sqrt{\lambda^2-k^2}}J_0(\rho\lambda)J_1(a\lambda)d\lambda\,, \tag{1.13}$$

putting $z=h=0$ in Eq. 1.12. Hence the vertical component of the field at the center ($\rho=0$, $z=0$) of the loop will become

$$H_{0z} = \frac{aI}{2}\int_0^\infty \frac{2\lambda^2}{\lambda+\sqrt{\lambda^2-k^2}} J_1(a\lambda)d\lambda\,, \tag{1.14}$$

where

$$k^2 = j\omega\mu_0\sigma\,, \tag{1.15}$$

provided $\mu_r=1$.

Using the relation

$$-\frac{\partial}{\partial a} J_0(a\lambda) = \lambda J_1(a\lambda)\,, \tag{1.16}$$

we can write Eq. 1.14 as

$$H_{0z} = -\frac{aI}{2}\frac{\partial}{\partial a}\int_0^\infty \frac{2\lambda}{\lambda+\sqrt{\lambda^2-k^2}} J_0(a\lambda)d\lambda\,, \tag{1.17}$$

and by means of the relations

$$\int_0^\infty \frac{2\lambda}{\lambda+\sqrt{\lambda^2-k^2}} J_0(a\lambda)d\lambda = \frac{2}{k^2}\int_0^k \eta d\eta \int_0^\infty \frac{J_0(a\lambda)}{\sqrt{\lambda^2-\eta^2}}\lambda d\lambda$$

and

$$\int_0^\infty \frac{J_0(a\lambda)}{\sqrt{\lambda^2-\eta^2}}\lambda d\lambda = \frac{\varepsilon^{ja\eta}}{a}\,,$$

we get the following result:

$$\frac{2aH_{0z}}{I} = -\frac{2}{k^2a^2}[3+\varepsilon^{jka}(k^2a^2+3jka-3)]\,. \tag{1.18}$$

1.2. Conductivity of the Earth

If we put

$$jka = -(1-j)\xi \tag{1.19}$$

or

$$\xi^2 = \frac{\omega\mu_0\sigma}{2}a^2\,, \tag{1.19'}$$

we can separate the real and imaginary parts of Eq. 1.18:

$$\left.\begin{aligned} R_e\left(\frac{2aH_{0z}}{I}\right) &= -\frac{\varepsilon^{-\xi}}{\xi^2}[(2\xi^2+3\xi)\cos\xi-(3\xi+3)\sin\xi]\,,\\ I_m\left(\frac{2aH_{0z}}{I}\right) &= \frac{1}{\xi^2}[3-\varepsilon^{-\xi}\{(3\xi+3)\cos\xi+(2\xi^2+3\xi)\sin\xi\}]\,. \end{aligned}\right\} \tag{1.20}$$

When the parameter ξ is small or large enough, we obtain the following approximate expressions:

$$\left.\begin{aligned} R_e\left(\frac{2aH_{0z}}{I}\right) &= 1-\frac{4}{15}\xi^2, & \xi \ll 1 ; \\ &= -2\xi^2\varepsilon^{-\xi}\cos\xi , & \xi \gg 1 , \end{aligned}\right\} \tag{1.21}$$

$$\left.\begin{aligned} I_m\left(\frac{2aH_{0z}}{I}\right) &= \frac{\xi^2}{2}, & \xi \ll 1 ; \\ &= \frac{3}{\xi^2}, & \xi \gg 1 , \end{aligned}\right\} \tag{1.22}$$

from the second equation of Eq. 1.20 we can calculate the value of $I_m\left(\frac{2aH_{0z}}{I}\right)$ as shown in Fig. 3; and if we can measure the value of $I_m(H_{0z})$, the off-phase component

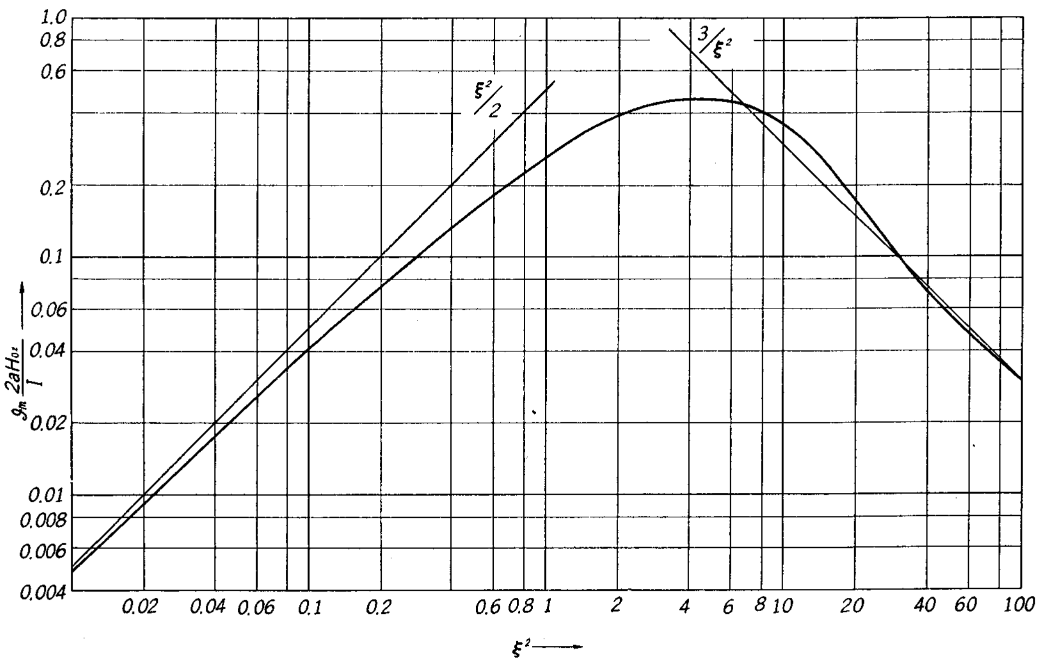

Fig. 3

of the secondary magnetic field, the conductivity σ of the earth can be determined by means of the graph. In the special case, where $\xi \ll 1$, the first equation of Eq. 1.22 will become

$$\frac{1}{\sigma} = \frac{\pi\mu_0}{4}\frac{\nu a I}{I_m(H_{0z})}, \tag{1.23}$$

where ν is the frequency in cycles per second.

In Fig. 4, the phase angle φ and its tangent are plotted against ξ, using the relation

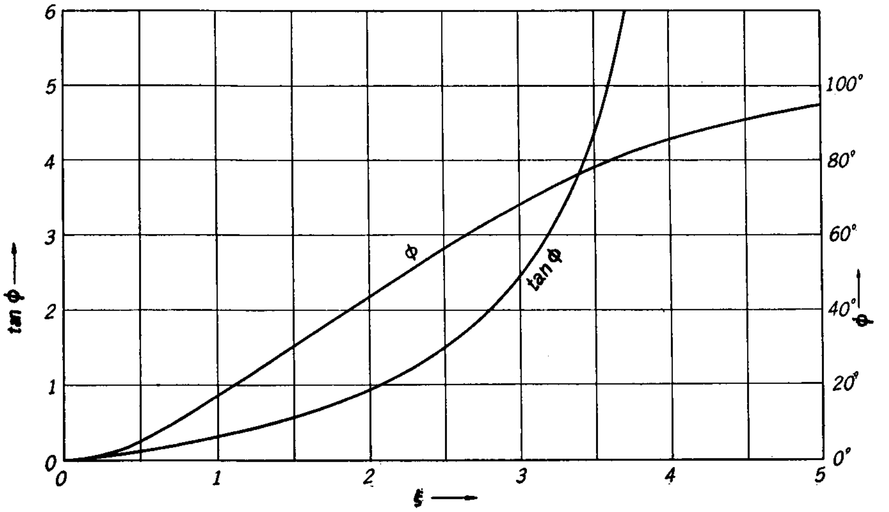

Fig. 4

$$\varphi = \tan^{-1}\frac{I_m(H_{0z})}{R_e(H_{0z})}. \tag{1.24}$$

1. 3. The Function *M* of Stefanescu

In the case of vertical sounding by the C.I.M., it will be more convenient to use the function M of Stefanescu defined as follows

$$M = -j\lim_{\nu\to 0}\frac{\partial H}{\partial \nu}. \tag{1.25}$$

Although the magnetic field H is generally a complex number, it can be readily shown that

$$\lim_{\nu\to 0}\frac{\partial}{\partial \nu}R_e(H_{0z}) = 0, \tag{1.26}$$

if the earth is uniform. Therefore the vertical magnetic component at the center of the loop may be written as

$$M_{0z} = \lim_{\nu\to 0}\frac{\partial}{\partial \nu}I_m(H_{0z}). \tag{1.27}$$

In case the medium is uniform, we obtain from Eq. 1.12

$$H_{0z} = \frac{aI}{2}\int_0^\infty [\varepsilon^{-z\lambda}+F_0(\lambda, k)\varepsilon^{-(2h-z)\lambda}]J_0(\rho\lambda)J_1(a\lambda)\lambda d\lambda, \tag{1.28}$$

where

$$F_0(\lambda, k) = \frac{\lambda-\sqrt{\lambda^2-k^2}}{\lambda+\sqrt{\lambda^2-k^2}}. \tag{1.29}$$

Differentiating Eq. 1.29 with respect to ν, we obtain

$$\lim_{\nu \to 0} \frac{\partial F_0}{\partial \nu} = j \frac{\pi \mu_0 \sigma}{2\lambda^2}.$$

Therefore, we get finally

$$M_{0z} = -j \lim_{\nu \to 0} \frac{\partial H_{0z}}{\partial \nu}$$
$$= \frac{\pi \mu_0 I}{4} \sigma a \int_0^\infty J_0(\rho\lambda) J_1(a\lambda) \frac{\varepsilon^{-(2h-z)\lambda}}{\lambda} d\lambda . \quad (1.30)$$

At any point on the z axis, we have

$$M_{0z} = \frac{\pi \mu_0 I}{4} \sigma a \int_0^\infty J_1(a\lambda) \frac{\varepsilon^{-(2h-z)\lambda}}{\lambda} d\lambda , \quad \rho = 0 . \quad (1.31)$$

By the formula

$$\int_0^\infty J_m(at) \varepsilon^{-bt} \frac{dt}{t} = \frac{[\sqrt{a^2+b^2}-b]^m}{ma^m}, \quad (1.31')$$

Eq. 1. 31 yields

$$M_{0z} = \frac{\pi \mu_0 I}{4} \sigma [\sqrt{a^2+(2h-z)^2} - (2h-z)] . \quad (1.32)$$

At the center of the loop, we get

$$M_{0z} = \frac{\pi \mu_0 I}{4} \sigma (\sqrt{a^2+4h^2} - 2h) . \quad (1.33)$$

When the loop is placed at the earth's surface, we get

$$M_{0z} = \frac{\pi \mu_0 I}{4} \sigma a \quad (1.34)$$

from which we can calculate the earth's conductivity σ as follows

$$\frac{1}{\sigma} = \frac{\pi \mu_0}{4} \frac{aI}{M_{0z}} . \quad (1.34')$$

This formula is the same to the limit of Eq. 1. 23 as $\nu \to 0$.

2. Vertical Sounding by C.I.M.

2. 1. Function M_{0z} in the case of Two or Three Layers

In the case of two layers, the magnetic components will become as follows, by a procedure similar to that in the preceding section, (see Fig. 5)

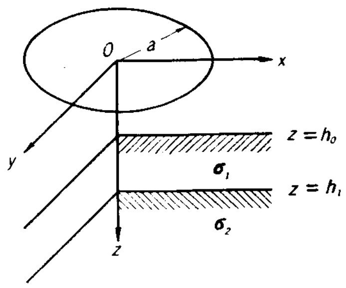

Fig. 5

$$\left.\begin{aligned} H_{0\rho} &= H_{e\rho}+H_{0\rho}^{*} \\ &= \frac{aI}{2}\int_0^\infty \varepsilon^{-z\lambda}J_1(a\lambda)J_1(\rho\lambda)\lambda d\lambda-\frac{1}{j\omega\mu_0}\int_0^\infty L_0(\lambda)\varepsilon^{z\lambda}J_1(\rho\lambda)\lambda d\lambda\,, \\ H_{0z} &= H_{ez}+H_{0z}^{*} \\ &= \frac{aI}{2}\int_0^\infty \varepsilon^{-z\lambda}J_1(a\lambda)J_0(\rho\lambda)\lambda d\lambda+\frac{1}{j\omega\mu_0}\int_0^\infty L_0(\lambda)\varepsilon^{z\lambda}J_0(\rho\lambda)\lambda d\lambda\,;\quad 0<z<h_0 \end{aligned}\right\} \tag{2. 1a}$$

$$\left.\begin{aligned} H_{1\rho} &= \frac{1}{j\omega\mu_0}\int_0^\infty [L_{-1}(\lambda)\varepsilon^{-\gamma_1 z}-L_{+1}(\lambda)\varepsilon^{+\gamma_1 z}]J_1(\rho\lambda)\gamma_1 d\lambda\,, \\ H_{1z} &= \frac{1}{j\omega\mu_0}\int_0^\infty [L_{-1}(\lambda)\varepsilon^{-\gamma_1 z}+L_{+1}(\lambda)\varepsilon^{+\gamma_1 z}]J_0(\rho\lambda)\lambda d\lambda\,;\quad h_0<z<h_1 \end{aligned}\right\} \tag{2. 1b}$$

$$\left.\begin{aligned} H_{2\rho} &= \frac{1}{j\omega\mu_0}\int_0^\infty L_2(\lambda)\varepsilon^{-\gamma_2 z}J_1(\rho\lambda)\gamma_2 d\lambda\,, \\ H_{2z} &= \frac{1}{j\omega\mu_0}\int_0^\infty L_2(\lambda)\varepsilon^{-\gamma_2 z}J_0(\rho\lambda)\lambda d\lambda\,;\quad h_1<z\,, \end{aligned}\right\} \tag{2. 1c}$$

where

$$\left.\begin{aligned} \gamma_1 &= \sqrt{\lambda^2-k_1^2}\,, & \gamma_2 &= \sqrt{\lambda^2-k_2^2}\,; \\ k_1^2 &= j2\pi\nu\mu_0\gamma_1\,, & k_2^2 &= j2\pi\nu\mu_0\gamma_2\,, \end{aligned}\right\} \tag{2. 2}$$

and it is assumed that $\mu_r=1$ as before.

In this case, the boundary conditions are

$$\left.\begin{aligned} H_{0\rho} &= H_{1\rho}\,, & H_{0z} &= H_{1z}\,; & z &= h_0\,, \\ H_{1\rho} &= H_{2\rho}\,, & H_{1z} &= H_{2z}\,; & z &= h_1\,. \end{aligned}\right\} \tag{2. 3}$$

Substituting Eq. 2. 1 into Eq. 2. 3, we get the vertical magnetic component H_{0z} at the earth's surface ($h_0=0$) :

$$H_{0z} = \frac{aI}{2}\int_0^\infty (1+B_0)\varepsilon^{z\lambda}J_1(a\lambda)J_0(\rho\lambda)\lambda d\lambda\,,\quad z\leqslant 0\,, \tag{2. 4}$$

where

$$B_0 = \frac{K_0+K_1\varepsilon^{-2\gamma_1 h_1}}{1+K_0K_1\varepsilon^{-2\gamma_1 h_1}}\,, \tag{2. 5}$$

$$K_0 = \frac{\lambda-\gamma_1}{\lambda+\gamma_1},\quad K_1 = \frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\,. \tag{2. 6}$$

At the center ($z=0$, $\rho=0$) of the loop the vertical component will become

$$H_{0z} = \frac{aI}{2}\int_0^\infty (1+B_0)J_1(a\lambda)\lambda d\lambda\,, \tag{2. 7}$$

from which we obtain

$$\begin{aligned} M_{0z} &= -j\lim_{\nu\to 0}\frac{\partial H_{0z}}{\partial\nu} \\ &= \frac{\pi\mu_0 I}{4}a\int_0^\infty [\sigma_1+(\sigma_2-\sigma_1)\varepsilon^{-2h_1\lambda}]\varepsilon^{z\lambda}J_1(a\lambda)\frac{d\lambda}{\lambda}\,. \end{aligned} \tag{2. 8}$$

This can be written as

$$M_{0z} = \frac{\pi\mu_0 I}{4}[a\sigma_1+(\sigma_2-\sigma_1)(\sqrt{a^2+4h_1^2}-2h_1)] \tag{2. 9}$$

by means of Eq. 1. 31′.

Eq. 2. 9 is the required formula in the case of two layers and is similar to Eq. 1. 34 for the uniform medium.

In the case of three layers, we obtain the function B_0 as follows

$$B_0 = \frac{K_0 + K_1 \varepsilon^{-2\gamma_1 h_1} + K_2 \varepsilon^{-2\gamma_1 h_1 - 2\gamma_1 (h_2 - h_1)} + K_0 K_1 K_2 \varepsilon^{-2\gamma_2 (h_2 - h_1)}}{1 + K_0 K_1 \varepsilon^{-2\gamma_1 h_1} + K_0 K_2 \varepsilon^{-2\gamma_1 h_1 - 2\gamma_2 (h_2 - h_1)} + K_1 K_2 \varepsilon^{-2\gamma_2 (h_2 - h_1)}}, \tag{2.10}$$

where

$$K_2 = \frac{\gamma_2 - \gamma_3}{\gamma_2 + \gamma_3}, \quad \gamma_3 = \sqrt{\lambda^2 - k_3^2}, \quad k_3^2 = j 2\pi\nu\mu_0\sigma_3. \tag{2.11}$$

Finally, the function M_{0z} in this case will become

$$M_{0z} = \frac{\pi\mu_0 I}{4}\left[a\sigma_1 + (\sigma_2 - \sigma_1)\left(\sqrt{a^2 + 4h_1^2} - 2h_1\right) + (\sigma_3 - \sigma_2)\left(\sqrt{a^2 + 4h_2^2} - 2h_2\right)\right]. \tag{2.12}$$

2. 2. Apparent Conductivity

As shown above, the conductivity of a semi-infinite uniform medium can be determined by Eq. 1. 34′. In a similar manner we can define the apparent conductivities fictitious earth which has the same value of two or more layers: namely the apparent conductivity is the conductivity of M_{0z} as that of an actual earth structure.

Substituting Eq. 2. 9 and Eq. 2. 12 into Eq. 1. 34′, the apparent conductivity becomes follows,

$$\sigma = \sigma_1 + (\sigma_2 - \sigma_1)\left[\sqrt{1 + (2h_1/a)^2} - (2h_1/a)\right], \quad n = 2 \tag{2.13}$$

for the two layer structure, and

$$\begin{aligned} \sigma = \sigma_1 &+ (\sigma_2 - \sigma_1)\left[\sqrt{1 + (2h_1/a)^2} - (2h_1/a)\right] \\ &+ (\sigma_3 - \sigma_2)\left[\sqrt{1 + (2h_2/a)^2} - (2h_2/a)\right], \quad n = 3 \end{aligned} \tag{2.14}$$

for the three layer structure.

Similarly, in the case of n layers we get

$$\sigma = \sigma_1 + \sum_{i=1}^{n-1} (\sigma_{i+1} - \sigma_i)\left[\sqrt{1 + (2h_i/a)^2} - (2h_i/a)\right]. \tag{2.15}$$

From Eq. 2. 15, we find that the conductivity of each layer is determined by an expression with an additive form, and this fact gives the C.I.M. an advantage over the resistivity method.

When the radius of the circular loop is small enough, i.e. $1 \ll (2h_1/a)^2$, the apparent conductivity is equal to that of the first layer as can be seen from Eq. 2. 15, namely

$$\sigma \simeq \sigma_1, \quad a \ll h_1. \tag{2.16a}$$

On the other hand, when the radius is so large that $2h_{n-1}/a \ll 1$, the apparent conductivity will become

$$\sigma = \sigma_1 + \sum_{i=1}^{n-1} (\sigma_{i+1} - \sigma_i) \,,$$

which yields

$$\sigma \simeq \sigma_n \,, \quad a \gg h_{n-1} \,. \tag{2.16b}$$

This equation indicates that we can determine the conductivity of the nth layer from the value of σ when the loop is very large.

2.3. Interpretation

In the C.I.M., we measure the value of $I_m(H_{0z})$ at the center of a circular loop as a function of the frequency ν, and determine the value of M_{0z} as follows

$$M_{0z} = \lim_{\nu \to 0} I_m(H_{0z}/\nu) \,. \tag{2.17}$$

Inserting this value into Eq. 1.34′, the apparent conductivity σ can be calculated. Repeating the above procedure, for various values of the radius of the circular loop, we plot the relation between σ and a graphically; then this curve is compared to the master curve to determine the thicknesses and conductivities of the horizontal layers. The master curve for this purpose is shown in Fig. 6, which gives the following function, in logarithmic scales (see Table 1):

$$S(t) = \frac{\sqrt{1+t^2}-1}{t} \,. \tag{2.18}$$

By using this function Eq. 2.15 can be expressed as follows

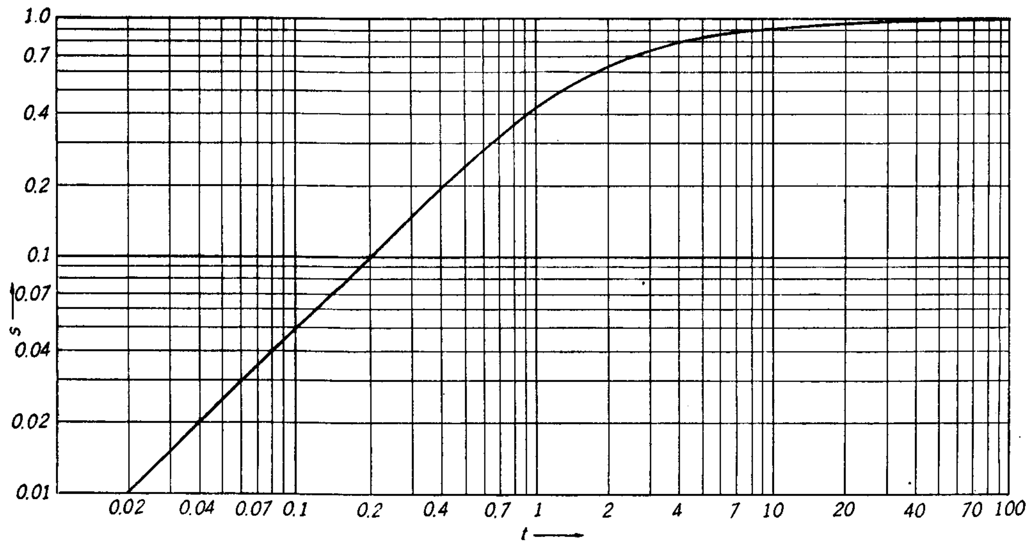

Fig. 6

Table 1.

t	$s(t)$	t	$s(t)$	t	$s(t)$
0.00	0.000 000	1.0	0.414 214	5	0.819 804
0.05	024 984	1.1	0.442 370	6	0.847 127
10	049 876	1.2	468 375	7	867 295
15	074 583	1.3	492 402	8	882 782
20	099 020	1.4	514 618	9	895 043
25	123 106	1.5	535 184	10	904 988
0.30	0.146 769	1.6	0.554 248	11	0.913 215
35	169 946	1.7	571 946	12	920 133
40	192 582	1.8	588 403	15	935 553
45	214 635	1.9	603 732	20	951 249
50	236 068	2.0	618 034	25	960 800
0.55	0.256 857	2.1	0.631 400	30	0.967 222
60	276 984	2.2	643 913	35	971 837
65	296 440	2.3	655 647	40	975 312
70	315 222	2.4	666 667	45	978 025
75	333 333	2.5	677 033	50	980 200
0.80	0.350 781	3.0	0.720 759	60	0.983 472
85	367 577	3.5	754 301	70	985 816
90	383 736	4.0	780 776	80	987 578
95	399 275	4.5	802 172	90	988 951
1.00	414 214	5.0	819 804	100	990 050

$$\sigma-\sigma_1 = (\sigma_2-\sigma_1)S\left(\frac{a}{2h_1}\right) + \sum_{i=2}^{n-1}(\sigma_{i+1}-\sigma_i)\left[\sqrt{1+\left(\frac{2h_i}{a}\right)^2}-\frac{2h_i}{a}\right]. \qquad (2.19a)$$

Since the conductivity of the first layer is determined by Eq. 2.16a, the left-hand side of Eq. 2.19a is known. When a is small, the value of the right-hand side of Eq. 2.19a is nearly equal to its first term, and therefore the curve of the measured value of $\sigma-a$ will fit the left part of the master curve, and vertical and horizontal displacements of the origin of the $\sigma-a$ curve will give the values of $\sigma-\sigma_1$ and h_1 respectively.

Using these results we obtain

$$(\sigma-\sigma_1)-(\sigma_2-\sigma_1)\,S\left(\frac{a}{2h_1}\right) = (\sigma_3-\sigma_2)\,S\left(\frac{a}{2h_2}\right) + \sum_{i=3}^{n-1}(\sigma_{i+1}-\sigma_i)\left[\sqrt{1+\left(\frac{2h_i}{a}\right)^2}-\frac{2h_i}{a}\right]. \qquad (2.19b)$$

comparing again this relation to the master curve we can determine $2h_2$ and $\sigma_3-\sigma_2$; and by following a similar procedure we can determine $2h_{n-1}$ and $\sigma_n-\sigma_{n-1}$ of an n layer structure.

3. C.I.M. by Rectangular Loops

Since in the field measurement, we can set up rectangular loops much more easily than circular loops, we will investigate here the method when using a rectangular loop.

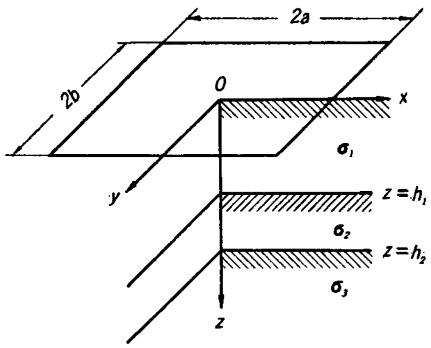

Fig. 7

3.1. The Function dM_{0z} with a Small Loop

Using Eq. 2.8 in the case of two layers, we discuss the value of M_{0z} when the radius of the circular loop is very small. From Eq. 2.4, we have

$$M_{0z} = \frac{\pi\mu_0 I}{4}\int_0^\infty [\sigma_1+(\sigma_2-\sigma_1)\varepsilon^{-2h_1\lambda}]\varepsilon^{z\lambda}J_0(\rho\lambda)\left[\frac{a}{\lambda}J_1(a\lambda)\right]d\lambda \tag{3.1}$$

and since we can write $aJ_1(a\lambda)/\lambda \simeq a^2/2$, when a becomes very small, the above equation yields

$$dM_{0z} = \frac{\pi\mu_0 I}{8}dS\int_0^\infty [\sigma_1+(\sigma_2-\sigma_1)\varepsilon^{-2h_1\lambda}]J_0(\rho\lambda)\varepsilon^{z\lambda}d\lambda\,, \tag{3.2}$$

where dS denotes the loop area, and dM_{0z} represents the value of M_{0z} due to the small loop. Using the formula

$$\int_0^\infty \varepsilon^{-b\lambda}J_0(c\lambda)d\lambda = \frac{1}{\sqrt{b^2+c^2}}\,,$$

Eq. 3.2 yields

$$dM_{0z} = \frac{\mu_0 I}{8}\left[\frac{\sigma_1}{\sqrt{\rho^2+z^2}}+\frac{\sigma_2-\sigma_1}{\sqrt{\rho^2+(z-2h_1)^2}}\right]dS\,.$$

In a similar manner, the following relation is obtained for the case of n layers:

$$dM_{0z} = \frac{\mu_0 I}{8}\left[\frac{\sigma_1}{\sqrt{\rho^2+z^2}}+\sum_{i=1}^{n-1}\frac{\sigma_{i+1}-\sigma_i}{\sqrt{\rho^2+(z-2h_i)^2}}\right]dS\,. \tag{3.3}$$

3.2. The Apparent Conductivity by the Rectangular Loop Method

From Eq. 3.3, dM_{0z} at any point $P(x, y, z)$ due to a small loop of the area $dx'dy'$ surrounding a point $Q(x', y', 0)$ shown in Fig. 7 will become

$$dM_{0z} = \frac{\mu_0 I}{8}\left[\frac{\sigma_1}{\sqrt{(x'-x)^2+(y'-y)^2+z^2}}+\sum_{i=1}^{n-1}\frac{\sigma_{i+1}-\sigma_i}{\sqrt{(x'-x)^2+(y'-y)^2+(z-2h_i)^2}}\right]dx'dy'\,, \tag{3.4}$$

and the resultant value of M_{0z} due to a loop of finite area becomes

$$M_{0z} = \frac{\mu_0 I}{8}\int_{-b}^{+b}dy'\int_{-a}^{+a}\left[\frac{\sigma_1}{\sqrt{(x'-x)^2+(y'-y)^2+z^2}}\right.$$
$$\left.+\sum_{i=1}^{n-1}\frac{\sigma_{i+1}-\sigma_i}{\sqrt{(x'-x)^2+(y'-y)^2+(z-2h_i)^2}}\right]dx'\,. \tag{3.4'}$$

If the point P is at the center of the loop, the above equation yields

$$M_{0z} = \frac{\mu_0 I}{4}\left[\sigma_1\left\{a\log\frac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}+b\log\frac{\sqrt{a^2+b^2}+a}{\sqrt{a^2+b^2}-a}\right\}\right.$$
$$+\sum_{i=1}^{n-1}(\sigma_{i+1}-\sigma_i)\left\{a\log\frac{\sqrt{a^2+b^2+4h_i^2}+b}{\sqrt{a^2+b^2+4h_i^2}-b}+b\log\frac{\sqrt{a^2+b^2+4h_i^2}+a}{\sqrt{a^2+b^2+4h_i^2}-a}\right.$$
$$\left.\left.-4h_i\tan^{-1}\frac{ab}{2h_i\sqrt{a^2+b^2+4h_i^2}}\right\}\right], \tag{3.5}$$

and when the loop is square, being $a=b$, it becomes

$$M_{0z} = \frac{\mu_0 I}{2}\left[\sigma_1 a\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right.$$
$$\left.+\sum_{i=1}^{n-1}(\sigma_{i+1}-\sigma_i)\left\{a\log\frac{\sqrt{2a^2+4h_i^2}+a}{\sqrt{2a^2+4h_i^2}-a}-2h_i\tan^{-1}\frac{a^2}{2h_i\sqrt{2a^2+4h_i^2}}\right\}\right]. \tag{3.6}$$

In the case of the uniform medium, we get

$$M_{0z} = \mu_0 I\sigma a\log(\sqrt{2}+1), \tag{3.7}$$

and in this case, the apparent conductivity is given by

$$\sigma = \frac{1}{\mu_0}\frac{1}{\log(\sqrt{2}+1)}\frac{M_{0z}}{aI}$$
$$= \frac{1}{\mu_0}\frac{1}{\log(\sqrt{2}+1)}\frac{1}{aI}\lim_{\nu\to 0}\frac{I_m(H_{0z})}{\nu}. \tag{3.8}$$

Substituting Eq. 3.6 into Eq. 3.8, the apparent conductivity for the n layers can be written as follows:

$$\sigma = \sigma_1+\frac{1}{2\log(\sqrt{2}+1)}\sum_{i=1}^{n-1}(\sigma_{i+1}-\sigma_i)$$
$$\times\left[\log\frac{\sqrt{2a^2+4h_i^2}+a}{\sqrt{2a^2+4h_i^2}-a}-\frac{2h_i}{a}\tan^{-1}\frac{a^2}{2h_i\sqrt{2a^2+4h_i^2}}\right]. \tag{3.9}$$

3.3. Master Curve

In a manner similar to the case of circular loop, we obtain the master curve for the rectangular loop method as shown in Fig. 8, which gives the function : (see Table 2)

$$p(t) = \frac{1}{2\log(\sqrt{2}+1)}\left[\log\frac{\sqrt{1+2t^2}+t}{\sqrt{1+2t^2}-t}-\frac{1}{t}\tan^{-1}\frac{t^2}{\sqrt{1+2t^2}}\right]. \tag{3.10}$$

4. Discussion

4.1. The Value of $I_m(H_{0z})$

From Eq. 3.7, M_{0z} becomes

$$M_{0z} = 1.108\times 10^{-6}a(m)\sigma(\mho/m)I(A), \tag{4.1}$$

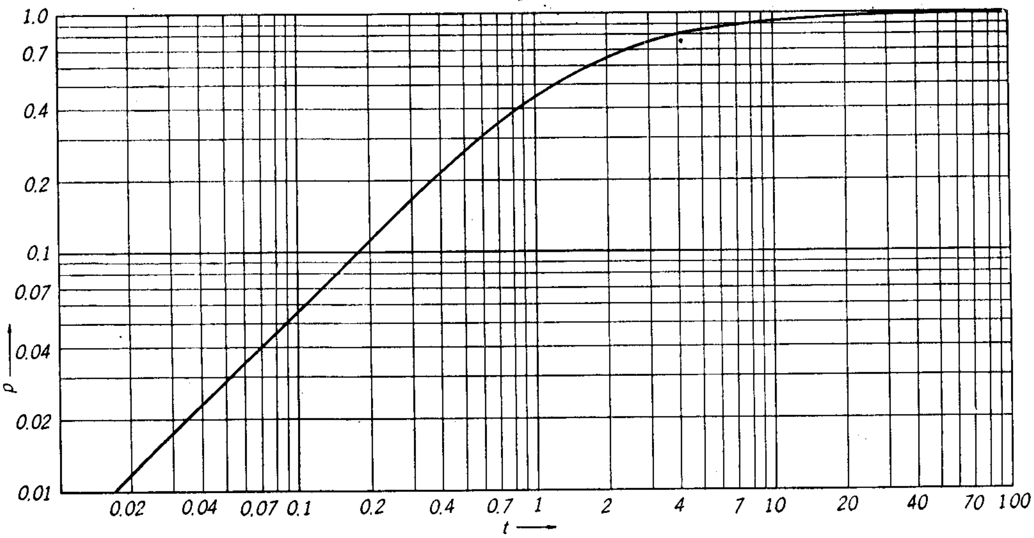

Fig. 8

Table 2.

t	$p(t)$	t	$p(t)$	t	$p(t)$
0.00	0.000 000	1.0	0.450 070	5	0.837 692
0.05	0.028 341	1.1	0.478 483	6	0.862 558
10	056 542	1.2	504 479	7	880 851
15	084 466	1.3	528 297	8	894 860
20	111 987	1.4	550 155	9	905 928
25	138 992	1.5	570 253	10	914 893
0.30	0.165 381	1.6	0.588 772	11	0.923 300
35	191 072	1.7	605 871	12	928 523
40	216 001	1.8	621 694	15	942 374
45	240 120	1.9	636 367	20	956 447
50	263 395	2.0	650 003	25	964 998
0.55	0.285 808	2.1	0.662 701	30	0.970 942
60	307 350	2.2	674 549	35	974 868
65	328 026	2.3	785 626	40	977 974
70	347 846	2.4	696 001	45	980 417
75	366 829	2.5	705 736	50	982 340
0.80	0.384 995	3.0	0.746 550	60	0.985 260
85	402 373	3.5	777 606	70	987 352
90	418 991	4.0	801 977	80	988 924
95	434 880	4.5	821 586	90	990 148
1.00	450 070	5.0	837 692	100	991 129

and when ν is small enough, $I_m(H_{0z})$ becomes

$$I_m(H_{0z}) = 1.108\times 10^{-6}\nu(c/s)a(m)\sigma(\mho/m)I(A)\,, \quad (AT/m)\,. \tag{4.2}$$

In practice, we use a search coil to measure the magnetic field. If the radius of the search coil is r, and its number of turns is n, the induced e.m.f. will become as follows,

$$e = 2.748\times10^{-11}\nu^2 nr^2(m^2)a(m)\sigma(\mho/m)I(A)\,, \quad (V)\,. \tag{4.3}$$

It shows that the higher the frequency, the larger the voltage will become. On the other hand, as we need the value of $\lim_{\nu\to0} I_m(H_{0z})/\nu$, we have to make measurements at low frequencies.

The frequencies to be used in practice will be estimated by considering a case of circular loops. From Fq. 1. 19′ and 1. 22 we get

$$I_m(H_{0z}) \simeq \frac{\pi\mu_0\nu\sigma aI}{4} \tag{4.4}$$

provided

$$\xi^2 = \pi\mu_0\nu\sigma a^2 \ll 1\,. \tag{4.4′}$$

From Eq. 4. 4, we find that the value $I_m(H_{0z})$ is proportional to the frequency. Since the condition of Eq. 4. 4′ can be modified as

$$\nu \ll \frac{2.5\times10^5}{\sigma(\mho/m)a^2(m^2)}\,, \tag{4.5}$$

which indicates that we can use considerably high frequencies providing σ and a are not too large.

4. 2. Proposal of a Method for Phase Difference Measurement

In the C.I.M. proposed by S. Stefanescu, the apparent conductivity is determined by Eq. 3. 8, using the measured values of $I_m(H_{0z})$ and I, but the apparent conductivity can be determined also by measuring the phase differences between the primary and the secondary magnetic fields. From Eq. 3. 8 we get

$$I_m(H_{0z}) = \mu_0 I\nu\sigma a \log(\sqrt{2}+1)\,, \tag{4.6}$$

on the other hand $R_e(H_{0z})$ i.e. the real component of the resultant magnetic field, will coincide with the vertical magnetic field H_{ez} of the primary one when the frequency is very low, and the latter field becomes

$$\lim_{\nu\to0} H_{ez} = \frac{\sqrt{2}I}{\pi a}\,. \tag{4.7}$$

Therefore we get

$$\tan\varphi = \frac{I_m(H_{0z})}{H_{ez}}$$

$$= \left\{\frac{\pi\mu_0}{\sqrt{2}}\log(\sqrt{2}+1)\right\} a^2\sigma\nu \tag{4.8}$$

when ν is small. From this equation we obtain the following result:

$$\sigma = 4.064\times10^5\frac{1}{a^2(m^2)}\lim_{\nu\to0}\left(\frac{\tan\varphi}{\nu}\right), \quad (\mho/m)\,. \tag{4.9}$$

In the case of circular loops, we get

$$\sigma = 5.066 \times 10^5 \frac{1}{a^2(m^2)} \lim_{\nu \to 0} \left(\frac{\tan \varphi}{\nu} \right), \quad (\mho/m) . \tag{4.10}$$

Since we can calculate the apparent conductivity using the measured values of the phase difference φ, we can plot the $\sigma - a$ curve and determine the structure of horizontally stratified earth by a method similar to that in the preceding section.

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