On the Electric Field due to Tides. III.

By

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(Received January 22, 1959)

In this paper the author will discuss briefly some results of theoretical work carried out by Dr. Longuet-Higgins, Dr. Malkus and others (Section 2). The theory of field due to streams of elliptical cross-sction will also be described (Section 3), and a method of generalization of this theory will be discussed (Section 4).

In Section 5 the results of numerical computation concerning the streams of elliptical cross-section will be shown, and in Section 6 the approximate relation between the mean velocity and potential difference will be discussed for two types of streams.

1. Introduction

The author has published two papers [1),2)] concerning the electric field induced by tidal streams, mainly to contribute to the theory of sheath currents in submarine cables[3),4)].

On the other hand, potential differences observed between moored or towed electrodes have been discussed by several authors in order to utilize this phenomenon for measurements of some oceanographic quantities[5)-9)]. In this connection, some theoretical results obtained by Dr. Longuet-Higgins, Dr. Malkus, Dr. Stommel, Dr. Stern and others will be discussed here briefly.

The theory of electric field induced by streams of elliptical cross-section, which has been established by Dr. Longuet-Higgins[10),11)], will also be presented, and a part of his theory will be generalized to a case where the stream velocity is not constant.

2. Electromagnetic Methods for Measuring the Velocity of Ocean Currents.

2.1. Principles of the Electromagnetic Methods[11)].

Let the stream velocity at any point be $\boldsymbol{v}$, and the earth's magnetic field be $\boldsymbol{H}$, then the e.m.f. per unit distance will be

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$$\boldsymbol{F} = \mu_0 \boldsymbol{v} \times \boldsymbol{H}, \tag{2.1}$$

As a result of this e.m.f., a current will flow through the sea water and sea bed, the density of which is given by

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \tag{2.2}$$

where σ is the conductivity of the medium and $\boldsymbol{E}$ the electric field. As shown in the previous paper[2], the field $\boldsymbol{E}$ can be divided into two parts, the irrotational partial field $\boldsymbol{E}'$ and the other partial field $\boldsymbol{E}''$ which is identical to the distributed e.m.f. $\boldsymbol{F}$:

$$\boldsymbol{E} = \boldsymbol{E}' + \boldsymbol{E}'', \tag{2.3}$$

$$\left.\begin{aligned} \boldsymbol{E}' &= -\text{grad}\, V', \\ \boldsymbol{E}'' &= \boldsymbol{F}. \end{aligned}\right\} \tag{2.4}$$

The potential V' can be determined by the velocity $\boldsymbol{v}$, the magnetic field $\boldsymbol{H}$, and the geometry of the stream and sea bed.

Eq. (2. 3) can be modified as follows:

$$\mu_0 \boldsymbol{v} \times \boldsymbol{H} = \boldsymbol{E} + \text{grad}\, V'. \tag{2.5}$$

This relation suggests the possibility of determining the velocity $\boldsymbol{v}$ from electrical measurements, assuming that the earth's magnetic field is known.

Generally, the relation between the observable electrical quantities and the velocity is rather complicated, but it will become very simple in the two limiting cases, where either essentially no current flows or the current is almost short circuited.

2. 2. Measurements by Electrodes at Rest.

(1) As we have seen in one of the previous papers[1], the potential difference $\varDelta V'$ observed between two point electrodes placed at arbitrary points is given by

$$\begin{aligned} \varDelta V' &= \int_A^B (\boldsymbol{F} - \boldsymbol{E}) \cdot d\boldsymbol{s} \\ &= \mu_0 \int_A^B (\boldsymbol{v} \times \boldsymbol{H}) \cdot d\boldsymbol{s} - \int_A^B \boldsymbol{E} \cdot d\boldsymbol{s}. \end{aligned} \tag{2.6}$$

If the stream velocity is almost uniform from the surface to the bottom, and the bed is effectively non-conducting, current will not flow, and the field $\boldsymbol{E}$ almost vanishes. In such a case Eq. (2. 6) will become

$$\varDelta V' \simeq \mu_0 \int_A^B (\boldsymbol{v} \times \boldsymbol{H}) \cdot d\boldsymbol{s}. \tag{2.7}$$

In the case where the velocity has no vertical component, and the two points A and B are at the same depth, Eq. (2. 7) can be written as follows:

$$\varDelta V' \simeq -\mu_0 H_y \int_A^B v_z\, dx + \mu_0 H_y \int_A^B v_x\, dz. \tag{2.7a}$$

Moreover, if the velocity is constant between the points A and B,

$$\Delta V' \simeq -\mu_0 H_y(v_z \Delta x - v_x \Delta z)\,. \tag{2.7b}$$

Hence the components v_x and v_z can be determined by measuring the potential differences along two different directions.

This method can be applied to the measurement of shoal water or tidal current ("the method of moored electrodes").

(2) When two electrodes are placed near the two shores of a channel, the potential difference between these electrodes is given by Eq. (2.6), or

$$V_B' - V_A' = -\mu_0 \int_A^B v_z H_y dx - \int_A^B \sigma j_x dx\,, \tag{2.8}$$

where the electrode line is taken as the x-axis, and the mean stream of the ocean current is assumed to be directed parallel to the z-axis.

Assuming H_y=const. and $\sigma=\sigma_1$=const., and integrating the above equation between $y=0$ and h, we get

$$\int_0^h V_A' dy - \int_0^h V_B' dy = \mu_0 H_y \int_{-c'}^{+c'} \int_0^h v_z dy dx + \sigma_1 \int_{-c'}^{+c'} \int_0^h j_x dy dx\,, \tag{2.9}$$

where $+c'$ and $-c'$ are the x-coordinates of the points A and B respectively, and $2c'$ is taken to be sufficiently larger than the width $2c$ of the stream.

If the bed is non-conducting ($\sigma_2=0$), the current flux flowing across a vertical plane x=const. must be zero:

$$\int_0^h j_x dx = 0\,. \tag{2.10}$$

On the other hand, the dependence of V' on y at $x=\pm c'$ ($c'-c \gg h$) is so small that we can write

$$\int_0^h V' dy \simeq hV'\,, \quad x = \pm c'\,.$$

Moreover, the transport T of the stream is given by

$$T = \int_{-c}^{+c} \int_0^h v_z dy dx = \int_{-c'}^{+c'} \int_0^h v_z dy dx\,. \tag{2.11}$$

Substituting these relations into Eq. (2.9), we obtain

$$T \simeq \frac{h}{\mu_0 H_y}(V_A' - V_B')\,. \tag{2.12}$$

Although this relation has been established by Dr. Malkus and Dr. Stern for a more general case[12], it should be noted that the assumption of a non-conducting bed is necessary in order that the relation of Eq. (2.10) be valid[11].

For a stream of rectangular cross-section the transport T is related to the mean velocity $\bar{v}$ by the following expression:

$$T = 2ch\bar{v}\,, \tag{2.13}$$

where $2c$ and h are the breadth and depth of the stream respectively. Substituting this expression into Eq. (2.12), we get

$$\bar{v} = \frac{1}{2c\mu_0 H_y} U, \tag{2.14}$$

where

$$U = V_{A}' - V_{B}'. \tag{2.15}$$

2.3. Method of Towed Electrodes[7),11)]

We shall consider here the potential difference between two electrodes which are towed in line behind a ship as shown in **Fig. 1.** Let the velocity of the ocean current be $\boldsymbol{v}$, and the velocity of the ship relative to the water be $\boldsymbol{v}_1$. Then the e.m.f. per unit distance in the stream will be

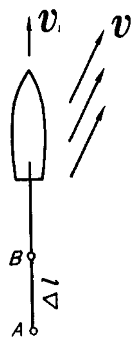

Fig. 1.

$$\boldsymbol{F} = \mu_0 \boldsymbol{v} \times \boldsymbol{H}. \tag{2.16}$$

On the other hand, since the ship and electrodes are in motion with velocity $\boldsymbol{v}+\boldsymbol{v}_1$ relative to the earth, an e.m.f. per unit distance of the electrode line will be induced, the intensity of which is given by

$$\boldsymbol{F}_s = \mu_0(\boldsymbol{v}+\boldsymbol{v}_1) \times \boldsymbol{H}. \tag{2.17}$$

Therefore, the potential difference observed between these two electrodes will be

$$\Delta V' = (\boldsymbol{F}-\boldsymbol{F}_s)\cdot\Delta \boldsymbol{l} - \boldsymbol{E}\cdot\Delta \boldsymbol{l}, \tag{2.18}$$

where $\Delta \boldsymbol{l}$ is the vector of the electrode line $\overline{AB}$.

Since $\boldsymbol{v}_1$ is parallel to $\Delta \boldsymbol{l}$, we have

$$(\boldsymbol{v}_1 \times \boldsymbol{H})\cdot\Delta \boldsymbol{l} = 0, \quad \text{and} \quad \boldsymbol{F}_s\cdot\Delta \boldsymbol{l} = \boldsymbol{F}\cdot\Delta \boldsymbol{l}.$$

and Eq. (2.18) becomes as follows:

$$\Delta V' = -\boldsymbol{E}\cdot\Delta \boldsymbol{l}. \tag{2.19}$$

That is to say, the potential difference between the towed electrodes of unit distance is equal to the electric field $\boldsymbol{E}$ in the water.

If the stream is limited to the shallow part of the ocean, and the stationary water makes a highly conducting return path for the current, the e.m.f. induced in the moving water will be short-circuited by the stationary water, the potential drop per unit distance $\boldsymbol{j}/\sigma$ will become nearly equal to the e.m.f. per unit distance $\boldsymbol{F}$. In this case, Eq. (2.19) can be written as follows:

$$\begin{aligned} \Delta V' &\simeq -\boldsymbol{F}\cdot\Delta \boldsymbol{l} \\ &= \mu_0(\boldsymbol{v}\times\boldsymbol{H})\cdot\Delta \boldsymbol{l}, \end{aligned} \tag{2.20}$$

from which we get

$$\Delta V' \simeq \mu_0 H_y (v_z \Delta x - v_x \Delta z). \tag{2.21}$$

This relation shows that the velocity (v_x, v_z) can be determined from two potential differences measured along two different directions, and is the basic formula for measuring the velocity by means of the Geomagnetic Electrokinetograph[7].

In this case it is assumed that the stream of water is shallow compared with the layer of water at rest; in other words, the current density is assumed to be so large, that the potential drop due to this current will become nearly equal to the e.m.f. However, since the e.m.f. per unit distance is always larger than the potential drop per unit distance, Eq. (2.21) may produce an error in some cases.

Dr. von Arx[7] defined the "k-factor" as a ratio of the e.m.f. between two electrodes to the observed potential difference:

$$k = \frac{F \cdot \Delta l}{\Delta V'} \tag{2.22}$$

or

$$k = \frac{\mu_0 H v}{j_e/\sigma}, \tag{2.22'}$$

where v is the horizontal component of stream velocity which meets the electrode line at right angle, and j_e is the component of current density along the electrode line. According to the observation by Dr. von Arx, the value of k is less than 1.10 in deep oceans, whereas it will become very large in shoal tidal reaches. Usually, tidal streams can be more easily measured by moored electrodes[11].

3. Theory of Electric Field due to a Stream of Elliptical Cross-section after Longuet-Higgins.

Dr. Longuet-Higgins has solved a problem of an electric field induced in a constant stream of elliptical cross-section[10),11)], as shown in **Fig. 2.**

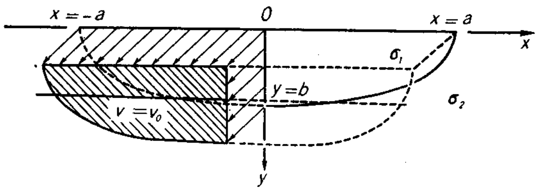

Fig. 2.

Since the method of fictitious current sources[2] is not adequate for this problem*, it will be better to treat it as a boundary value problem. For this purpose, elliptic cylindrical coordinates will be convenient:

$$\left.\begin{aligned} x &= c \cosh \xi \cos \eta, \\ y &= c \sinh \xi \sin \eta, \end{aligned}\right\} \tag{3.1}$$

* Although the distribution of the fictitious current sources can easily be found, it is difficult to obtain the potential due to these sources by means of integrals of Eq. (3.7) in Part II.

An ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{3.2}$$

can be expressed by $\xi=\xi_0=$const., where

$$\left.\begin{aligned} a = c\cosh\xi_0\,,\quad b = c\sinh\xi_0\,; \\ c=\sqrt{a^2-b^2}\,, \end{aligned}\right\} \tag{3.3}$$

Also in this case the electric field $\boldsymbol{E}_1$ inside the ellipse can be expressed as a sum of two partial electric fields $\boldsymbol{E}_1'$ and $\boldsymbol{E}_1''$:

$$\left.\begin{aligned} \boldsymbol{E}_1 &= \boldsymbol{E}_1'+\boldsymbol{E}_1''\,, \\ \boldsymbol{E}_1' &= -\text{grad}\ V_1'\,, \\ \boldsymbol{E}_1'' &= \boldsymbol{F}\,; \end{aligned}\right\} \tag{3.4}$$

and the field outside the ellipse is deduced from a potential V_2':

$$\left.\begin{aligned} \boldsymbol{E}_2 &= \boldsymbol{E}_2'\,, \\ \boldsymbol{E}_2' &= -\text{grad}\ V_2'\,. \end{aligned}\right\} \tag{3.5}$$

These fields must satisfy the boundary conditions:

$$\left.\begin{aligned} E_{1t}' &= E_{2t}'\,, \\ \sigma_1(E_{1n}'+E_{1n}'') &= \sigma_2 E_{2n}'\,; \end{aligned}\right\}\quad \text{when}\quad \xi=\xi_0\,, \tag{3.6}$$

and

$$E_{1y}=0\,,\quad E_{2y}=0\,;\qquad \text{when}\quad y=0\,. \tag{3.7}$$

As shown before, the partial field $\boldsymbol{E}_1''$ is identical with the distributed e.m.f. $\boldsymbol{F}$, and the ξ- and η-components of $\boldsymbol{F}$ can be expressed as follows:

$$\left.\begin{aligned} F_\xi &= E_{0x}\frac{\sinh\xi\cos\eta}{\sqrt{\cosh^2\xi-\cos^2\eta}}\,, \\ F_\eta &= -E_{0x}\frac{\cosh\xi\sin\eta}{\sqrt{\cosh^2\xi-\cos^2\eta}}\,, \end{aligned}\right\}\quad \text{due to } H_y\,; \tag{3.8}$$

$$\left.\begin{aligned} F_\xi &= -E_{0y}\frac{\cosh\xi\sin\eta}{\sqrt{\cosh^2\xi-\cos^2\eta}}\,, \\ F_\eta &= -E_{0y}\frac{\sinh\xi\cos\eta}{\sqrt{\cosh^2\xi-\cos^2\eta}}\,, \end{aligned}\right\}\quad \text{due to } H_x\,; \tag{3.9}$$

where

$$E_{ox}=\mu_0 v_0 H_y\,,\quad E_{ox}=\mu_0 v_0 H_x\,. \tag{3.10}$$

Laplace's equation from which the potentials V_1' and V_2' are obtained, can be written in the coordinates (ξ, η) as follows:

$$\frac{\partial^2 V'}{\partial\xi^2}+\frac{\partial^2 V'}{\partial\eta^2}=0\,, \tag{3.11}$$

the solutions of which are

$$V' = e^{\pm\alpha\xi}\cdot\begin{pmatrix}\cos\alpha\eta \\ \sin\alpha\eta\end{pmatrix}. \tag{3.12}$$

The normal and tangential components of the partial field $\boldsymbol{E}'$ are given by

$$\left.\begin{aligned} E_n' &= -\frac{\partial V'}{\partial n} = -\frac{1}{h_\xi}\frac{\partial V'}{\partial \xi}, \\ E_t' &= -\frac{\partial V'}{\partial t} = -\frac{1}{h_\eta}\frac{\partial V'}{\partial \eta}, \end{aligned}\right\} \tag{3.13}$$

where

$$h_\xi = h_\eta = c\sqrt{\cosh^2 \xi - \cos^2 \eta}\,. \tag{3.14}$$

3.1. Electric Field due to the Vertical Magnetic Field.

Since the partial field $\boldsymbol{E}_1''$ has no vertical component in this case, the conditions of Eq. (3.7) at the surface can be written as follows:

$$\left.\begin{aligned} E_{1\xi}' &= 0, \quad \text{when } \xi = 0,\ 0 > \eta > \pi, \\ E_{1\eta}' &= 0, \quad \text{when } \eta = 0 \text{ or } \pi,\ 0 > \xi > \xi_0; \\ E_{2\eta} &= 0, \quad \text{when } \eta = 0 \text{ or } \pi,\ \xi_0 < \xi. \end{aligned}\right\} \tag{3.15}$$

To satisfy these conditions, the potentials inside and outside the ellipse should take the following forms:

$$\left.\begin{aligned} V_1' &= \sum_{n=1}^{\infty} A_n \cosh n\xi \cos n\eta, \quad 0 < \xi < \xi_0; \\ V_2' &= \sum_{n=1}^{\infty} B_n e^{-n\xi} \cos n\eta, \quad \xi_0 < \xi. \end{aligned}\right\} \tag{3.16}$$

The boundary conditions of Eq. (3.6) can be modified as

$$\left.\begin{aligned} & V_1' = V_2', \\ & -\frac{\partial V_1'}{\partial \xi} + cE_0 \sinh \xi \cos \eta = -\kappa \frac{\partial V_2'}{\partial \xi}, \end{aligned}\right\} \quad \xi = \xi_0. \tag{3.17}$$

where

$$\kappa = \sigma_2/\sigma_1\,. \tag{3.18}$$

Substitution of Eq. (3.16) into Eq. (3.17) gives:

$$\left.\begin{aligned} V_1' &= cE_{0x} \frac{\sinh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \cosh \xi \cos \eta \\ &= E_{0x} \frac{\sinh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} x, \quad \xi < \xi_0; \\ V_2' &= cE_{0x} \frac{\sinh \xi_0 \cosh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} e^{-(\xi - \xi_0)} \cos \eta, \quad \xi_0 < \xi. \end{aligned}\right\} \tag{3.19}$$

Two components of the partial electric field $\boldsymbol{E}_1'$ inside the ellipse can be deduced from V_1':

$$\left.\begin{aligned} E_{1\xi}' &= -E_{0x} \frac{\sinh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \frac{c}{h_\xi} \sinh \xi \cos \eta, \\ E_{1\eta}' &= E_{0x} \frac{\sinh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \frac{c}{h_\xi} \cosh \xi \sin \eta, \end{aligned}\right\} \quad \xi < \xi_0, \tag{3.20}$$

or

$$\left.\begin{aligned} E'_{1x} &= -E_{0x}\frac{\sinh\xi_0}{\sinh\xi_0+\kappa\cosh\xi_0}, \\ E'_{1y} &= 0; \quad \xi<\xi_0, \end{aligned}\right\} \tag{3.21}$$

and the resultant field in the ellipse becomes

$$\left.\begin{aligned} E_{1x} &= E'_{1x}+E''_{1x} = E_{0x}\frac{\kappa\cosh\xi_0}{\sinh\xi_0+\kappa\cosh\xi_0}, \\ E_{1y} &= 0; \quad \xi<\xi_0. \end{aligned}\right\} \tag{3.22}$$

This means that the electric field inside the ellipse is uniform and horizontal.

The electric field outside the ellipse is

$$\begin{pmatrix} E_{2\xi} \\ E_{2\eta} \end{pmatrix} = E_{0x}\frac{\sinh\xi_0\cosh\xi_0}{\sinh\xi_0+\kappa\cosh\xi_0}\frac{c}{h_\xi}e^{-(\xi-\xi_0)}\cdot\begin{pmatrix}\cos\eta \\ \sin\eta\end{pmatrix}, \quad \xi_0<\xi. \tag{3.23}$$

In the special case of a uniform medium ($\kappa=1$), the potentials will become as follows:

$$\left.\begin{aligned} V_1' &= cE_{0x}\sinh\xi_0\cosh\xi\cos\eta, \quad \xi<\xi_0, \\ V_2' &= \frac{1}{2}cE_{0x}\sinh 2\xi_0\cdot e^{-\xi}\cos\eta, \quad \xi_0<\xi. \end{aligned}\right\} \tag{3.24}$$

3. 2. Electric Field due to the Horizontal Magnetic Field.

The partial electric field E_1'' in this case can be expressed as follows:

$$\left.\begin{aligned} E''_{1\xi} &= -E_{0y}\frac{c}{h_\xi}\frac{\partial}{\partial\xi}(\sinh\xi\sin\eta), \\ E''_{1\eta} &= -E_{0y}\frac{c}{h_\eta}\frac{\partial}{\partial\eta}(\sinh\xi\sin\eta), \end{aligned}\right\} \tag{3.25}$$

(see Eqs. (3. 4) and (3. 9)).

The condition, $E_{1y}=0$, to be satisfied at the surface can be modified as

$$\left.\begin{aligned} E'_{1\xi}+E''_{1\xi} &= 0, \quad \xi=0, \quad 0<\eta<\pi; \\ E'_{1\eta}+E''_{1\eta} &= 0, \quad \eta=0 \ \text{ or } \ \pi,\ 0<\xi<\xi_0, \end{aligned}\right\} \tag{3.26}$$

or, combining with Eqs. (3. 25),

$$\left.\begin{aligned} &\frac{\partial}{\partial\xi}(V_1'+cE_{0y}\sinh\xi\sin\eta)=0, \quad \text{when} \ \ \xi=0; \\ &\frac{\partial}{\partial\eta}(V_1'+cE_{0y}\sinh\xi\sin\eta)=0, \quad \text{when} \ \ \eta=0 \ \text{ or } \ \pi. \end{aligned}\right\} \tag{3.27}$$

Considering these conditions, we assume a general expression for the potential V_1' as

$$V_1'+cE_{0y}\sinh\xi\sin\eta=\sum_{n=0}^{\infty}A_n\cosh n\xi\cos n\eta. \tag{3.28}$$

As for the potential outside the ellipse, the following conditions must be satisfied:

$$\left.\begin{aligned} \frac{\partial V_2'}{\partial \eta} &= 0, \qquad \text{when} \quad \eta = 0 \quad \text{or} \quad \pi, \\ V_2' &= 0, \qquad \text{when} \quad \xi = \infty, \end{aligned}\right\} \tag{3.29}$$

for which we assume the form:

$$V_2' = \sum_{n=1}^{\infty} C_n e^{-n\xi} \cos n\eta. \tag{3.30}$$

Substituting Eqs. (3.28) and (3.30) into the boundary conditions of Eq. (3.6), or

$$\left.\begin{aligned} V_1' &= V_2', \\ \frac{\partial}{\partial \xi}(V_1' + cE_0 \sinh \xi \sin \eta) &= \kappa \frac{\partial V_2'}{\partial \xi}, \end{aligned}\right\} \quad \xi = \xi_0, \tag{3.31}$$

we get the relations:

$$\sum_{n=0}^{\infty} A_n \cosh n\xi_0 \cos n\eta - cE_0 \sinh \xi_0 \sin \eta = \sum_{n=1}^{\infty} C_n e^{-n\xi_0} \cos n\eta, \tag{i}$$

and

$$A_n \sinh n\xi_0 = -\kappa C_n e^{-n\xi_0}. \tag{ii}$$

The treatment of Eq. (i) is rather complicated, since it contains $\sin \eta$ as well as $\cos n\eta$. To avoid this difficulty, Dr. Longuet-Higgins employed the relation:

$$\sin \eta = \frac{2}{\pi}\left[1 - 2\sum_{m=1}^{\infty} \frac{1}{4m^2-1} \cos 2m\eta\right]. \tag{3.32}$$

Substituting this expression into Eq. (i), and comparing the coefficients of $\cos n\eta$ we get:

$$\begin{aligned} V_1' &= \frac{2}{\pi} cE_0 - cE_0 \sinh \xi \sin \eta \\ &\quad - \frac{4}{\pi} \kappa cE_0 \sinh \xi_0 \sum_{m=1}^{\infty} \frac{\cosh 2m\xi \cos 2m\eta}{(4m^2-1)(\sinh 2m\xi_0 + \kappa \cosh 2m\xi_0)}, \end{aligned} \tag{3.33}$$

$$V_2' = \frac{4}{\pi} cE_0 \sinh \xi_0 \sum_{m=1}^{\infty} \frac{\sinh 2m\xi_0 \cdot e^{-2m(\xi-\xi_0)} \cos 2m\eta}{(4m^2-1)(\sinh 2m\xi_0 + \kappa \cosh 2m\xi_0)}, \tag{3.34}$$

$$\left.\begin{aligned} E_{1\xi} &= \frac{4}{\pi} \kappa E_0 \frac{c}{h_\xi} \sinh \xi_0 \sum_{m=1}^{\infty} \frac{2m}{4m^2-1} \frac{\sinh 2m\xi \cos 2m\eta}{\sinh 2m\xi_0 + \kappa \cosh 2m\xi_0}, \\ E_{1\xi}' &= E_{1\xi} + E_0 \frac{c}{h_\xi} \cosh \xi \sin \eta. \end{aligned}\right\} \tag{3.35}$$

$$\left.\begin{aligned} E_{1\eta} &= -\frac{4}{\pi} \kappa E_0 \frac{c}{h_\eta} \sinh \xi_0 \sum_{m=1}^{\infty} \frac{2m}{4m^2-1} \frac{\cosh 2m\xi \sin 2m\eta}{\sinh 2m\xi_0 + \kappa \cosh 2m\xi_0}, \\ E_{1\eta}' &= E_{1\eta} + E_0 \frac{c}{h_\eta} \sinh \xi \cos \eta. \end{aligned}\right\} \tag{3.36}$$

$$E_{2\xi} = \frac{4}{\pi} E_0 \frac{c}{h_\xi} \sinh \xi_0 \sum_{m=1}^{\infty} \frac{2m}{4m^2-1} \frac{\sinh 2m\xi_0 \cdot e^{-2m(\xi-\xi_0)} \cos 2m\eta}{\sinh 2m\xi_0 + \kappa \cosh 2m\xi_0}, \tag{3.37}$$

$$E_{2\eta} = -\frac{4}{\pi} E_0 \frac{c}{h_\eta} \sinh \xi_0 \sum_{m=1}^{\infty} \frac{2m}{4m^2-1} \frac{\sinh 2m\xi_0 \cdot e^{-2m(\xi-\xi_0)} \sin 2m\eta}{\sinh 2m\xi_0 + \kappa \cosh 2m\xi_0}. \tag{3.38}$$

In the case where the conductivity is uniform ($\sigma_2=\sigma_1$, $\kappa=1$), these components of the fields can be expressed in closed forms*:

$$\left.\begin{aligned} E_{1\xi} &= \frac{1}{2\pi}E_0\frac{c}{h_\xi}\sinh\xi_0[L(\xi_0+\xi-i\eta)+L(\xi_0+\xi+i\eta) \\ &\qquad -L(\xi_0-\xi-i\eta)-L(\xi_0-\xi+i\eta)], \\ E_{1\eta} &= -\frac{1}{i\,2\pi}E_0\frac{c}{h_\eta}\sinh\xi_0[L(\xi_0+\xi-i\eta)-L(\xi_0+\xi+i\eta) \\ &\qquad +L(\xi_0-\xi-i\eta)-L(\xi_0-\xi+i\eta)]. \end{aligned}\right\} \quad (3.39)$$

$$\left.\begin{aligned} E_{2\xi} &= \frac{1}{2\pi}E_0\frac{c}{h_\xi}\sinh\xi_0[L(\xi+\xi_0-i\eta)+L(\xi+\xi_0+i\eta) \\ &\qquad -L(\xi-\xi_0-i\eta)-L(\xi-\xi_0+i\eta)], \\ E_{2\eta} &= -\frac{1}{i\,2\pi}E_0\frac{c}{h_\eta}\sinh\xi_0[L(\xi+\xi_0-i\eta)-L(\xi+\xi_0+i\eta) \\ &\qquad +L(\xi-\xi_0-i\eta)-L(\xi-\xi_0+i\eta)], \end{aligned}\right\} \quad (3.40)$$

where

$$L(\zeta) = \cosh\zeta\cdot\log\tanh(\zeta/2)\,. \quad (3.41)$$

The horizontal components on the surface will become as follows[11]:

$$E_x = -E_{1\eta} = \frac{1}{\pi}E_0\frac{\sinh\xi_0}{\sin\eta}\Big[\sinh\xi_0\sin\eta\log\frac{\cosh\xi_0-\cos\eta}{\cosh\xi_0+\cos\eta}$$

$$+2\cosh\xi_0\cos\eta\tan^{-1}\frac{\sin\eta}{\sinh\xi_0}\Big], \quad y=0, \quad |x|<c, \quad (3.42a)$$

$$E_x = E_{1\xi} = \frac{1}{\pi}E_0\frac{\sinh\xi_0}{\sinh\xi}[L(\xi_0+\xi)-L(\xi_0-\xi)], \quad y=0, \quad c<|x|<a, \quad (3.42b)$$

$$E_x = E_{2\xi} = \frac{1}{\pi}E_0\frac{\sinh\xi_0}{\sinh\xi}[L(\xi+\xi_0)-L(\xi-\xi_0)], \quad y=0, \quad a<|x|. \quad (3.42c)$$

4. Electric Field due to a Stream of Elliptical Cross-section in which the Velocity Diminishes toward the Boundary.

4.1. Uniform Medium

We shall consider the case where the stream velocity in the elliptical cross-section diminishes toward the boundary, assuming that the magnetic field is vertical and the conductivity is uniform ($\sigma_2=\sigma_1$). If the velocity distribution is given by a function

$$v = v_0 f(\xi')\,, \qquad 0<\xi'<\xi_0\,, \quad (4.1)$$

the distributed e.m.f. is expressed by

$$\left.\begin{aligned} F_x &= E_0 f(\xi')\,, \quad F_y = 0\,, \\ E_0 &= \mu_0 v_0 H_y\,. \end{aligned}\right\} \quad (4.2)$$

---

* For this purpose the following relations are employed:

$$\frac{2m}{4m^2-1} = \frac{1}{2}\left(\frac{1}{2m+1}+\frac{1}{2m-1}\right), \quad (i)$$

$$\sum_{m=1}^{\infty}\frac{e^{-(2m-1)\alpha}}{2m-1} = -\frac{1}{2}\log\tanh\frac{\alpha}{2}\,. \quad (ii)$$

If we divide the velocity v into infinitesimal velocities dv, which is constant inside an ellipse $\xi=\xi'$, the elementary potential dV' due to the elementary velocity dv will become as follows:

$$dV' = \frac{1}{2}\,c\,dF_x(1-e^{-2\xi'})\cosh\xi\cos\eta\,,\qquad \xi<\xi'\,,$$
$$= \frac{1}{4}\,c\,dF_x(e^{2\xi'}-e^{-2\xi'})\,e^{-\xi}\cos\eta\,,\qquad \xi'<\xi\,,$$

(see Eq. (3.24)), where

$$dF_x = \mu_0 H_y dv$$
$$= E_0 f'(\xi')\,d\xi'\,.$$

Integrating the above equations, we obtain the potentials due to the stream with a velocity distribution given by Eq. (4.1):

$$V_1' = \frac{1}{4}\,cE_0 e^{-\xi}\cos\eta\int_{\xi}^{0}(e^{2\xi'}-e^{-2\xi'})\,f'(\xi')\,d\xi'$$
$$+\frac{1}{2}\,cE_0\cosh\xi\cos\eta\int_{\xi_0}^{\xi}(1-e^{-2\xi'})\,f'(\xi')\,d\xi'\,,\qquad 0<\xi<\xi_0\,,\tag{4.3}$$
$$V_2' = \frac{1}{4}\,cE_0 e^{-\xi}\cos\eta\int_{\xi_0}^{0}(e^{2\xi'}-e^{-2\xi'})\,f'(\xi')\,d\xi'\,,\qquad \xi_0<\xi\,.\tag{4.4}$$

If we assume a velocity distribution such that

$$\left.\begin{aligned} f(\xi') &= 1\,, && 0\leqslant\xi'\leqslant\xi_1\,,\\ &= \frac{e^{\xi_0}-e^{\xi'}}{e^{\xi_0}-e^{\xi_1}}\,, && \xi_1\leqslant\xi'\leqslant\xi_0\,, \end{aligned}\right\}\tag{4.5}$$

as shown in **Fig. 3**, the potential will become as follows:

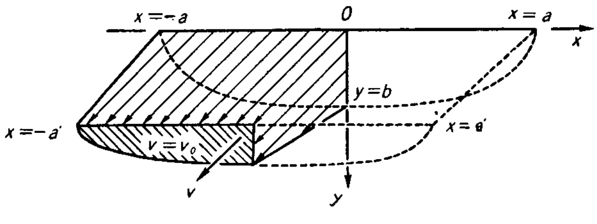

Fig. 3.

$$V_1' = \frac{1}{2}\,cE_0(1-e^{-\xi_0-\xi_1})\cosh\xi\cos\eta\,,\qquad 0<\xi<\xi_1\,,$$
$$= \frac{1}{2}\,cE_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\cosh\xi\cos\eta+\frac{1}{4}\,cE_0\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\,\xi_1)\,e^{-\xi}\cos\eta\,,$$
$$\xi_1<\xi<\xi_0\,,\tag{4.6}$$
$$V_2' = \frac{1}{4}\,cE_0K(\xi_0,\,\xi_1)\,e^{-\xi}\cos\eta\,,\qquad \xi_0<\xi\,;\tag{4.7}$$

where

$$K(p,q) = \frac{1}{3}(e^{2p}+e^{p+q}+e^{2q})-e^{-p-q}\,.\tag{4.8}$$

From these expressions we obtain the ξ- and η-components of the electric fields:

$$\left.\begin{aligned} E_{1\xi} &= E'_{1\xi}+E''_{1\xi}, \\ E_{1\eta} &= E'_{1\eta}+E''_{1\eta}, \end{aligned}\right\} \quad 0<\xi<\xi_0; \tag{4.9}$$

$$\left.\begin{aligned} E'_{1\xi} &= -\frac{1}{2}E_0(1-e^{-\xi_0-\xi_1})\frac{c}{h_\xi}\sinh\xi\cos\eta, \\ E'_{1\eta} &= \frac{1}{2}E_0(1-e^{-\xi_0-\xi_1})\frac{c}{h_\eta}\cosh\xi\sin\eta, \end{aligned}\right\} \quad 0<\xi<\xi_1; \tag{4.10}$$

$$\left.\begin{aligned} E'_{1\xi} &= -\frac{1}{2}E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\frac{c}{h_\xi}\sinh\xi\cos\eta \\ &\quad +\frac{1}{4}E_0\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\frac{c}{h_\xi}e^{-\xi}\cos\eta, \\ E'_{1\eta} &= \frac{1}{2}E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\frac{c}{h_\eta}\cosh\xi\sin\eta \\ &\quad +\frac{1}{4}E_0\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\frac{c}{h_\eta}e^{-\xi}\sin\eta, \end{aligned}\right\} \quad \xi_1<\xi<\xi_0; \tag{4.11}$$

$$E''_{1\xi} = E_0\frac{c}{h_\xi}\sinh\xi\cos\eta\cdot\begin{cases} 1, & 0<\xi<\xi_1, \\ \dfrac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}, & \xi_1<\xi<\xi_0; \end{cases} \tag{4.12}$$

$$E''_{1\eta} = -E_0\frac{c}{h_\eta}\cos\xi\sin\eta\cdot\begin{cases} 1, & 0<\xi<\xi_1, \\ \dfrac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}, & \xi_1<\xi<\xi_0; \end{cases} \tag{4.13}$$

$$\left.\begin{aligned} E_{2\xi} &= \frac{1}{4}E_0K(\xi_0,\ \xi_1)\frac{c}{h_\xi}e^{-\xi}\cos\eta, \\ E_{2\eta} &= \frac{1}{4}E_0K(\xi_0,\ \xi_1)\frac{c}{h_\eta}e^{-\xi}\sin\eta, \end{aligned}\right\} \quad \xi_0<\xi. \tag{4.14}$$

It should be noted that the electric field in the region of constant velocity ($0<\xi<\xi_1$) is uniform and horizontal, i.e.

$$\left.\begin{aligned} E_{1x} &= E_{1\xi}\frac{c}{h_\xi}\sinh\xi\cos\eta-E_{1\eta}\frac{c}{h_\eta}\cosh\xi\sin\eta \\ &= \frac{1}{2}E_0(1+e^{-\xi_0-\xi_1}), \\ E_{1y} &= E_{1\xi}\frac{c}{h_\xi}\cosh\xi\sin\eta+E_{1\eta}\frac{c}{h_\eta}\sinh\xi\cos\eta \\ &= 0. \end{aligned}\right\} \quad 0<\xi<\xi_1. \tag{4.15}$$

4.2. Two-Layer Problem.

If the conductivity σ_2 of the medium outside the ellipse $\xi=\xi_0$ is different from that of the water inside the ellipse, the interior and exterior potentials will be changed from the values given by Eqs. (4.6) and (4.7).

Let the secondary potential inside the ellipse be V_1^*, and the resultant potential outside the ellipse be V_2. These will be expressed as follows:

$$\left.\begin{aligned} V_1{}^* &= A\cosh\xi\cos\eta\,, \\ V_2 &= Be^{-\xi}\cos\eta\,. \end{aligned}\right\} \tag{4.16}$$

The ξ- and η- components of the secondary field inside the ellipse are:

$$\left.\begin{aligned} E^*_{1\xi} &= -\frac{A}{h_\xi}\sinh\xi\cos\eta\,, \\ E^*_{1\eta} &= \frac{A}{h_\eta}\cosh\xi\sin\eta\,, \end{aligned}\right\} \tag{4.17}$$

and the components of the resultant field outside the ellipse are:

$$\left.\begin{aligned} E_{2\xi} &= \frac{B}{h_\xi}e^{-\xi}\cos\eta\,, \\ E_{2\eta} &= \frac{B}{h_\eta}e^{-\xi}\sin\eta\,. \end{aligned}\right\} \tag{4.18}$$

The expressions (4.11), obtained for the field in the case where the conductivity is uniform, can be used as the primary field in this problem:

$$\left.\begin{aligned} E'_{p\xi} &= -\frac{1}{2}E_0\Big[\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\sinh\xi \\ &\quad -\frac{1}{2}\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\,e^{-\xi}\Big]\frac{c}{h_\xi}\cos\eta\,, \\ E'_{p\eta} &= \frac{1}{2}E_0\Big[\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\cosh\xi \\ &\quad +\frac{1}{2}\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\,e^{-\xi}\Big]\frac{c}{h_\eta}\sin\eta\,. \end{aligned}\right\} \tag{4.19}$$

These must satisfy the boundary conditions:

$$\left.\begin{aligned} E'_{p\eta}+E^*_{1\eta} &= E_{2\eta}\,, \\ E'_{p\xi}+E^*_{1\xi} &= \kappa E_{2\xi}\,, \end{aligned}\right\}\quad \xi = \xi_0\,, \tag{4.20}$$

where

$$\kappa = \sigma_2/\sigma_1\,.$$

It should be noted that the partial field $E''_{p\xi}$ which is equal to $E''_{1\xi}$ of Eq. (4.12) vanishes when $\xi=\xi_0$.

Substitution of Eqs. (4.17), (4.18) and (4.19) into Eq. (4.20) will give:

$$\left.\begin{aligned} A &= \frac{1}{4}cE_0\frac{(1-\kappa)\,e^{-\xi_0}}{\sinh\xi_0+\kappa\cosh\xi_0}K(\xi_0,\ \xi_1)\,, \\ B &= \frac{e^{2\xi_0}}{1-\kappa}A\,. \end{aligned}\right\} \tag{4.21}$$

Using these constants, we obtoin the following results:

$$V_1 = V_p'+V_1{}^*\,, \tag{4.22}$$

$$\left.\begin{aligned}
V_p' &= \frac{1}{2}cE_0(1-e^{-\xi_0-\xi_1})\cosh\xi\cos\eta\,, && 0<\xi<\xi_0;\\
&= \frac{1}{2}cE_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\cosh\xi\cos\eta\\
&\quad+\frac{1}{4}cE_0\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\,e^{-\xi}\cos\eta\,, && \xi_1<\xi<\xi_0;
\end{aligned}\right\}\tag{4.23}$$

$$V_1^* = \frac{1}{4}cE_0\frac{(1-\kappa)\,e^{-\xi_0}}{\sinh\xi_0+\kappa\cosh\xi_0}K(\xi,\ \xi_1)\cosh\xi\cos\eta\,,\qquad 0<\xi<\xi_0;\tag{4.24}$$

$$V_2 = \frac{1}{2}cE_0\frac{1}{\sinh\xi_0+\kappa\cosh\xi_0}K(\xi_0,\ \xi_1)\,e^{-(\xi-\xi_0)}\cos\eta\,,\qquad \xi_0<\xi;\tag{4.25}$$

$$E_{1\xi} = E'_{p\xi}+E''_{p\xi}+E^*_{1\xi};\tag{4.26}$$

$$\left.\begin{aligned}
E'_{p\xi} &= -\frac{1}{2}E_0(1-e^{-\xi_0-\xi_1})\frac{c}{h_\xi}\sinh\xi\cos\eta\,, && 0<\xi<\xi_1,\\
&= -\frac{1}{2}E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\frac{c}{h_\xi}\sinh\xi\cos\eta\\
&\quad+\frac{1}{4}E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\frac{c}{h_\xi}e^{-\xi}\cos\eta\,, && \xi_1<\xi<\xi_0;
\end{aligned}\right\}\tag{4.27}$$

$$\left.\begin{aligned}
E''_{p\xi} &= E_0\frac{c}{h_\xi}\sinh\xi\cos\eta\,, && 0<\xi<\xi_1,\\
&= E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}\frac{c}{h_\xi}\sinh\xi\cos\eta\,, && \xi_1<\xi<\xi_0;
\end{aligned}\right\}\tag{4.28}$$

$$E^*_{1\xi} = -\frac{1}{4}E_0\frac{(1-\kappa)\,e^{-\xi_0}K(\xi_0,\ \xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}\frac{c}{h_\xi}\sinh\xi\cos\eta\,,\qquad 0<\xi<\xi_0;\tag{4.29}$$

$$E_{1\eta} = E'_{p\eta}+E''_{p\eta}+E^*_{1\eta};\tag{4.30}$$

$$\left.\begin{aligned}
E'_{p\eta} &= \frac{1}{2}E_0(1-e^{-\xi_0-\xi_1})\frac{c}{h_\eta}\cosh\xi\sin\eta\,, && 0<\xi<\xi_1,\\
&= \frac{1}{2}E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1-e^{-\xi_0-\xi})\frac{c}{h_\eta}\cosh\xi\sin\eta\\
&\quad+\frac{1}{4}E_0\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}K(\xi,\ \xi_1)\frac{c}{h_\eta}e^{-\xi}\sin\eta\,, && \xi_1<\xi<\xi_0;
\end{aligned}\right\}\tag{4.31}$$

$$\left.\begin{aligned}
E''_{p\eta} &= \frac{1}{2}E_0\frac{c}{h_\eta}\cosh\xi\sin\eta\,, && 0<\xi<\xi_1,\\
&= -E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}\frac{c}{h_\eta}\cosh\xi\sin\eta\,, && \xi_1<\xi<\xi_0;
\end{aligned}\right\}\tag{4.32}$$

$$E^*_{1\eta} = \frac{1}{4}E_0\frac{(1-\kappa)\,e^{\xi_0}K(\xi_0,\ \xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}\frac{c}{h_\eta}\cosh\xi\sin\eta\,,\qquad 0<\xi<\xi_1;\tag{4.33}$$

$$\left.\begin{aligned}
E_{2\xi} &= \frac{1}{4}E_0\frac{K(\xi_0,\ \xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}\frac{c}{h_\eta}e^{-(\xi-\xi_0)}\cos\eta\,,\\
E_{2\eta} &= \frac{1}{4}E_0\frac{K(\xi_0,\ \xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}\frac{c}{h_\eta}e^{-(\xi-\xi_0)}\sin\eta\,,
\end{aligned}\right\}\ \xi_0<\xi.\tag{4.34}$$

Also in this case, the resultant field in the region $0<\xi<\xi_1$ is uniform and horizontal, i.e.

$$\left.\begin{aligned} E_{1x} &= \frac{1}{2}E_0(1+e^{-\xi_0-\xi_1})-\frac{1}{4}E_0\frac{(1-\kappa)\,e^{-\xi_0}K(\xi_0,\ \xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}\,, \\ E_{1y} &= 0\,, \qquad 0<\xi<\xi_1\,. \end{aligned}\right\} \tag{4.35}$$

5. Numerical Examples

5.1. Current Lines.

The current lines can be drawn by computing the current flux through a surface η=const.:

$$\left.\begin{aligned} J &= -\sigma_1\int_0^{\xi} E_{1\eta}\,h_\xi\,d\xi\,, & 0<\xi<\xi_0\,, \\ &= -\sigma_1\int_0^{\xi_0} E_{1\eta}\,h_\xi\,d\xi-\sigma_2\int_{\xi_0}^{\xi} E_{2\eta}\,h_\xi\,d\xi\,, & \xi_0<\xi\,. \end{aligned}\right\} \tag{5.1}$$

In the case of uniform velocity and vertical magnetic field we get:

$$\left.\begin{aligned} J &= J_{\max}\frac{\sinh\xi}{\sinh\xi_0}\sin\eta\,, & 0<\xi<\xi_0\,, \\ &= J_{\max}e^{-(\xi-\xi_0)}\sin\eta\,, & \xi_0<\xi\,, \end{aligned}\right\} \tag{5.2}$$

where $J_{\max}$ is the total current which circulates about the point $x=0$, $y=b$:

$$J_{\max} = c\,\sigma_1 E_0\frac{\kappa\sinh\xi_0\cosh\xi_0}{\sinh\xi_0+\kappa\cosh\xi_0}\,. \tag{5.3}$$

The current lines inside the ellipse are uniform and horizontal, as can be seen from Eq. (5.2), or

$$J = \sigma_1 E_{1x}\,y\,, \tag{5.4}$$

where E_{1x} has been given in Eq. (3.22).

Although the total current $J_{\max}$ depends on the ratio of conductivities ($\kappa=\sigma_2/\sigma_1$), the shape of the current lines does not depend on the ratio κ (see Eq. (5.2)).

Fig. 4 shows a set of current lines inside and outside the stream of elliptical cross-section. The curves are similar to those of the rectangular cross-section shown in Fig. 4(a) of Part II[2].

If the stream velocity varies in the manner shown in Fig. 3, and the conductivity of the stationary medium is equal to that of the moving medium, the integrals of Eq. (5.1) will become as follows:

$$\begin{aligned} J &= \frac{c}{2}\sigma_1 E_0(1+e^{-\xi_0-\xi_1})\sinh\xi\sin\eta\,, \qquad 0<\xi<\xi_1\,, \\ &= \frac{c}{2}\sigma_1 E_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}(1+e^{-\xi_0-\xi})\sinh\xi\sin\eta \\ &\quad +\frac{c}{4}\sigma_1 E_0\frac{e^{\xi}-e^{\xi_1}}{e^{\xi_0}-e^{\xi_1}}e^{-\xi}K(\xi,\ \xi_1)\sinh\eta\,, \quad \xi_1<\xi<\xi_0\,. \end{aligned} \tag{5.5}$$

$$J = \frac{c}{4}\sigma_1 E_0 e^{-\xi}K(\xi_0,\ \xi_1)\sin\eta\,, \qquad \xi_0<\xi\,. \tag{5.6}$$

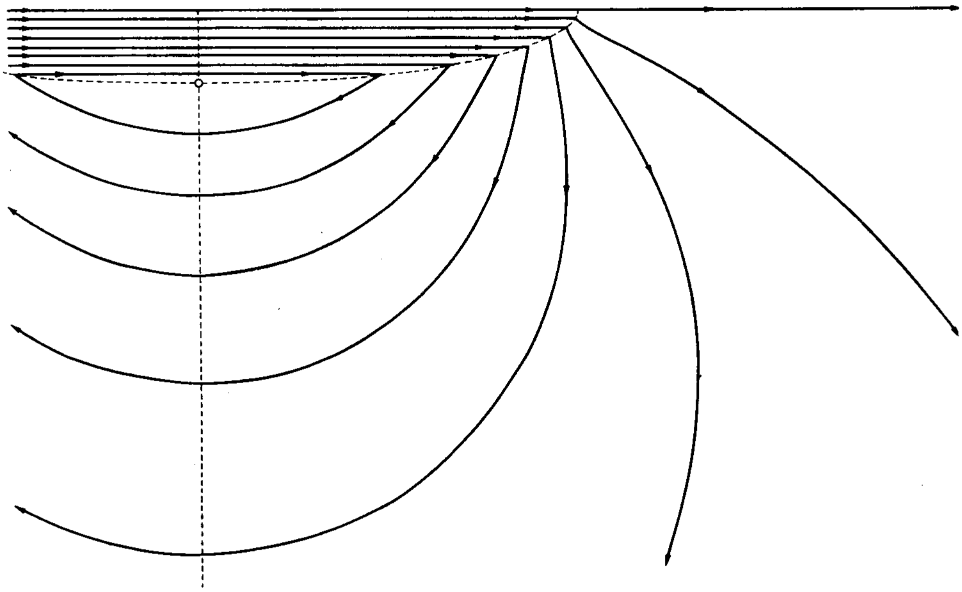

Fig. 4.

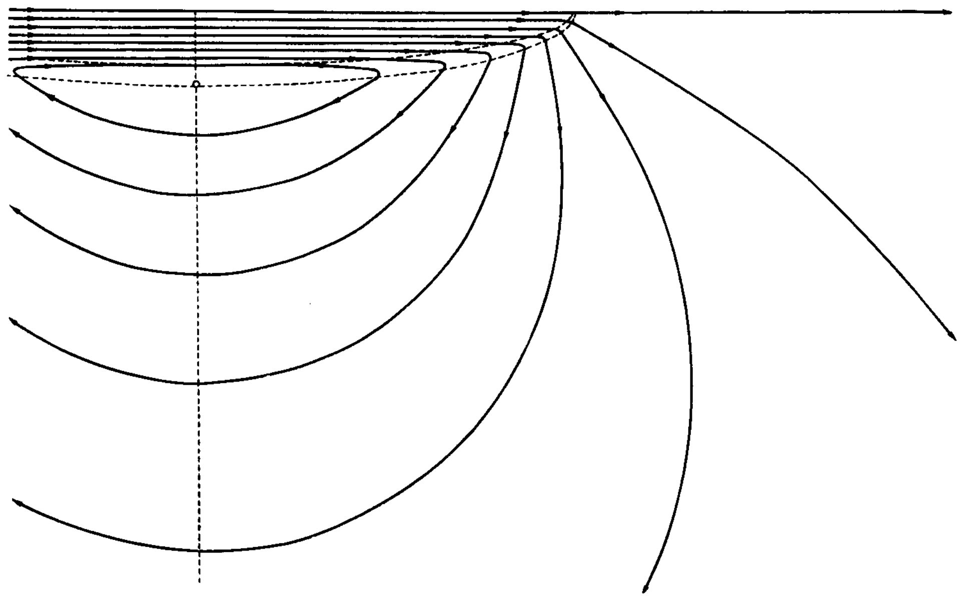

Fig. 5.

In this case the center of the circulation of current will not be on the ellipse $\xi=\xi_0$, but on its minor axis ($\xi_1<\xi<\xi_0$, $\eta=\pi/2$). However, as can be seen from Eqs. (5.2) and (5.6), the expressions for the current flux outside the ellipse take the same form in both cases where the velocity is constant over the elliptical cross-section, and the velocity varies according to the function shown in Eq. (4.5), i.e.

$$J = \text{const.}\, e^{-\xi} \sin \eta\,, \qquad \xi_0 < \xi\,; \tag{5.7}$$

which means that the shape of the current lines outside the ellipse does not depend on the parameter ξ_1 (or b').

In **Fig. 5** a set of current lines are shown for the case where the stream velocity diminishes toward the boundary. Although these curves outside the ellipse are almost the same in their general shape to that of Fig. 4, the current lines inside the ellipse are not uniform in the annular domain near the boundary, where the stream velocity is not constant.

5.2. Electric Field on the Surface and the Bottom of a Stream.

If the stream velocity is uniform and the magnetic field is vertical, the electric field on the surface can be expressed as follows (see Eqs. (3.22) and (3.23)):

$$\left.\begin{aligned} E_{1x} &= E_0 \frac{\kappa \cosh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0}, \qquad -a < x < +a\,, \quad y = 0\,, \\ E_{2x} &= E_0 \frac{\cosh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} e^{-(\xi - \xi_0)}, \qquad a < |x|, \quad y = 0\,. \end{aligned}\right\} \tag{5.8}$$

The tangential component of the electric field along the boundary (ellipse) of the stream is given by:

$$\left.\begin{aligned} E_{1\eta} &= -E_0 \frac{\kappa \cosh^2 \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \frac{c}{h_{\xi_0}} \sin \eta\,, \\ E_{2\eta} &= E_0 \frac{\sinh \xi_0 \cosh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \frac{c}{h_{\xi_0}} \sin \eta\,, \end{aligned}\right\} \quad \xi = \xi_0\,, \tag{5.9}$$

where

$$h_{\xi_0} = c\sqrt{\cosh^2 \xi_0 - \cos^2 \eta}\,. \tag{5.10}$$

The difference between $E_{1\eta}$ and $E_{2\eta}$ at $\xi = \xi_0$ is equal to $E''_{1\eta}$ (or F_η).

In the case of variable velocity, these fields on the surface become as follows:

$$\begin{aligned} E_{1x} &= \frac{1}{2} E_0 (1 + e^{-\xi_0 - \xi_1}) - \frac{1}{4} E_0 (1 - \kappa) \frac{K(\xi_0, \xi_1)}{\sinh \xi_0 + \kappa \cosh \xi_0}, \quad y = 0\,, \quad 0 < |x| < a'\,, \\ &= \frac{1}{2} E_0 \frac{e^{\xi_0} - e^{\xi}}{e^{\xi_0} - e^{\xi_1}} (1 + e^{-\xi_0 - \xi}) - \frac{1}{4} E_0 (1 - \kappa) \frac{K(\xi_0, \xi_1)}{\sinh \xi_0 + \kappa \cosh \xi_0} \\ &\quad + \frac{1}{4} E_0 \frac{e^{\xi} - e^{\xi_1}}{e^{\xi_0} - e^{\xi_1}} \frac{K(\xi, \xi_1)}{e^{2\xi} - 1}, \qquad y = 0\,, \quad a' < |x| < a\,; \end{aligned} \tag{5.11}$$

$$E_{2x} = \frac{1}{2} E_0 \frac{K(\xi_0, \xi_1)}{\sinh \xi_0 + \kappa \cosh \xi_0} \frac{e^{\xi_0}}{1 - e^{-2\xi}}, \qquad y = 0\,, \quad a < |x|\,. \tag{5.12}$$

The tangential component of the electric field along the boundary is given by:

$$E_{1\eta} = E_{2\eta} = \frac{1}{4} E_0 \frac{K(\xi_0, \xi_1)}{\sinh \xi_0 + \kappa \cosh \xi_0} \frac{c}{h_\xi} \sin \eta\,, \qquad \xi = \xi_0\,. \tag{5.13}$$

Fig. 6 shows electric fields on the surface ($y=0$) and along the bottom ($\xi=\xi_0$) for three values of the conductivity ratio κ. As mentioned above, the relative shape of each curve is equal to that in the case of uniform conductivity ($\kappa=1$).

Fig. 7 shows electric fields for the case where the velocity diminishes toward the boundary of the stream. It can be seen that the tangential components of the electric field along the bottom and on the surface outside the ellipse are very weak compared to those for the case of constant velocity (Fig. 6). In **Fig. 8** the electric fields at the surface near the transition region are shown in detail.

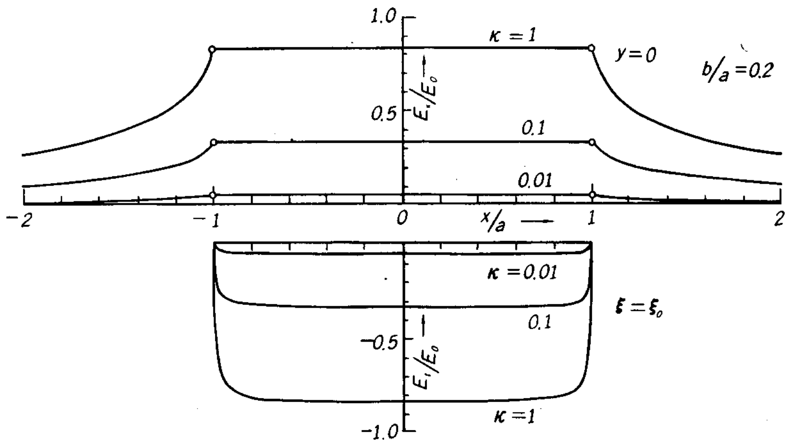

Fig. 6.

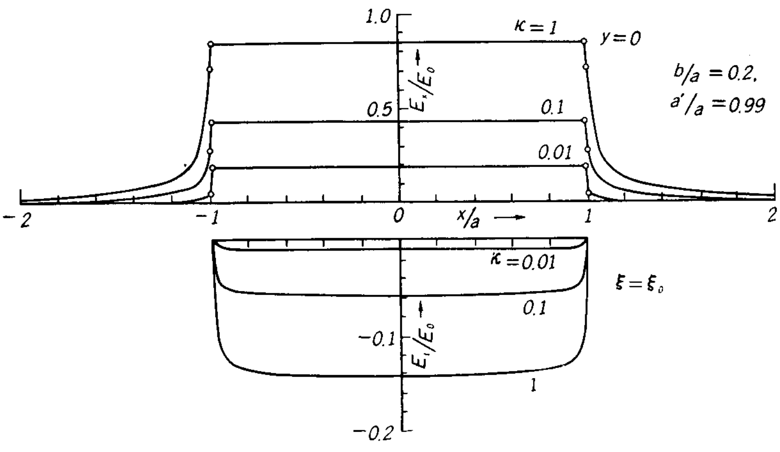

Fig. 7.

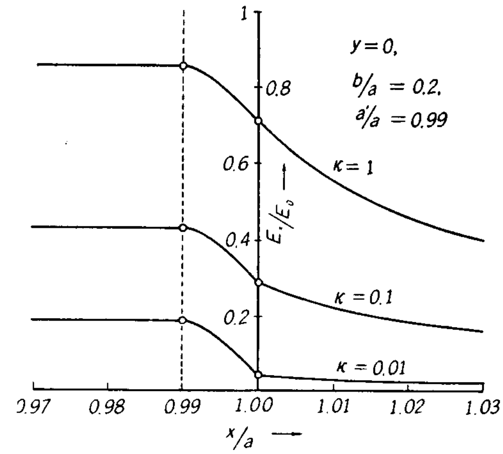

Fig. 8.

6. Mean Velocity.

We have seen in Section 2.2 that the potential difference between the two shores of a stream will be nearly proportional to the total transport or to the mean velocity of the stream. We shall discuss here this approximation for some special cases.

6.1. Rectangular Cross-Section.

If the stream velocity is a function of depth only (Case II in Table I of Part II[2)]), and the medium is uniform ($\sigma_2=\sigma_1$), the potential difference between the two shores is given by:

$$U = \frac{E_0}{\pi(h-g)}[F(0, h)-F(0, g)-F(2c, h)+F(2c, g)], \tag{6.1}$$

where

$$F(\xi, \eta) = \frac{1}{2}(\xi^2-\eta^2)\log(\xi^2+\eta^2)-2\xi\eta\tan^{-1}(\eta/\xi). \tag{6.2}$$

For simplicity, if we put $g=0$ (see **Fig. 9**), Eq. (6.1) becomes as follows:

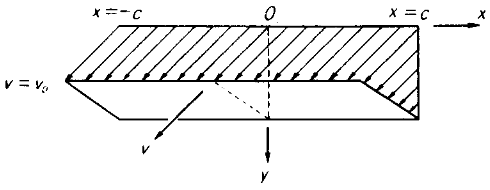

Fig. 9.

$$U = \frac{2}{\pi}cE_0\left[\frac{2c}{h}\log 2c-\frac{h}{2c}\log h-\left(\frac{c}{h}-\frac{h}{4c}\right)\log(4c^2+h^2)+2\tan^{-1}\frac{h}{2c}\right], \tag{6.3}$$

where

$$E_0 = \mu_0 v_0 H_y;$$

and v_0 is the maximum velocity at the surface.

The mean velocity $\bar{v}$ and the mean e.m.f. $\bar{E}_0$ per unit distance in this case will be:

$$\bar{v}=\frac{1}{2}v_0,\quad \bar{E}_0=\frac{1}{2}E_0. \tag{6.4}$$

On the other hand, if we picture a stream of cross-section equal to that in the above case, but assume that the velocity is constant and equal to $\bar{v}$, the potential difference between the shores of this stream will be given by:

$$\bar{U}=\frac{\bar{E}_0}{\pi}[H(2c,h)-H(0,h)], \tag{6.5}$$

or

$$\bar{U}=\frac{4}{\pi}c\bar{E}_0\Big[-\frac{h}{2c}\log h+\frac{h}{4c}\log(4c^2+h^2)+\tan^{-1}\frac{h}{2c}\Big]. \tag{6.6}$$

Substituting the value of $\bar{E}_0$ of Eq. (6.4) into Eq. (6.6), and comparing with Eq. (6.3), we obtain:

$$U=\bar{U}-\frac{2}{\pi}cE_0\Big[\frac{c}{h}\log\Big\{1+\Big(\frac{h}{2c}\Big)^2\Big\}-\tan^{-1}\Big(\frac{h}{2c}\Big)\Big], \tag{6.7}$$

or

$$U\simeq\bar{U}+\frac{1}{\pi}cE_0\Big[\Big(\frac{h}{2c}\Big)+\frac{1}{6}\Big(\frac{h}{2c}\Big)^3\Big]. \tag{6.8}$$

This means that the potential difference U can be replaced by $\bar{U}$, when $h/(2c)$ is small ; in other words, U can be calculated assuming that the velocity is uniform and its value is equal to the mean velocity $\bar{v}$ of the stream.

6.2. Elliptical Cross-Section.

If we put $\xi=\xi_0$ in Eq. (4.25), we get:

$$V_2=\frac{1}{4}cE_0\frac{K(\xi_0,\xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}\cos\eta,\qquad \xi=\xi_0. \tag{6.9}$$

Hence, the potential difference between the shores ($\eta=0$ and π) will be

$$U=\frac{1}{2}cE_0\frac{K(\xi_0,\xi_1)}{\sinh\xi_0+\kappa\cosh\xi_0}. \tag{6.10}$$

Since the velocity distribution in this case has been assumed as

$$\left.\begin{aligned} v(\xi)&=v_0, && 0<\xi<\xi_1;\\ &=v_0\frac{e^{\xi_0}-e^{\xi}}{e^{\xi_0}-e^{\xi_1}}, && \xi_1<\xi<\xi_0,\end{aligned}\right\} \tag{6.11}$$

the mean velocity on the minor axis ($\eta=\pi/2$) of the ellipse can be calculated by:

$$\bar{v}=\frac{\int_0^{\xi_0}v(\xi)\,h_\xi\,d\xi}{\int_0^{\xi_0}h_\xi\,d\xi}, \tag{6.12}$$

where

$$h_\xi = c \cosh \xi\,, \quad \eta = \pi/2\,. \tag{6.13}$$

From Eq. (6.12) we get:

$$\bar{v} = v_0 \frac{1}{2 \sinh \xi_0} \left[\frac{1}{2} (e^{\xi_0} + e^{\xi_1}) - \frac{\xi_0 - \xi_1}{e^{\xi_0} - e^{\xi_1}} \right]. \tag{6.14}$$

Therefore, in the case where the velocity is $\bar{v}$ (const.), the potential difference between the two ends of the major axis of the ellipse is given by:

$$\begin{aligned} \bar{U} &= -2cE_0 \frac{\sinh \xi_0 \cosh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \\ &= cE_0 \frac{\cosh \xi_0}{\sinh \xi_0 + \kappa \cosh \xi_0} \left[\frac{1}{2} (e^{\xi_0} + e^{\xi_1}) - \frac{\xi_0 - \xi_1}{e^{\xi_0} - e^{\xi_1}} \right]. \end{aligned} \tag{6.15}$$

Eqs. (6.10) and (6.15) can be modified as follows:

$$\left. \begin{aligned} U &= U_0 \frac{1}{4 \cosh \xi_0} \frac{K(\xi_0, \xi_1)}{\sinh \xi_0 + \kappa \cosh \xi_0}, \\ \bar{U} &= U_0 \frac{1}{2} \left(\frac{e^{\xi_0} + e^{\xi_1}}{2} - \frac{\xi_0 - \xi_1}{e^{\xi_0} - e^{\xi_1}} \right) \frac{1}{\sinh \xi_0 + \kappa \cosh \xi_0}, \end{aligned} \right\} \tag{6.16}$$

where U_0 is the potential difference in the case where the velocity is constant $(v = v_0)$ inside the ellipse $(\xi = \xi_0)$, and the medium outside the ellipse is non-conducting $(\kappa = 0)$;

$$U_0 = 2aE_0 = 2c \cosh \xi_0 \cdot E_0\,. \tag{6.17}$$

It can be easily shown that the ratio of two values of U for $\kappa = 0$ and $\kappa = 1$, and the ratio of two values of $\bar{U}$ for $\kappa = 0$ and $\kappa = 1$ are given by the formula:

$$\frac{U(\kappa=0)}{U(\kappa=1)} = \frac{\bar{U}(\kappa=0)}{\bar{U}(\kappa=1)} = 1 + \frac{a}{b}, \qquad b \leqslant a, \tag{6.18}$$

where a and b are the semi-major and semi-minor axes of the ellipse respectively.

It will also be evident that the relative error

$$\frac{\Delta U}{U} = \frac{\bar{U} - U}{U} \tag{6.19}$$

which arose from the approximation of using the value $\bar{U}$ in place of U does not depend on the conductivity ratio κ.

In **Fig. 10** the true value U and the approximate value $\bar{U}$ of the potential differences are compared for the two extreme cases where $\kappa = 0$ and $\kappa = 1$, varying the parameter ξ_1 (or b') of the velocity distribution.

Fig. 11 shows the results for a similar comparison between U and $\bar{U}$ for the case where $\xi_1 = 0$, varying the shape of ellipse.

From these figures it can be seen that the error of $\bar{U}$ is quite small if a/b exceeds 4 or 5.

Acknowledgement.

The author wishes to express his sincere thanks to Dr. Longuet-Higgins who suggested some of the problems treated in this paper.

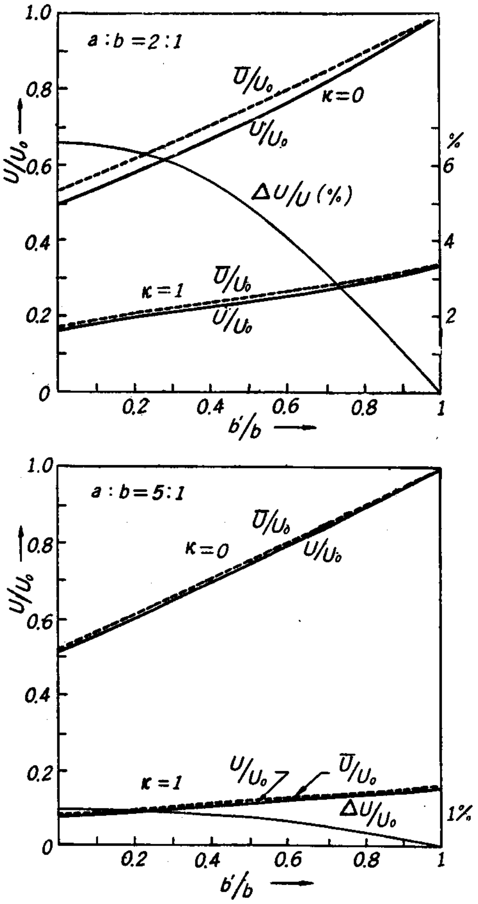

Fig. 10.

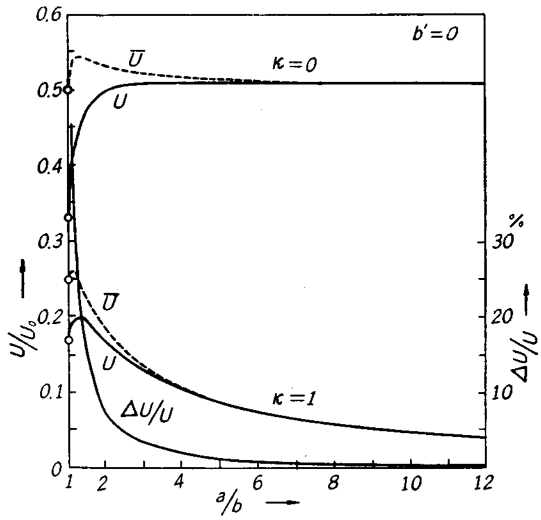

Fig. 11.

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