A Graphical Method for the Evaluation of the Statistical Responses of Non-Linear Control Systems to a Stationary Random Signal Considering Random Noise

By

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This paper describes a graphical method for the analysis of automatic control systems containing a non-linear element of zero-memory type with stationary random inputs. The starting point of the analysis is the use of the statistical equivalent gain of the non-linear element for the input, by which the non-linear feedback control system reduces to an equivalent linear one. When the input to the non-linear feedback control system consists of a random noise and a random signal and if these are statistically independent of each other, we can use two different equivalent gains of the non-linear element which can be found by the "R.M.S.-Error Criterion" for each one of these. Then the non-linear system may be treated as an equivalent linear feedback control system with random noise or with random signal. It is easy, therefore, to find the responses of the non-linear feedback control system to such inputs. In the special case when the system input is only a sinusoidal wave, this method reduces to the "describing-function" method.

In order to illustrate the procedure of the graphical analysis in detail, we considered three control systems as numerical examples. As for the results we were able to investigate the effects of random noise on the harmonic response of the non-linear control system and the jump phenomenon which can be seen in the relation curve between the variance of the random signal and the one of the system response.

1. Introduction

In our previous paper[1], we proposed the statistical linearization method for of a non-linear element impressed by random inputs and derived analytically the statistical, equivalent gains of the non-linear element of zero-memory type. The outline of this method is as follows.

Assume that a non-linear element of zero-memory type is expressed by

$$y(t) = f\{z(t)\} \tag{1-1}$$

where $y(t)$ and $z(t)$ are output and input signals of the non-linear element respectively.

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If the input $z(t)$ is a sinusoidal function of a constant amplitude B_z with constant frequency ω_0 contaminated by a stationary random time function $z_2(t)$, the magnitude of which is normally distributed with mean zero and the variance σ_z;

$$z(t) = B_z \cos(\omega_0 t+\theta)+z_2(t)\,, \tag{1-2}$$

then the optimum linear representations (in a mean-square error sense) of the non-linear element are κ for the sinusoidal portion and α for the random portion of the input. We call κ and α the equivalent gains of the non-linear element to such an input. The representations of κ and α are as follows,

$$\kappa = \frac{2j}{\pi B_z}\int_{C+} F_+(jw)\exp\left(-\frac{\sigma_z}{2}w^2\right)J_1(B_z w)dw \tag{1-3}$$

$$\alpha = \frac{1}{\pi}\int_{C+} F_+(jw)(jw)\exp\left(-\frac{\sigma_z}{2}w^2\right)J_0(B_z w)dw \tag{1-4}$$

where J_1 and J_0 are the Bessel functions, and the function $F_+(jw)$ is given by the following expression;

$$F_+(jw) = \left[\int_0^\infty f(z)\exp(-sz)dz\right]_{s=jw} \tag{1-5}$$

and c_+ is the Bromwich integral path in the s-plane (see Appendix 1).

When σ_z tends to zero, the input to the non-linear element becomes only the sinusoidal wave and we have the describing-function κ_0 from (1-3);

$$\begin{aligned}\lim_{\sigma_z\to 0}\kappa = \kappa_0 &= \frac{2j}{\pi B_z}\int_{C+} F_+(jw)J_1(B_z w)dw \\ &= \frac{1}{\pi B_z}\int_0^{2\pi} f(B_z\sin\varphi)\sin\varphi\, d\varphi \end{aligned}\tag{1-6}$$

where $\varphi=\omega_0 t+\theta$, (see Appendix 2).

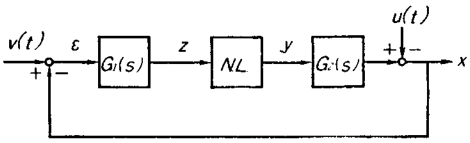
On the other hand, when B_z tends to zero, the input becomes only the random time function and we have the following equivalent gain α_0 from (1-4);

$$\begin{aligned}\lim_{B_z\to 0}\alpha &= \alpha_0 \\ &= \frac{1}{\pi}\int_{C+}(jw)F_+(jw)\exp\left(-\frac{\sigma_z}{2}w^2\right)dw\,. \end{aligned}\tag{1-7}$$

Since the equivalent gains, κ and α, both depend on the statistical properties of the input, it is difficult to find analytically the response of a feedback control system containing such non-linear element impressed by a random input. Therefore, a graphical method is proposed for the analysis of the statistical response of the non-linear feedback control system to the random inputs.

2. Graphical Method for the Evaluation of the Response of a Non-linear Feedback Control System to Sinusoidal and Stationary Random Inputs

In order to describe the graphical method in detail, a non-linear feedback control system impressed by a sinusoidal signal and a stationary random noise as shown in Fig. 1, will be considered. In Fig. 1, we assume that the input to the system is composed of the sinusoidal signal $v(t)=B_v \cos \omega_0 t$ and the random noise $u(t)$ with a normal distribution function.



N.L.: Non-Linear Element of Zero-Memory Type

Fig. 1. Block Diagram for Typical Non-Linear Control System.

In the feedback control system, the application of the equivalent gains of the non-linear element derived in the preceding section is difficult because the input to the nonlinear element depends on the response of the feedback control system, and the determination of the equivalent gains requires knowledge of the statistical properties of the input to the non-linear element in the feedback loop.

Now, if the input to a linear feedback control system has a sinusoidal signal and a random noise with a normal distribution function, the response is also composed of a sinusoidal wave and a random time function with a normal distribution function. Since the superposition principle does not hold in the non-linear feedback system, the response of the system to such an input is too complicated to be analyzed rigorously. From the fact that a controlled system usually is of the nature of a low-pass filter, it will be assumed that the input to the non-linear element also is composed of a sinusoidal wave $z_1(t)$ and a random time function $z_2(t)$ with a normal distribution function, that is

$$
\begin{aligned}
z(t) &= z_1(t)+z_2(t) \\
&= B_z \cos(\omega_0 t+\theta)+z_2(t) \qquad (2\text{-}1)
\end{aligned}
$$

where it is again assumed that θ is a random variable with a uniform distribution function.

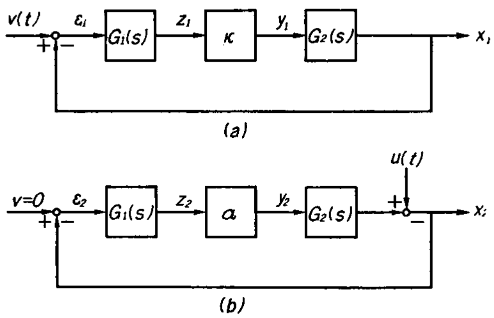


Fig. 2. Two Equivalent Linear Systems for the Non-Linear System as shown in Fig. 1.

Although the values of the equivalent gains, κ and α, are not yet known, the non-linear system can be analyzed as two linear systems containing two parameters as shown in Fig. 2(a) and (b) respectively. From Fig. 2(a) we have

$$B_z^2 = \left| \frac{G_1(j\omega_0)}{1+\kappa G(j\omega_0)} \right|^2 B_v^2 \tag{2-2}$$

where

$$G(j\omega) = G_1(j\omega)G_2(j\omega)\,.$$

On the other hand, from Fig. 2 (b) we have

$$\sigma_z = \frac{1}{4\pi}\int_{-\infty}^{\infty} \left| \frac{G_1(j\omega)}{1+\alpha G(j\omega)} \right|^2 S_u(\omega)d\omega \tag{2-3}$$

where $S_u(\omega)$ is the spectral density of the random noise $u(t)$. Therefore, we can determine the unknowns κ, α, B_z^2 and σ_z from simultaneous equations of the four relations (1-3), (1-4), (2-2) and (2-3). But these solutions are not determined analytically, because equations (1-3) and (1-4) are complicated functions of the arguments B_z^2 and σ_z. Accordingly, it is convenient to use the following graphical method.

By substituting the particular values of the system parameters, B_v, ω_0 and $S_u(\omega)$ into (2-2) and (2-3), these equations are rewritten as follows,

$$B_z^2 = g_1(\kappa) \tag{2-4}$$

$$\sigma_z = g_2(\alpha)\,. \tag{2-5}$$

On the other hand, the relations between κ and B_z^2 with parameter σ_z are obtained from (1-3), and the relations between α and σ_z with parameter B_z^2 are obtained from (1-4). These results are shown in Fig. 3 (a) and (b).

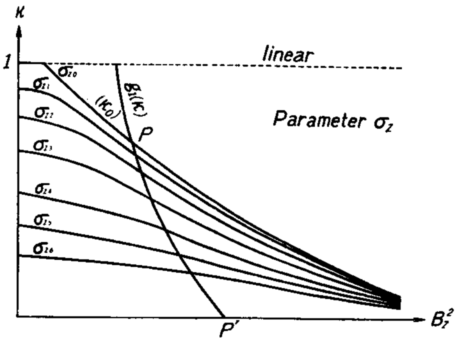


Fig. 3 (a). Graphical Procedure of Step (1).

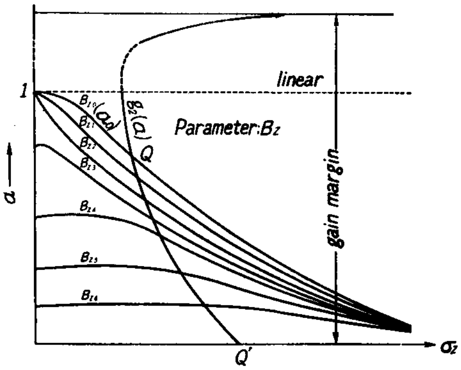


Fig. 3 (b). Graphical Procedure of Step (2).

(1) The relation curve between B_z^2 and κ satisfying (2-4) is drawn in Fig. 3 (a).
(2) The relation curve between σ_z and α satisfying (2-5) is drawn in Fig. 3 (b).
(3) The relation curve between B_z^2 and σ_z obtained from the intersections of κ curves and $g_1(\kappa)$ curve, and the relation curve between the same variables obtained

from the intersections of α curves and $g_2(\alpha)$ curve, are drawn in the $(B_z^2-\sigma_z)$-plane as shown in Fig. 4.

(4) From the intersection of the above two curves, we can find the required values of B_z^2 and σ_z. At the same time, from Fig. 3 (a) and (b) the values of κ and α can be obtained.

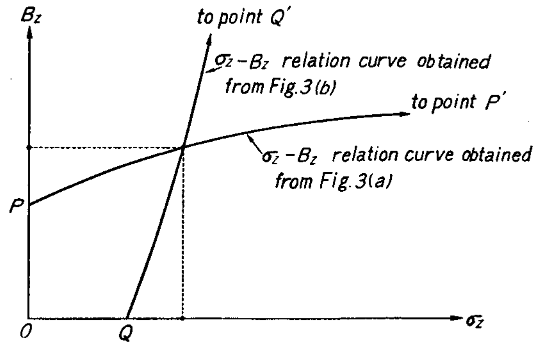


Fig. 4. Graphical Procedure of Step (3).

Special Cases

I. If the system is linear, then κ and α are independent of B_z^2 and σ_z, and are shown by straight lines parallel to the abscissa as shown in Fig. 3 (a) and (b). If the value of σ_z becomes infinite at a certain large value of α in (2-5), then this value corresponds to the gain margin of the linear control system, because we can obtain from (2-3) the following relation;

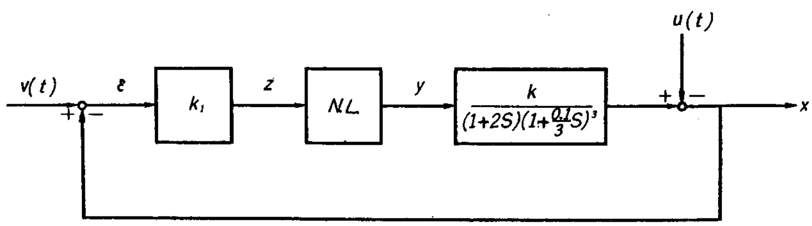
$$\alpha=\frac{1}{|G(s)|}.$$

II. When the input to the non-linear control system is only a random noise with a normal distribution function, the required response σ_z can be obtained from the procedure as shown in Fig. 3 (b) only. On the other hand, when the input to the non-linear control system is only a sinusoidal wave, the required harmonic response is obtained from the procedure as shown in Fig. 3 (a) only.

3. Numerical Examples

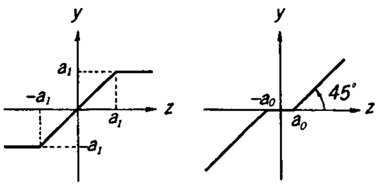
3-1. Harmonic Response of a Non-linear Automatic Control System with a random Disturbance

Let us consider the system as shown in Fig. 5 containing a non-linear element as shown in Fig. 6 (a) or (b). The values of equivalent gains[2], κ and α, for the saturation as shown in Fig. 6 (a) can be calculated from (1-3) and (1-4) as follows,



N.L.: Non-Linear Element as shown in Fig. 6.

Eig. 5. Block Diagram for Non-Linear Process Control System.



(a) Saturation (b) Neutral Zone

Fig. 6. Non-Linear Element of Zero-memory Type.

$$\left.\begin{aligned}\kappa &= \sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{2a_1^2}{\sigma_z}\right)^{m+\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}-m\right)} \\ &\quad\times{}_1F_1\left(\frac{1}{2}+m,\ 2;\ -B_z^2/2\sigma_z\right) \\ \alpha &= \sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{2a_1^2}{\sigma_z}\right)^{m+\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}-m\right)} \\ &\quad\times{}_1F_1\left(\frac{1}{2}+m,\ 1;\ -B_z^2/2\sigma_z\right).\end{aligned}\right\} \qquad (3\text{-}1)$$

The values of equivalent gains[3)], κ and α, for the neutral zone as shown in Fig. 6 (b) can be calculated from the same equations as follows,

$$\left.\begin{aligned}\kappa &= 1-\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{2a_0^2}{\sigma_z}\right)^{m+\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}-m\right)}{}_1F_1\left(\frac{1}{2}+m,\ 2;\ -B_z^2/2\sigma_z\right) \\ \alpha &= 1-\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{2a_0^2}{\sigma_z}\right)^{m+\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}-m\right)}{}_1F_1\left(\frac{1}{2}+m,\ 1;\ -B_z^2/2\sigma_z\right)\end{aligned}\right\} \qquad (3\text{-}2)$$

where ${}_1F_1(a, b;\ -\xi)$ is the confluent hypergeometric function defined as follows,

$${}_1F_1(a, b;\ -\xi) = 1-\frac{a}{b}\frac{\xi}{1!}+\frac{a(a+1)}{b(b+1)}\frac{\xi^2}{2!}-\cdots\cdots$$

and Γ is a Gamma function. In Fig. 7 and Fig. 8, the values of such equivalent gains are plotted in the semi-logarithmic scale.

In Fig. 5, we assume that the disturbance $u(t)$ is a stationary random process with a normal distribution function, and write the auto-correlation function of the disturbance in the following form;

$$R_u(\tau) = \sigma_u e^{-|\tau|} \qquad (3\text{-}3)$$

where σ_u is the variance of the disturbance $u(t)$. Then the spectral density becomes as follows;

$$S_u(\omega) = \frac{4\sigma_u}{1+\omega^2}. \qquad (3\text{-}4)$$

Therefore, from (2-3) we have

$$\sigma_z = \frac{k_1^2}{4\pi}\int_{-\infty}^{\infty}\left|\frac{(1+2j\omega)(1+\frac{0.1}{3}j\omega)^3}{(1+2j\omega)(1+\frac{0.1}{3}j\omega)^3+k_1k\alpha}\right|^2\frac{4\sigma_u}{1+\omega^2}d\omega. \qquad (3\text{-}5)$$

On the other hand, a sinusoidal signal $v(t)$ is applied to the set point of the control system;

$$v(t) = B_v\cos\omega_0 t, \qquad (3\text{-}6)$$

From (2-2) we have

$$\frac{B_z^2}{a_i^2}=\frac{B_\eta^2}{a_i^2}\frac{(1+4\omega_0^2)(1+\frac{0.01}{9}\omega_0^2)^3k_1^2}{(1+4\omega_0^2)(1+\frac{0.01}{3}\omega_0^2)^3+(k_1k\kappa)^2+2k_1k\kappa\left\{1-\frac{0.61}{3}\omega_0^2+\frac{0.002}{27}\omega_0^4\right\}} \tag{3-7}$$

where a_i $(i=0,1)$ is a characteristic parameter of the non-linear element as shown in Fig. 6.

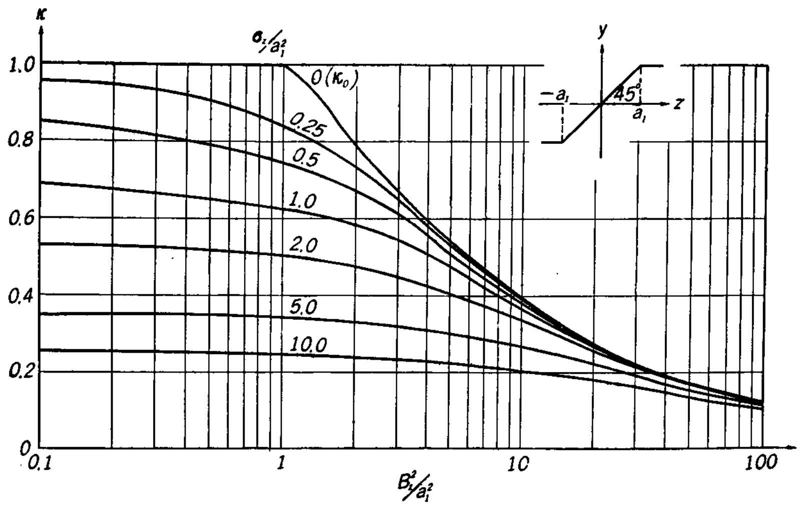


Fig. 7 (a). Equivalent Gains for Sinusoidal Signal of Saturation.

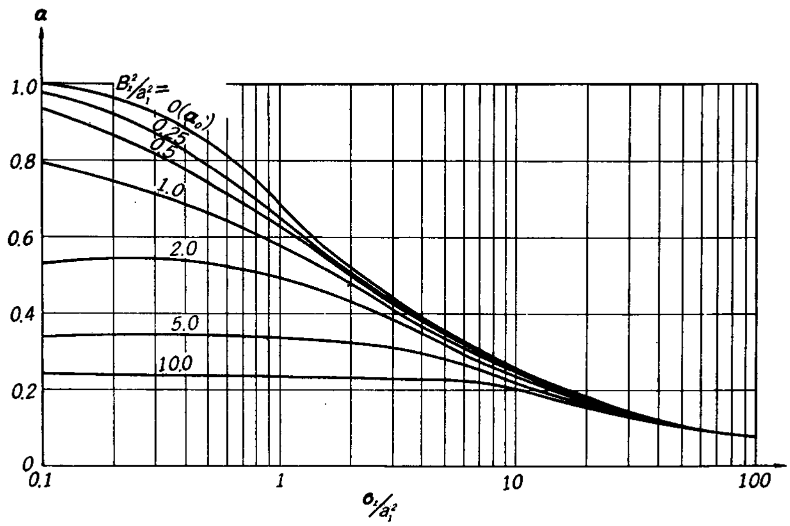


Fig. 7 (b). Equivalent Gains for Random Noise of Saturation.

The evaluation of the integral (3-5) leads to the following result using the table computed by G. R. MacLane[4],

$$\frac{\sigma_z}{a_i^2} = \frac{k_1^2\sigma_u}{a_i^2}\,\frac{-1.81(k_1k\alpha)^3+556.9(k_1k\alpha)^2+20189.3(k_1k\alpha)+30047}{(1+k_1k\alpha)\{-163.8(k_1k\alpha)^2+9824.5(k_1k\alpha)+30047\}}\,. \tag{3-8}$$

As an example, let us put $k_1=k=1$, then equations (3-7) and (3-8) become as follows,

$$\frac{B_z^2}{a_i^2} = \frac{B_v^2}{a_i^2}\,\frac{2}{2+\kappa^2+1.9\kappa} \qquad \text{for} \quad \omega_0 = 0.5 \tag{3-9}$$

$$\frac{\sigma_z}{a_i^2} = \frac{\sigma_u}{a_i^2}\,\frac{-1.81\alpha^3+556.9\alpha^2+20189.3\alpha+30047}{(1+\alpha)(-163.8\alpha^2+9824.5\alpha+30047)}\,. \tag{3-10}$$

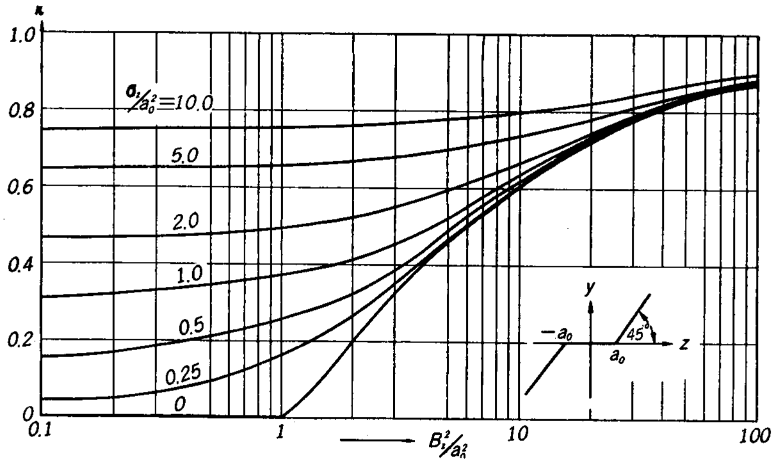


Fig. 8 (a). Equivalent Gains for Sinusoidal Signal of Neutral Zone.

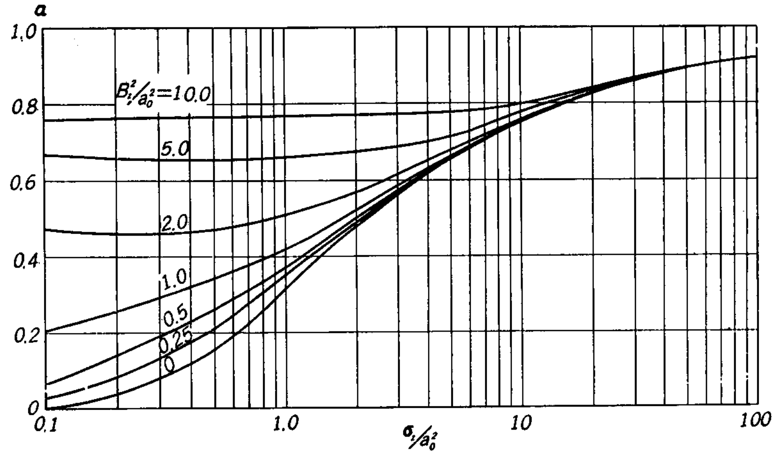
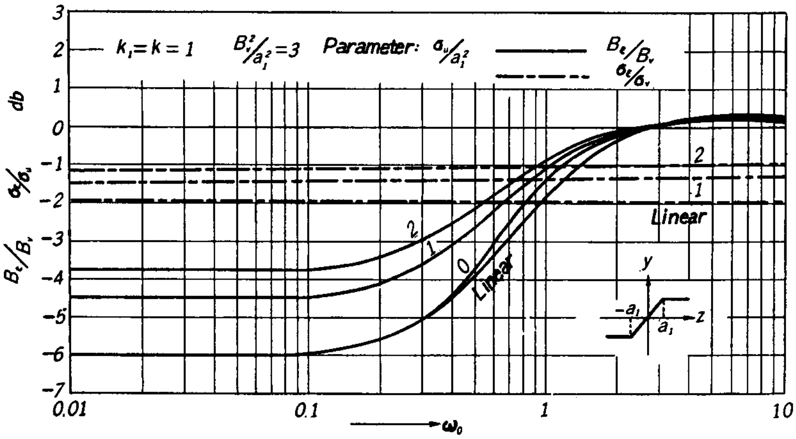


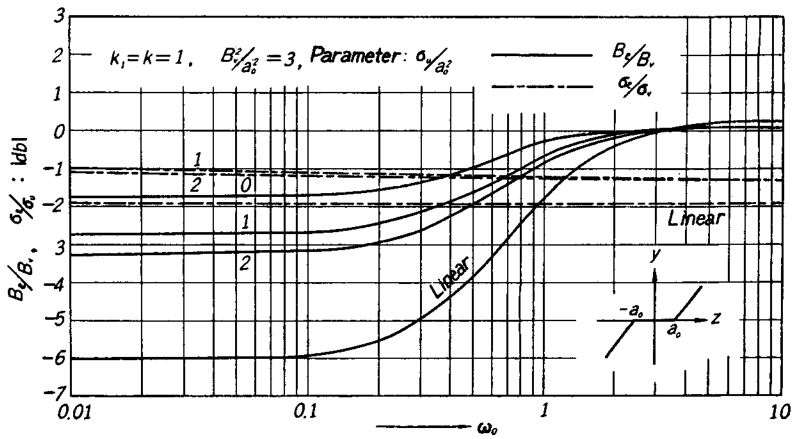
Fig. 8 (b). Equivalent Gains for Random Noise of Neutral Zone.

For the particular values of the system parameter $k_1=k=1$, and the amplitude of sinusoidal input $B_v^2/a_i^2=3$, the obtained values, B_ε/B_v and $\sigma_\varepsilon/\sigma_u$, are shown in Fig. 9 as a function of ω_0 with a parameter σ_u/a_i^2. For the particular values of the same system parameter and the variance of the disturbace $\sigma_u/a_i^2=1$, the obtained value B_x/B_z is shown in Fig. 10 as a function of ω_0 with a parameter B_v^2/a_i^2.

Moreover, the obtained values of B_x/B_v and σ_x/σ_u are shown in Fig. 11 as a function of the gain of the controlled system k, for the particular values of $B_v/a_i=1$, $\omega_0=0.5$ and $\sigma_u/a_i^2=1$.



(a) When N.L. is Saturation



(b) When N.L. is Neutral Zone

Fig. 9. Effects of the Random Disturbance on the Harmonic Response and Variance of Error Signal against the Frequency of Impressing Sinusoidal Signal.

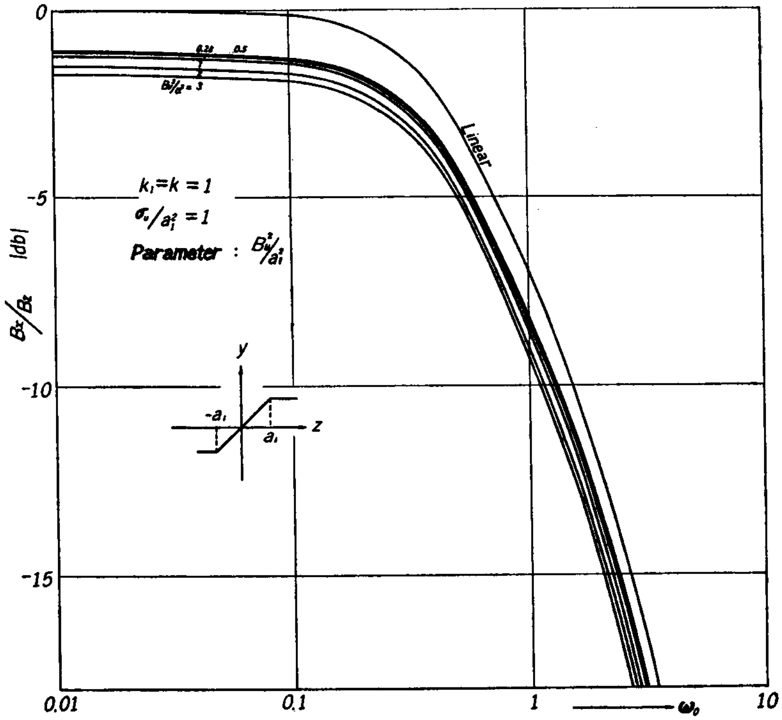


Fig. 10 (a). Harmonic Responses of Controlled System Containing Saturation in the Feedback Loop.

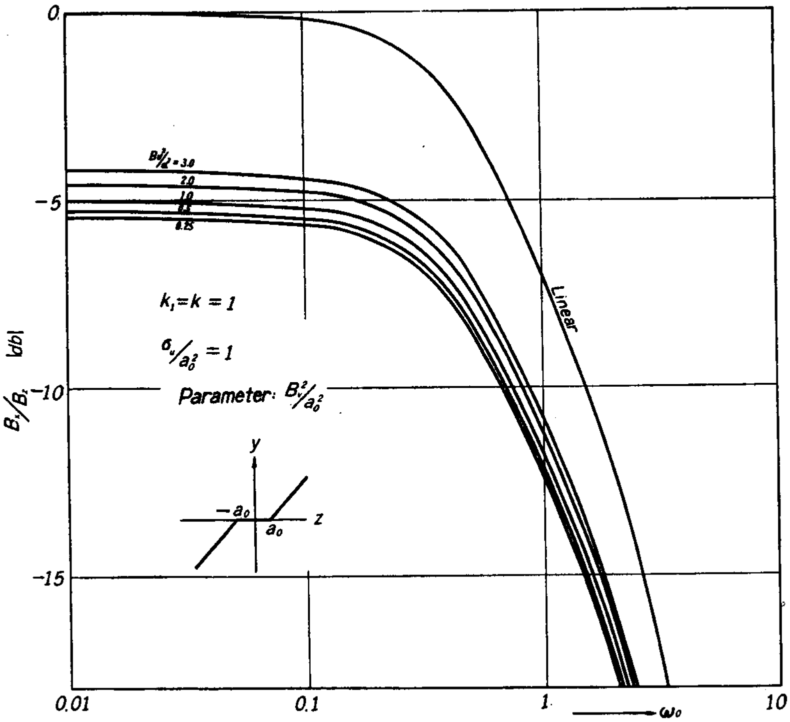


Fig. 10 (b). Harmonic Responses of Controlled System Containing Neutral Zone in the Feedback Loop.

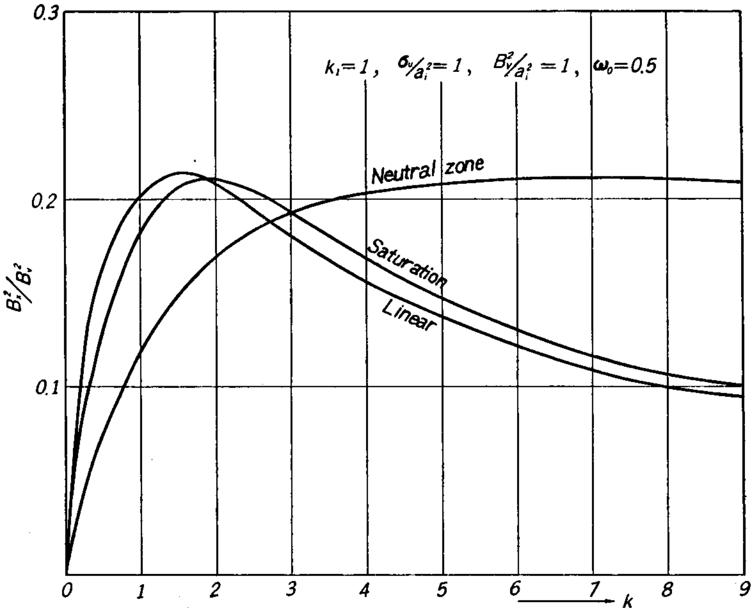


Fig. 11 (a). Evaluation of Sinusoidal portion of Output Signal against the Gain of Controlled System.

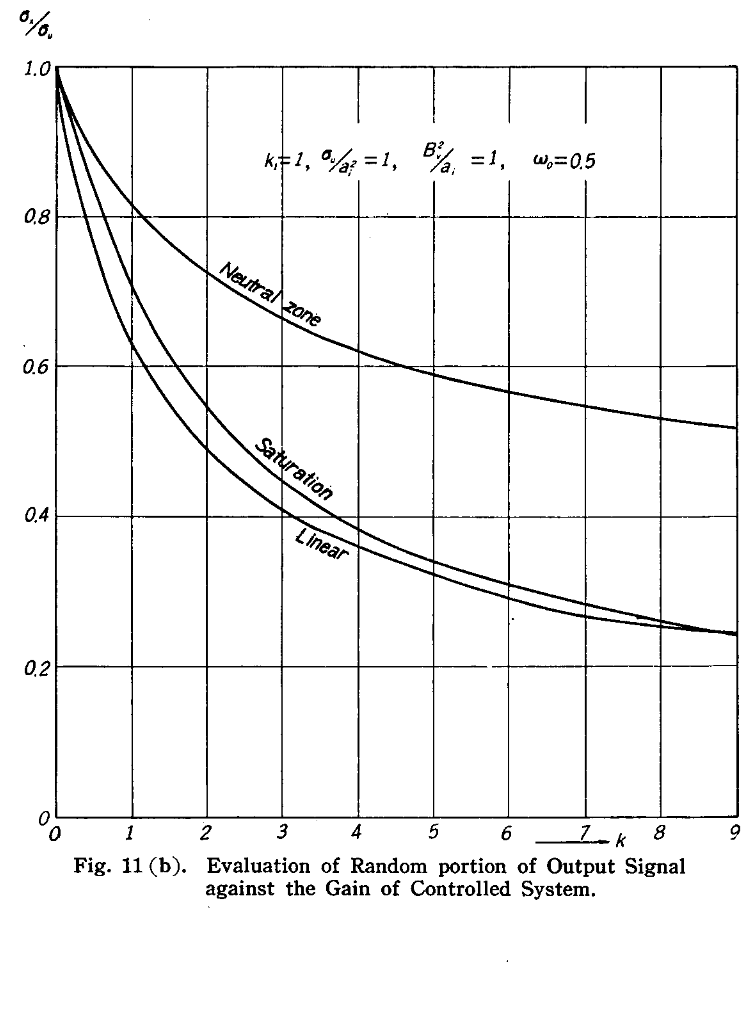


Fig. 11 (b). Evaluation of Random portion of Output Signal against the Gain of Controlled System.

3–2. Jump Phenomena Observed in the Respone of Some Non-linear Control Systems

R. C. Booton[5] has reported that, with the help of an analog computer, when an automatic control system containing a saturation is excited by a stationary random signal with a normal distribution function, a discontinuous jump in the system response is observed for a slow increase in variance of the random signal. If the variance of the measured error response σ_ε is plotted as a function of variance of the exciting signal σ_u, the curve as shown in Fig. 12 is obtained. As the variance σ_u increases from zero, the variance σ_ε follows the curve through points 1, 2 and 3, but at point 3 an incremental increase in variance σ_u results in a discontinuous jump in response to point 4, after which the further increase in variance σ_u leads to values along the curve 4–5. If the variance σ_u is decreased, the curve 5–4–6 is followed, and a jump occurs from point 6 to point 2, and the variance σ_ε follows the curve 2–1. The over-all curve exhibits the familiar hysteresis, or jump phenomenon. The above mentioned jump phenomenon is similar to the well-known jump-resonance phenomenon observed in the harmonic response of a non-linear control system[6].

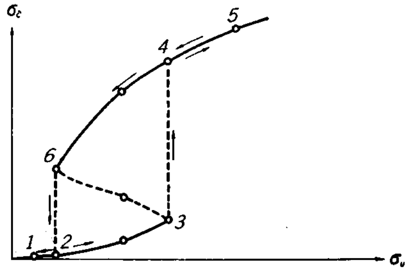


Fig. 12. Jump Phenomenon observed in Non-Linear Control System impressed by Random Signal.

For the purpose of explaining the occurrence of such jump phenomenon analytically, two non-linear control systems are analyzed.

As a first example, we consider a process control system of Fig. 13 containing a saturation with neutral zone as shown in the same figure. In Fig. 13, the values of the system parameters are purposely chosen to yield the multivalued response. Let us assume that the random disturbance $u(t)$ have the same properties as ones in the previous section; its auto-correlation function be expressed by (3–3) and its spectral density by (3–4). From (2–3) we have

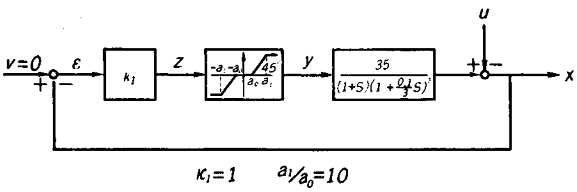


Fig. 13. Block Diagram for Non-Linear Control System Containing Saturation with Neutral Zone.

$$\frac{\sigma_z}{a_0^2}=\frac{\sigma_u}{a_0^2}\frac{-\alpha_0^3+13.3\alpha_0^2+2.89\alpha_0+0.14}{(1+35\alpha_0)(-2.6\alpha_0^2+1.97\alpha_0+0.14)}=\frac{\sigma_u}{a_0^2}g_3(\alpha_0)\,. \tag{3–11}$$

The equivalent gain α_0 for the saturation with neutral zone[7] is calculated from (1–7) as follows,

$$\alpha_0 = 2\{\phi^{-1}(a_1/\sqrt{\sigma_z})-\phi^{-1}(a_0/\sqrt{\sigma_z})\} \tag{3-12}$$

where $\phi^{-1}(\xi)$ is the error function defined as follows,

$$\phi^{-1}(\xi) = \frac{1}{\sqrt{2\pi}}\int_0^{\xi} \exp(-\eta^2/2)d\eta .$$

In Fig. 14, the equivalent gain α_0 is plotted as a function of σ_z/a_0^2 in the semi-logarithmic scale. By the graphical method shown in Fig. 14, the result shown in Fig. 15 is obtained easily.

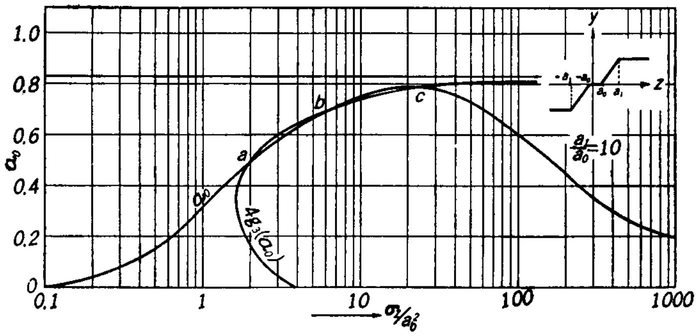


Fig. 14. Equivalent Gain for Random Noise of Saturation with Neutral Zone and Graphical Procedure Evaluating Response.

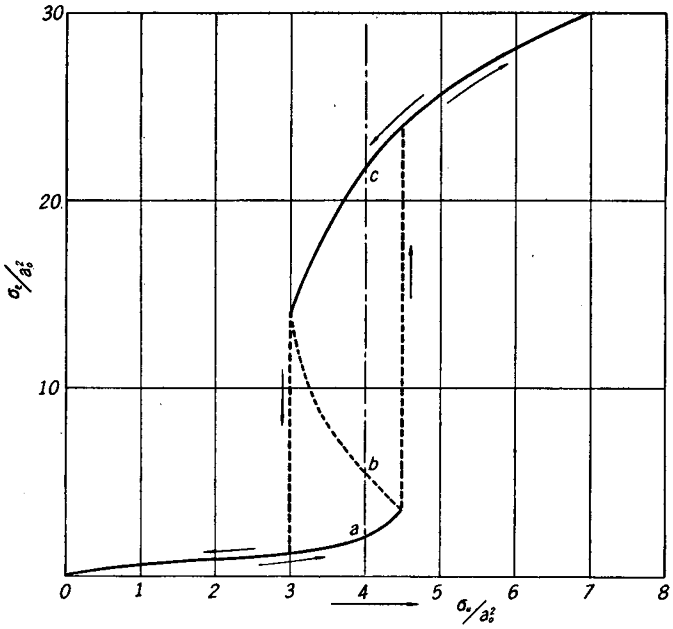


Fig. 15. Jump-Phenomenon observed in the System as shown in Fig. 13.

As a second example, we consider the simple servo-system containing a saturation as shown in Fig. 16. Let us assume that the input $v(t)$ is a random time function with a normal distribution function and its auto-correlation function is expressed in the following form;

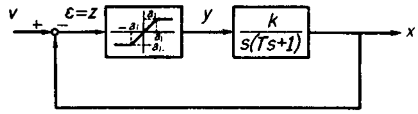


Fig. 16. Block Diagram for Simple Servosystem Containing Saturation.

$$R_v(\tau) = \sigma_v e^{-\beta|\tau|} \cos \omega_0 \tau \tag{3-13}$$

where σ_v is the variance of $v(t)$. Then the spectral density becomes

$$S_v(\omega) = \frac{2\sigma_v\beta}{\beta^2+(\omega-\omega_0)^2}+\frac{2\sigma_v\beta}{\beta^2+(\omega+\omega_0)^2}. \tag{3-14}$$

Accordingly, from (2-3) we have

$$\frac{\sigma_z}{a_1^2} = \frac{\sigma_v}{a_1^2}\frac{T\beta(k\alpha_0)^2+\beta(2T\beta+T^2\omega_0^2+T^2\beta^2+1)k\alpha_0+(\beta^2+\omega_0^2)(T^2\omega_0^2+T^2\beta^2+2T\beta+1)}{(k\alpha_0)^2+2(\beta+T\beta^2-T\omega_0^2)k\alpha_0+(\beta^2+\omega_0^2)(T^2\omega_0^2+T^2\beta^2+2T\beta+1)}. \tag{3-15}$$

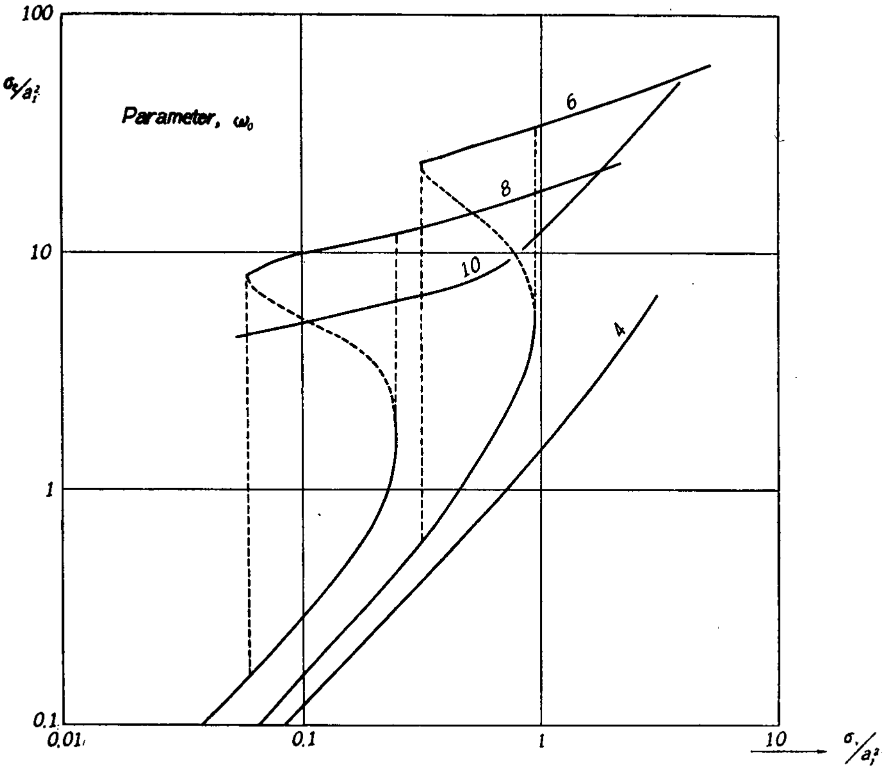


Fig. 17. Relation Curves between Variance of Error Signal and one of Impressing Signal.

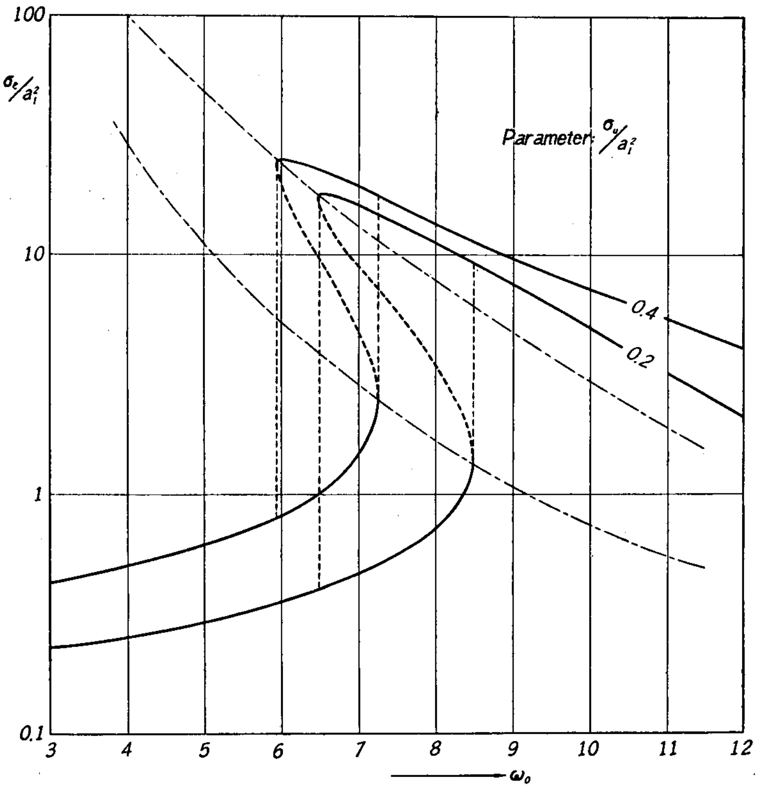


Fig. 18. Relation Curves between Variance of Error Signal and Central Frequency of Impressing Random Signal.

The equivalent gain α_0 for the saturation is calculated from (1-7),

$$\alpha_0 = 2\phi^{-1}(a_1/\sqrt{\sigma_x}) \tag{3-16}$$

and this is a special case of Fig. 7 (a). In similar to the first example, for the particular values;

$$k = 575, \quad T = 2.5, \quad \beta = 0.4,$$

the obtained values of σ_ε/a_1^2 are shown in Fig. 17 as a function of σ_v/a_1^2 with a parameter of ω_0. In Fig. 18, the variance of the error response σ_ε/a_1^2 is shown as a function of ω_0 with a parameter of σ_v/a_1^2.

4. Conclusions

Since the non-linear automatic control systems encountered in practice are usually impressed by many random inputs, the response is actually very complicated and the precise analysis for such inputs is almost impossible. However, the problem is very important from the view-point of control engineering, because the optimum characteristic of a non-linear element should be selected for the improvement of the control performance of control systems impressed by random inputs. In this paper, we described the graphical method of analysis for the above mentioned problem and the three examples of control system were treated (numerically) for the purpose of illustrating the usefulness of this graphical method. As the results of calculations, harmonic response of a non-linear control system with a random disturbance and jump phenomena in some non-linear control systems were explained theoretically. It may be expected that the graphical method will be further developed and applied to the synthetical problems in non-linear control systems in the future.

Acknowledgement

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Refercnces

1) Y. Sawaragi and N. Sugai; On a Statistical Method for Linearization of Non-linear Control System with Random Inputs. Proc. 8th Japan National Congress for Appl. Mech. (1958).
2), 3) Y. Sawaragi and S. Takahashi; Fachtagung Regelungstechnik Heidelberg (1956).
4) H. M. James, N. B. Nichols and R. S. Phillips; Theory of Servomechanisms. McGraw-Hill Book Co., INC. (1947).
5) R. C. Booton; Non-linear System with Statistical Inputs. M.I.T., D.A.C.L. Report No. 61, (1952).
6) J. G. Truxal; Control System Synthesis. McGraw-Hill Book Co. (1955).
7) S. Takahashi; Statistical Analysis of Non-linear Control System with Random Inputs. Memoires of the Faculty of Industrial Arts, Kyoto Technical Univ., scie. and tech. Vol. 4. (1955).
8) Ref. 6, p. 422.

Appendix 1. The Derivation of Equivalent Gains

Assume that the input to a non-linear element of zero-memory type with the transfer characteristic $f(z)$ is expressed by

$$z(t) = z_1(t)+z_2(t) \tag{A1-1}$$

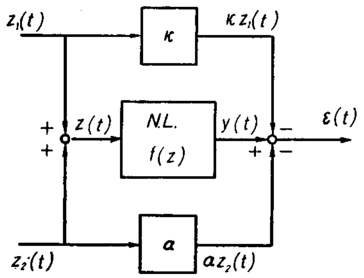

Fig. 19.

where $z_1(t)$ and $z_2(t)$ are both stationary random time functions and are statistically independent of each other. Then, the mean-square difference between the actual output $y(t)$ of the non-linear element and its approximate output $\kappa z_1+\alpha z_2$ (as shown in Fig. 19) will become minimum, if the relations[1]

$$\left.\begin{aligned} R_{z_1y}(\tau) &= \kappa R_{z_1z_1}(\tau) \\ R_{z_2y}(\tau) &= \alpha R_{z_2z_2}(\tau) \end{aligned}\right\} \tag{A1-2}$$

are satisfied where

$R_{z_1y}(\tau)$: cross-correlation function of $z_1(t)$ with $y(t)$,
$R_{z_1z_1}(\tau)$: auto-correlation function of $z_1(t)$,
$R_{z_2y}(\tau)$: cross-correlation function of $z_2(t)$ with $y(t)$,
$R_{z_2z_2}(\tau)$: auto-correlation function of $z_2(t)$.

For the purpose of evaluating $R_{z_1y}(\tau)$ and $R_{z_2y}(\tau)$, it is convenient to use the two-sided Laplace transform of $f(z)$ with respect to the argument z;

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(z)e^{-sz}dz \\ &= \int_0^{\infty} f_+(z)e^{-sz}dz+\int_{-\infty}^{0} f_-(z)e^{-sz}dz \\ &= F_+(s)+F_-(s) \end{aligned} \tag{A1-3}$$

where

$$\left.\begin{aligned} f_+(z) &= f(z) \quad z\geq 0 \\ f_+(z) &= 0 \quad z<0 \end{aligned}\right) \quad \left.\begin{aligned} f_-(z) &= f(z) \quad z\leq 0 \\ f_-(z) &= 0 \quad z>0 \end{aligned}\right) \tag{A1-4}$$

and

$$\begin{aligned} F_+(s) &= \int_0^{\infty} f_+(z)e^{-sz}dz \\ F_-(s) &= \int_{-\infty}^{0} f_-(z)e^{-sz}dz = \int_0^{\infty} f_-(-z)e^{-(-s)z}dz. \end{aligned} \tag{A1-5}$$

Examining the derivatives of $F_+(s)$ and $F_-(s)$ with respect to s, we find that $F_+(s)$ is analytic in the right half-plane and $F_-(s)$ is analytic in the left half-plane. Therefore, $F_+(s)$ reduces to the usual one-sided Laplace transform of the function $f_+(z)$, and

$F_-(s)$ can be determined by finding the usual one-sided Laplace transform of the function $f_-(z)$ reflected about the $z=0$ axis and then replacing s by $-s$.[8]
Accordingly, we have

$$f(z) = \frac{1}{2\pi j}\int_{C-j\infty}^{C+j\infty} F(s)e^{sz}dz$$

$$= \frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty} F_+(s)e^{sz}dz + \frac{1}{2\pi j}\int_{-\gamma-j\infty}^{-\gamma+j\infty} F_-(s)e^{sz}dz\,, \quad \begin{pmatrix}\delta>0\\ \gamma>0\end{pmatrix} \tag{A 1-6}$$

When the transfer characteristic of the non-linear element is an odd function of its argument;

$$f_-(z) = -f_+(-z) \tag{A 1-7}$$

we have from (A 1-5)

$$F_-(s) = -F_+(-s)\,. \tag{A 1-8}$$

If the transfer characteristic $f(z)$ is linear;

$$f(z) = z \tag{A 1-9}$$

we have from (A 1-5)

$$F_+(s) = \frac{1}{s^2}\,, \quad F_-(s) = -\frac{1}{s^2}\,. \tag{A 1-10}$$

Therefore, from (A 1-6) we get

$$z = \lim_{\substack{\delta\to 0\\ \gamma\to 0}} \left[\frac{1}{2\pi j}\int_{\delta-j\infty}^{\delta+j\infty} \frac{1}{s^2}e^{sz}dz + \frac{1}{2\pi j}\int_{-\gamma+j\infty}^{-\gamma-j\infty} \frac{1}{s^2}e^{sz}dz\right]$$

$$= \frac{1}{2\pi j}\oint \frac{1}{s^2}e^{sz}dz \tag{A 1-11}$$

where $\oint$ is the integral along the closed path of a small circle with the origin as its center.

Now, if we put $s=jw$, equations (A 1-6) and (A 1-11) become as follows,

$$y = f(z) = \frac{1}{2\pi}\int_{C+} F_+(jw)e^{jzw}dw + \frac{1}{2\pi}\int_{C-} F_-(jw)e^{jzw}dw$$

$$= \frac{1}{2\pi}\int_{C} F(jw)e^{jzw}dw \tag{A 1-12}$$

$$z = \frac{1}{2\pi}\oint \frac{1}{(jw)^2}e^{jzw}dw \tag{A 1-13}$$

where c_+ and c_- are the integral paths along the straight lines from $-j\delta-\infty$ to $-j\delta+\infty$ and from $j\gamma-\infty$ to $j\gamma+\infty$ respectively.

By substituting (A 1-12) and (A 1-13) into (A 1-2), the left-hand side of (A 1-2) becomes

$$R_{z_1y}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z_1(t)y(t+\tau)dt$$

$$= \frac{1}{4\pi^2}\oint (jw_1)^{-2}dw_1\int_C F(jw_2)\varphi(w_1, w_2\,;\,\tau)dw_2 \qquad \text{(A 1-14)}$$

$$R_{z_2y}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z_2(t)y(t+\tau)dt$$

$$= \frac{1}{4\pi^2}\oint (jw_1)^{-2}dw_1\int_C F(jw_2)\varphi'(w_1, w_2\,;\,\tau)dw_2 \qquad \text{(A 1-15)}$$

where

$$\left.\begin{aligned}\varphi(w_1, w_2\,;\,\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\exp\{jw_1z_1(t)+jw_2z_1(t+\tau)+jw_2z_2(t+\tau)\}dt\\ \varphi'(w_1, w_2\,;\,\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\exp\{jw_1z_2(t)+jw_2z_2(t+\tau)+jw_2z_1(t+\tau)\}dt\,.\end{aligned}\right\} \qquad \text{(A 1-15)}$$

By the Ergodic hypothesis and the independency between $z_1(t)$ and $z_2(t)$, the above time averages equal the ensemble averages;

$$\left.\begin{aligned}\varphi(w_1, w_2\,;\,\tau) &= \varphi_1(w_1, w_2\,;\,\tau)\varphi_2(w_2)\\ \varphi'(w_1, w_2\,;\,\tau) &= \varphi_1'(w_1, w_2\,;\,\tau)\varphi_2'(w_2)\end{aligned}\right\} \qquad \text{(A 1-16)}$$

where $\varphi_1(w_1, w_2\,;\,\tau)$ is the joint characteristic function of the two random variables $z_1(t)$ and $z_1(t+\tau)$, and $\varphi_1'(w_1, w_2\,;\,\tau)$ is the one of two random variables $z_2(t)$ and $z_2(t+\tau)$;

$$\left.\begin{aligned}\varphi_1(w_1, w_2\,;\,\tau) &= \langle\exp\{jw_1z_1(t)+jw_2z_1(t+\tau)\}\rangle_{av}\\ \varphi_1'(w_1, w_2\,;\,\tau) &= \langle\exp\{jw_1z_2(t)+jw_2z_2(t+\tau)\}\rangle_{av}\end{aligned}\right\} \qquad \text{(A 1-17)}_1$$

and $\varphi_2(w_2)$ is the characteristic function of one random variable $z_2(t)$, and $\varphi_2'(w_2)$ is that of one random variable $z_1(t)$;

$$\left.\begin{aligned}\varphi_2(w_2) &= \langle\exp\{jw_2z_2(t)\}\rangle_{av}\\ \varphi_2'(w_2) &= \langle\exp\{jw_2z_1(t)\}\rangle_{av}\,.\end{aligned}\right\} \qquad \text{(A 1-17)}_2$$

If $z_1(t)$ is a sinusoidal wave of a constant amplitude B_z with a constant frequency ω_0 and a random phase angle θ uniformly distributed over the range from 0 to 2π, and $z_2(t)$ is a random time function normally distributed with mean zero and the variance σ_z, then we have

$$\varphi_1(w_1, w_2\,;\,\tau) = \langle\exp\{jB_z\sqrt{w_1^2+w_2^2+2w_1w_2\cos\omega_0\tau}\,\sin\theta_1\}\rangle_{av} \qquad \text{(A 1-18)}$$

$$\varphi_1'(w_1, w_2\,;\,\tau) = \exp\left\{-\frac{\sigma_z}{2}(w_1^2+w_2^2)+R_{z_2z_2}(\tau)w_1w_2\right\} \qquad \text{(A 1-19)}$$

$$\varphi_2(w_2) = \exp\left(-\frac{\sigma_z}{2}w_2^2\right) \qquad \text{(A 1-20)}$$

$$\varphi_2'(w_2) = \langle\exp\{jB_zw_2\sin\theta_2\}\rangle_{av} \qquad \text{(A 1-21)}$$

where

$$\left.\begin{aligned}\theta_1 &= \omega_0 t+\theta-\tan^{-1}\frac{w_1+w_2\cos\omega_0\tau}{w_2\sin\omega_0\tau}\\ \theta_2 &= \omega_0 t+\theta+\omega_0\tau+\frac{\pi}{2}\end{aligned}\right\} \qquad \text{(A 1-22)}$$

θ_1 and θ_2 are again random variables and are uniformly distributed over the range from 0 to 2π. By the definition of the characteristic function, equations (A 1-18) and (A 1-21) are expressed as;

$$\begin{aligned}\varphi_1(w_1, w_2; \tau) &= \frac{1}{2\pi}\int_0^{2\pi}\exp\{jB_z\sqrt{w_1^2+w_2^2+2w_1w_2\cos\omega_0\tau}\sin\theta_1\}d\theta_1\\ &= J_0[B_z\sqrt{w_1^2+w_2^2+2w_1w_2\cos\omega_0\tau}] \qquad \text{(A 1-23)}\end{aligned}$$

$$\varphi_2'(w_2) = \frac{1}{2\pi}\int_0^{2\pi}\exp(jB_z w_2\sin\theta_2)d\theta_2 = J_0(B_z w_2) \qquad \text{(A 1-24)}$$

where $J_0(w)$ is a Bessel function of the 0th order defined as follows,

$$J_0(w) = \frac{1}{2\pi}\int_0^{2\pi}\exp(jw\sin\varphi)d\varphi\,. \qquad \text{(A 1-25)}$$

Substituting (A 1-23), (A 1-20), (A 1-19) and (A 1-24) into (A 1-16) and using the relations;

$$\begin{aligned}&J_0[B_z\sqrt{w_1^2+w_2^2+2w_1w_2\cos\omega_0\tau}]\\ &\qquad = \sum_{k=0}^{\infty}\varepsilon_k I_k(B_z jw_1)\, I_k(B_z jw_2)\cos k\omega_0\tau \qquad \text{(A 1-26)}\end{aligned}$$

where

$$I_k(jB_z w) = (j)^k J_k(B_z w) \qquad \text{(A 1-27)}$$

$$\varepsilon_0 = 1\,, \quad \varepsilon_k = 2 \qquad (k = 1, 2, \cdots) \qquad \text{(A 1-28)}$$

and

$$\exp\{-R_{z_2z_2}(\tau)w_1w_2\} = \sum_{n=0}^{\infty}\frac{1}{n!}(jw_1)^n(jw_2)^n R^n_{z_2z_2}(\tau) \qquad \text{(A 1-29)}$$

equatlon (A 1-16) can be written;

$$\begin{aligned}&\varphi(w_1, w_2; \tau)\\ &\quad = \sum_{k=0}^{\infty}\varepsilon_k I_k(B_z jw_1) I_k(B_z jw_2)\exp\left(-\frac{\sigma_z}{2}w_2^2\right)\cos k\omega_0\tau \qquad \text{(A 1-30)}\\ &\varphi'(w_1, w_2; \tau)\\ &\quad = \sum_{n=0}^{\infty}\frac{1}{n!}(jw_1)^n(jw_2)^n J_0(B_z w_2) R^n_{z_2z_2}(\tau)\,. \qquad \text{(A 1-31)}\end{aligned}$$

Therefore, equations (A 1-13) and (A 1-14) are expressed as follows,

$$R_{z_1y}(\tau) = \sum_{k=0}^{\infty}\varepsilon_k\beta_k\alpha_k\cos k\omega_0\tau \qquad \text{(A 1-32)}$$

$$R_{z_2y}(\tau) = \sum_{n=0}^{\infty}\frac{1}{n!}\beta_n'\alpha_n' R^n_{z_2z_2}(\tau) \qquad \text{(A 1-33)}$$

where

$$\beta_k = \frac{1}{2\pi}\oint (jw)^{-2} I_k(B_z jw) dw \qquad (A1\text{-}34)_1$$

$$\alpha_k = \frac{1}{2\pi}\int_C F(jw) I_k(B_z jw) \exp\left(-\frac{\sigma_z}{2}w^2\right) dw \qquad (A1\text{-}35)$$

$$\beta_n' = \frac{1}{2\pi}\oint (jw)^{n-2} \exp\left(-\frac{\sigma_z}{2}w^2\right) dw \qquad (A1\text{-}36)_1$$

$$\alpha_n' = \frac{1}{2\pi}\int_C F(jw)(jw)^n J_0(B_z w) \exp\left(-\frac{\sigma_z}{2}w^2\right) dw . \qquad (A1\text{-}37)$$

If we expand $I_k(B_z jw)$ and $\exp\left(-\frac{\sigma_z}{2}w^2\right)$ in powers of (jw) ;

$$I_k(B_z jw) = \sum_{h=0}^{\infty} \frac{(B_z/2)^{2h+k}}{h!(h+k)!}(jw)^{2h+k}$$

$$\exp\left(-\frac{\sigma_z}{2}w^2\right) = \sum_{l=0}^{\infty} \frac{(\sigma_z/2)^l}{l!}(jw)^{2l}$$

equations $(A1\text{-}34)_1$ and $(A1\text{-}36)_1$ are expressed as follows,

$$\beta_k = \sum_{h=0}^{\infty} \frac{(B_z/2)^{2h+k}}{h!(h+k)!}\frac{1}{2\pi}\oint (jw)^{2h+k-2} dw \qquad (A1\text{-}34)_2$$

$$\beta_k' = \sum_{l=0}^{\infty} \frac{(\sigma_z/2)^l}{l!}\frac{1}{2\pi}\oint (jw)^{2l+n-2} dw . \qquad (A1\text{-}36)_2$$

Accordingly we have

$$\beta_1 = \frac{B_z}{2}, \quad \beta_k = 0 \qquad (k = 0, 2, 3, \cdots)$$

$$\beta_1' = 1 \ , \quad \beta_n' = 0 \qquad (n = 0, 2, 3, \cdots)$$

Thus, equations (A1-32) and (A1-33) become

$$\left.\begin{aligned} R_{z_1 y}(\tau) &= B_z \alpha_1 \cos\omega_0\tau \\ R_{z_2 y}(\tau) &= \alpha_1' R_{z_2 z_2}(\tau) . \end{aligned}\right\} \qquad (A1\text{-}38)$$

On the other hand, from (A1-2) we have

$$\left.\begin{aligned} R_{z_1 y}(\tau) &= \kappa\frac{B_z^2}{2}\cos\omega_0\tau \\ R_{z_2 y}(\tau) &= \alpha R_{z_2 z_2}(\tau) . \end{aligned}\right\} \qquad (A1(39)$$

By the principle of statistical linearization, from (A1-38) and (A1-39) we obtain

$$\begin{aligned} \kappa = \frac{2}{B_z}\alpha_1 &= \frac{2}{B_z}\frac{1}{2\pi}\int_C F(jw) I_1(B_z jw) \exp\left(-\frac{\sigma_z}{2}w^2\right) dw \\ &= \frac{2}{B_z}\left\{\frac{1}{2\pi}\int_{C+} F_+(jw) I_1(B_z jw) \exp\left(-\frac{\sigma_z}{2}w^2\right) dw \right. \\ &\quad \left. + \frac{1}{2\pi}\int_{C-} F_-(jw) I_1(B_z jw) \exp\left(-\frac{\sigma_z}{2}w^2\right) dw \right\} \end{aligned} \qquad (A1\text{-}40)$$

$$\alpha = \alpha_1' = \frac{1}{2\pi}\int_C F(jw)(jw)J_0(B_z w)\exp\left(-\frac{\sigma_z}{2}w^2\right)dw$$

$$= \frac{1}{2\pi}\int_{C+} F_+(jw)(jw)J_0(B_z w)\exp\left(+\frac{\sigma_z}{2}w^2\right)dw$$

$$+\frac{1}{2\pi}\int_{C-} F_-(jw)(jw)J_0(B_z w)\exp\left(-\frac{\sigma_z}{2}w^2\right)dw\,. \qquad \text{(A1-41)}$$

Since we assume a symmetrical non-linear characteristic, by the use of the relation (A1-8) the above equations are finally expressed as follows,

$$\kappa = \frac{2}{B_z\pi}\int_{C+} F_+(jw)I_1(B_z jw)\exp\left(-\frac{\sigma_z}{2}w^2\right)dw \qquad \text{(A1-42)}$$

$$\alpha = \frac{1}{\pi}\int_{C+} F_+(jw)(jw)J_0(B_z w)\exp\left(-\frac{\sigma_z}{2}w^2\right)dw\,. \qquad \text{(A1-43)}$$

Appendix 2. Statistical Representation of the "describing-function"

Suppose now that the input to a non-linear element is only a sinusoidal wave;

$$z(t) = B_z\cos(\omega_0 t+\theta)\,. \qquad \text{(A2-1)}$$

Then from (A1-40) we have

$$\lim_{\sigma_z\to 0}\kappa = \kappa_0 = \frac{1}{\pi B_z}\int_{C+} F_+(jw)I_1(B_z jw)dw$$

$$+\frac{1}{\pi B_z}\int_{C-} F_-(jw)I_1(B_z jw)dw \qquad \text{(A2-2)}$$

where

$$I_1(B_z jw) = jJ_1(B_z w)$$

$$= \frac{j}{2\pi}\int_0^{2\pi}\exp(jB_z w\sin\varphi - j\varphi)d\varphi\,. \qquad \text{(A2-3)}$$

By substituting (A2-3) into (A2-2) and changing the order of integration, equation (A2-2) becomes

$$\kappa_0 = \frac{j}{\pi B_z}\int_0^{2\pi}\exp(-j\varphi)\left[\frac{1}{2\pi}\int_{C+} F_+(jw)\exp(jB_z w\sin\varphi)dw\right.$$

$$\left.+\frac{1}{2\pi}\int_{C-} F_-(jw)\exp(jB_z w\sin\varphi)\,dw\right]d\varphi\,. \qquad \text{(A2-4)}$$

By the help of the expression (A1-12) the above equation may be written as follows,

$$\kappa_0 = \frac{j}{\pi B_z}\int_0^{2\pi}\exp(-j\varphi)f(B_z\sin\varphi)d\varphi$$

$$= \frac{1}{\pi B_z}\int_0^{2\pi} f(B_z\sin\varphi)\sin\varphi d\varphi + j\frac{1}{\pi B_z}\int_0^{2\pi} f(B_z\sin\varphi)\cos\varphi d\varphi\,. \qquad \text{(A2-5)}$$

The last expression of the right-hand side of (A2-5) is well-known as the "describing-function".