A Theoretical Treatment of the Frequency Entrainment of a Reflex Klystron

By

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The frequency entrainment of a reflex klystron is studied in a fundamental way by setting up the differential equation which describes the oscillation of the klystron cavity under application of an external signal. Under such a condition that the oscillator runs at its normal operation with no electronic susceptance and with matched loading, the problem is shown to be essentially analogous to the simmilar one in an ordinary vacuum-tube oscillator of lower frequency range. Within this limitation on the operation condition the bandwidth of frequency entrainment is estimated in relation to the power ratio of the externally applied signal and the output of the normally operating klystron oscillator. Phase retardation of the cavity oscillation in the entrained klystron from the phase of the externally applied signal is investigated.

In connection with this subject, the self-excited oscillation of the reflex klystron is also introductively treated from the standpoint of the theory of non-linear oscillation.

1. Introduction

The phenomenon of frequency entrainment in a vacuum-tube oscillator impressed by a periodic force has hitherto been discussed and successfully analysed as an interesting problem in the theory of non-linear oscillation.[1]

Although a klystron oscillator apparently has a principle of operation different from that of an ordinary vacuum-tube oscillator, the differential equations which express the self-excited oscillation of each oscillator prove to be quite analogous. Thus it is easily supposed that a reflex klystron can be entrained by application of a sinusoidal signal whose frequency is very near the frequency of oscillation of the klystron. Slater[2] treated this problem by using a quasi-stationary approximation to give some interpretations of physical situation. But, in order to be able to compare the theory with the experimental results, it seemed to be necessary to present a more detailed study.

The authors have derived a differential equation which describes the

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oscillation of the cavity of a reflex klystron impressed by an external signal, and analysed the phenomenon of frequency entrainment by use of a technique which is familiar in the theory of non-linear oscillation.

2. Setting up of the Differential Equations

Though, in a cavity of finite Q, normal oscillations cannot be defined strictly as in a lossless one, nevertheless the electromagnetic field in the former may roughly be considered as composed of normal modes when fairly sharp resonances are expected. The usual method of analysing the oscillation of the cavity is one in which we expand the oscillating field in the cavity in terms of normal modes of the adequately chosen lossless cavity and then treat the differential equations obeyed by the amplitude of each mode. In the case of the oscillating cavity of a reflex klystron, it is supposed that a certain mode is dominantly excited, so that we may discuss the cavity oscillation using one equation.

In our problem, the coupled system of a klystron oscillator with a load may be shown schematically as in Fig. 1. In the first place, we must assume the lossless cavity which we wish to take as a reference and in which the normal modes are to be defined. If we suppose that there exist no losses throughout the system except the load and no electron beam in the cavity, and if we impose

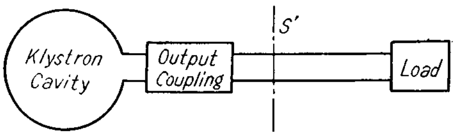

Fig. 1. Schematic diagram of a klystron oscillator coupled to a load.

the open-circuited condition on a certain sectional plane S'* of the output guide, that is to say, if no tangential component of the magnetic field is on S', the hollow space in the left of S' will qualify as the lossless cavity which we call the "reference cavity". In the rest of this article, we use the term "cavity" to mean this space region.

Now, if we denote the normal modes of the cavity by $\boldsymbol{E}_n$'s and $\boldsymbol{H}_n$'s ($n=1, 2, \cdots$) for electric and magnetic fields, respectively, these are shown to be the solutions of the wave equations:

* In principle, the plane S' is determined in the following way. Supposing that the a-th resonant mode is the dominantly excited one in our klystron cavity, S' is chosen on the guide as any one of the positions of the loop of VSW which could be obtained when the TE_{10} wave of frequency ω_a was introduced into the cavity through the output waveguide.

$$\nabla^2 \boldsymbol{E}_n + k_n^2 \boldsymbol{E}_n = 0 \quad \text{and} \quad \nabla^2 \boldsymbol{H}_n + k_n^2 \boldsymbol{H}_n = 0\,, \tag{2.1}$$

where k_n is the eigenvalue determined from the boundary conditions

$$\left.\begin{array}{ll} \boldsymbol{n}\times\boldsymbol{E}_n = 0 & (\text{on } S) \\ \boldsymbol{n}\times\boldsymbol{H}_n = 0 & (\text{on } S') \end{array}\right\}, \tag{2.2}$$

in which S denotes the boundary surface of the cavity other than S' and $\boldsymbol{n}$ denotes the outer normal to S or S'. As is well known, the angular frequency of the n-th resonant mode, ω_n, is connected with k_n by the following relation:

$$k_n^2 = \omega_n^2 \varepsilon\mu\,. \tag{2.3}$$

$\boldsymbol{E}_n$'s and $\boldsymbol{H}_n$'s, as is easily shown, form an orthogonal systems respectively, so that these may be taken to satisfy the orthogonality and normalization conditions

$$\int \boldsymbol{E}_m\cdot\boldsymbol{E}_n dv = \delta_{mn} \quad \text{and} \quad \int \boldsymbol{H}_m\cdot\boldsymbol{H}_n dv = \delta_{mn}\,, \tag{2.4}$$

where integration is carried out over the whole volume of the cavity.

Now let us consider the cavity oscillation occurring in an operating reflex klystron. If the a-th mode oscillation is dominantly excited, we can discuss our problem by deriving the differential equation which reveals the relations among the a-th mode componets of electric field $\boldsymbol{E}$, magnetic field $\boldsymbol{H}$ and current density $\boldsymbol{J}$:

$$e_a = \int \boldsymbol{E}\cdot\boldsymbol{E}_a dv\,, \qquad h_a = \int \boldsymbol{H}\cdot\boldsymbol{H}_a dv$$

and

$$J_a = \int \boldsymbol{J}\cdot\boldsymbol{E}_a dv\,.$$

Taking the scalar product of both sides of each of Maxwell's equations,

$$\text{rot}\,\boldsymbol{E} + \mu\frac{\partial \boldsymbol{H}}{\partial t} = 0 \quad \text{and} \quad \text{rot}\,\boldsymbol{H} - \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J}\,,$$

by multiplying by $\boldsymbol{H}_a$ and $\boldsymbol{E}_a$ respectively, and integrating over the whole volume of the cavity, we obtain

$$\mu\frac{dh_a}{dt} + k_a e_a = -\int_S (\boldsymbol{n}\times\boldsymbol{E})\cdot\boldsymbol{H}_a dS \tag{2.5}$$

and

$$\varepsilon\frac{de_a}{dt} - k_a h_a = -J_a + \int_{S'} (\boldsymbol{n}\times\boldsymbol{H})\cdot\boldsymbol{E}_a dS'\,. \tag{2.6}$$

Or, combining these two equations, we get the separate equation for e_a:

$$\varepsilon\mu\frac{d^2 e_a}{dt^2} + k_a^2 e_a = -\mu\frac{d}{dt}\left\{J_a - \int_{S'}(\boldsymbol{n}\times\boldsymbol{H})\cdot\boldsymbol{E}_a dS'\right\} - k_a\int_S (\boldsymbol{n}\times\boldsymbol{E})\cdot\boldsymbol{H}_a dS \tag{2.7}$$

and a similar equation for h_a. Eq. (2.7) was fully discussed by Slater, and forms the basis for treatment of cavity oscillation. The existence of surface integrals in this equation, the meaning of which will later become clear, causes, together with the term J_a, a deviation of the cavity oscillation from the harmonic one.

In the following we rewrite Eq. (2.7) in the more convenient form. First, the current density $\boldsymbol{J}$ consists of the beam current $\boldsymbol{J}'$ and the ohmic current due to the electric conductivity σ of the medium. That is

$$J_a = J_a' + \sigma e_a , \tag{2.8}$$

where

$$J_a' = \int \boldsymbol{J}' \cdot \boldsymbol{E}_a dv .$$

As the second term of (2.8) results in dielectric loss, introducing the quantity $Q_d = \omega_a \varepsilon / \sigma$ which is the quality factor corresponding to dielectric loss, Eq. (2.8) can be written

$$J_a = J_a' + \frac{\omega_a \varepsilon}{Q_d} e_a . \tag{2.8$'$}$$

Next the surface integral over S' in Eq (2.7) is shown in the following to be a quantity proportional to the current which flows into or out of the cavity through the plane S'. If only the dominant guide mode (TE_{10} mode) exists at S', as is realized by chosing S' distant from the output coupling, $\boldsymbol{n}\times\boldsymbol{H}$ and $\boldsymbol{E}$ in the integrand can be connected with the electric field of the dominant guide mode $\boldsymbol{E}_g$ in the form:

$$\boldsymbol{E}_a = \kappa_{ag} \boldsymbol{E}_g \qquad (\text{at } S') \tag{2.9}$$

and

$$-\boldsymbol{n}\times\boldsymbol{H} = i_g \boldsymbol{E}_g \qquad (\text{at } S') \tag{2.10}$$

The coefficient κ_{ag} in (2.9) is the coupling constant between the excited mode of the cavity and the dominant guide mode. Here we note that the external Q of the cavity, Q_{ext}, is defined from κ_{ag} by the following equation:

$$\kappa_{ag}^2 = \frac{\omega_a \varepsilon Z_0}{Q_{ext}} , \tag{2.11}$$

where Z_0 is the characteristic impedance of the waveguide. The quantity i_g which appears in Eq. (2.10), as shown later, may be interpreted as the current associated with the guide mode, the sign of which is taken to be positive when it flows out. By use of the relations (2.9) and (2.10), the surface integral can be written as

$$-\int_{S'} (\boldsymbol{n}\times\boldsymbol{H}) \cdot \boldsymbol{E}_a dS' = \kappa_{ag} i_g , \tag{2.12}$$

provided that $\boldsymbol{E}_g$ is normalized according to $\int_{S'} \boldsymbol{E}_g^2 dS' = 1$.

Thirdly, the surface integral over S which appears in Eq (2.7) is the quantity related to wall loss. Since we are supposing that the a-th mode oscillation is dominantly excited, using approximations $\boldsymbol{H} \simeq h_a \boldsymbol{H}_a$ and $k_a h_a \simeq \varepsilon de_a/dt$, it may be approximated as follows:

$$k_a \int_S (\boldsymbol{n} \times \boldsymbol{E}) \cdot \boldsymbol{H}_a dS = (1+j)\, \omega_a \varepsilon \mu \frac{1}{Q_w} \frac{de_a}{dt} \tag{2.13}$$

with

$$\frac{1}{Q_w} = \frac{\delta}{2} \int_S \boldsymbol{H}_a^2 dS ,$$

where δ is the skin depth of the wall material. In deriving (2.13) it was assumed that $\boldsymbol{E}$ and $\boldsymbol{H}$ have the time factor $e^{j\omega_a t}$. But practically we do not know how the time variation of $\boldsymbol{E}$ and $\boldsymbol{H}$ is until we solve the differential equations such as (2.7). Then it will be somewhat strange that we use the approximate formula of Eq. (2.13) which includes the factor $(1+j)$. We shall recover the formal consistency, however, if ω_a is replaced by

$$\omega_a' = \omega_a \left(1 - \frac{1}{2Q_w}\right), \tag{2.14}$$

and if the term $(j\omega_a \varepsilon \mu / Q_w) de_a/dt$ is omitted (see Appendix).

Thus Eq. (2.7) becomes, using the notation ω_a in the meaning of ω_a' hereafter,

$$\frac{d^2 e_a}{dt} + \omega_a^2 e_a + \frac{\omega_a}{Q_a} \frac{de_a}{dt} + \frac{1}{\varepsilon} \frac{d}{dt} (J_a' + \kappa_{ag} i_g) = 0 \tag{2.15}$$

with

$$\frac{1}{Q_a} = \frac{1}{Q_w} + \frac{1}{Q_d} .$$

This is the fundamental equation which describes the excited mode oscillation in a klystron cavity, although it cannot be solved unless the current term, $J_a' + \kappa_{ag} i_g$, is known in connection with e_a.

Now in a reflex klystron, it is known that e_a and J_a' are connected with the gap voltage, v, and the fundamental wave component of beam current, i, respectively as follows:

$$v = \sqrt{\frac{\varepsilon}{C}}\, e_a \quad \text{and} \quad i = \sqrt{\frac{C}{\varepsilon}}\, J_a', \tag{2.16}$$

where C is the constant interpreted as the equivalent capacity of the gap. Substituting these relations into Eq. (2.15), we have

$$\frac{d^2 v}{dt^2} + \omega_a^2 v + \frac{\omega_a}{Q_a} \frac{dv}{dt} + \frac{1}{C} \frac{di}{dt} + \frac{m}{C} \frac{di_g}{dt} = 0 \tag{2.17}$$

with

$$m^2 = \frac{C}{\varepsilon} \kappa_{a0}^2 = \frac{C \omega_a Z_0}{Q_{ext}} . \tag{2.18}$$

The quantity m here introduced has a meaning such that i_g is m times as effective as i on the cavity resonance. The last term in the left side of Eq. (2.17) may be interpreted in such a manner that it expresses the decay of oscillation due to power flow from the cavity in the normally operating klystron, and that, when an external signal is applied into the cavity, it includes the term of excitation besides the decay term. Thus we proceed to treat first the self-oscillatory system and next the system with an external signal which we call the "entrained system".

(i) Self-oscillatory System

We introduced a current i_g associated with $\boldsymbol{H}$ at Eq. (2.10). Now let us introduce a voltage, v_g, associated with $\boldsymbol{E}$. This quantity may be defined by

$$\boldsymbol{E} = v_g \boldsymbol{E}_g \qquad (\text{at } S') , \tag{2.19}$$

since the electric field at S' is expressible by $\boldsymbol{E}_g$.

The interpretations of v_g as a voltage associated with $\boldsymbol{E}$ and of i_g as previously mentioned is allowable because we have

$$\int_{S'} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n} dS' = v_g i_g$$

and

$$|\boldsymbol{E}| / |\boldsymbol{H}| = v_g / i_g ,$$

combining Eq. (2.19) with Eq. (2.10).

As the electric field $\boldsymbol{E}$ is expressed by $\boldsymbol{E} \simeq e_a \boldsymbol{E}_a = e_a \kappa_{ag} \boldsymbol{E}_g$ on the other hand, comparing this with Eq. (2.19), we have

$$v_g = \kappa_{ag} e_a ,$$

or, using Eqs. (2.16) and (2.18), we have

$$v_g = mv . \tag{2.20}$$

Since the time variation of i_g and v_g is considered to be nearly sinusoidal, the ratio of i_g to v_g, which generally depends upon time, may be defined as the admittance in a wide sense.

That is

$$Y^{(0)} Y_0 = i_g / v_g , \tag{2.21}$$

in which Y_0 is the characteristic admittance of the waveguide. Superscript (0) is used to indicate that the quantity is related to the self-oscillatory system. From Eqs. (2.20) and (2.21) we have

$$i_g = m Y_0 Y^{(0)} v . \tag{2.22}$$

Substituting (2.22) into (2.17), and using (2.18), we obtain the differential equation of the self-oscillatory system as follows:

$$\frac{d^2 v}{dt^2} + \omega_a^2 v + \frac{\omega_a}{Q_a} \frac{dv}{dt} + \frac{\omega_a}{Q_{ext}} \frac{d(Y^{(0)} v)}{dt} + \frac{1}{C} \frac{di}{dt} = 0 . \tag{2.23}$$

(ii) The Entrained System

If, besides i, a small external signal is applied as a excitation source of the cavity oscillation, quantities i_g and v_g will become somewhat different. We suppose that these are composed of two components, one coming from the incidental signal and the other being the remainder.

Namely we may write this as

$$\left.\begin{aligned} i_g &= i_g' + i_e \\ v_g &= v_g' + v_e \end{aligned}\right\}, \tag{2.24}$$

where i_e and v_e denote, respectively, the current and the voltage associated with the incidental signal. If the time variation of i_e and v_e is sinusoidal,then that of i_g' and v_g' will be quasi-sinusoidal, because i_g and v_g are assumed to vary quasi-sinusoidally with time. Thus we may define

$$YY_0 = i_g'/v_g' \tag{2.25}$$

in a way similar to the defining of $Y^{(0)}$. Then, eliminating i_g' and v_g' from Eqs. (2.24) and (2.25), we find

$$i_g = YY_0 v_g + (1+Y) i_e,$$

where the relation $v_e = -i_e/Y_0$ was used. Multiplying both sides of this equation by m, we have

$$m i_g = \frac{C\omega_a}{Q_{ext}} Yv + (1+Y) m i_e .$$

Thus Eq. (2.17) becomes

$$\frac{d^2v}{dt^2} + \omega_a^2 v + \frac{\omega_a}{Q_a}\frac{dv}{dt} + \frac{\omega_a}{Q_{ext}}\frac{d(Yv)}{dt} + \frac{1}{C}\frac{di}{dt} + \frac{m}{C}\frac{d}{dt}[(1+Y)i_e] = 0. \tag{2.26}$$

Comparison between the last two terms in the left hand side of this equation shows that application of i_e on S' is equivalent to the current flowing $m\ (1+Y)i_e$ in the gap region.

From Eqs. (2.23) and (2.26), we may write the equivalent circuit of the self-oscillatory and the entrained system respectively. For the latter the equivalent circuit is shown in Fig. 2, in which the following notations are used:

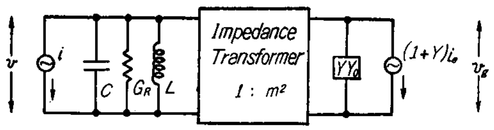

Fig. 2. Equivalent circuit of the entrained system.

$$L = \frac{1}{C\omega_a^2} \quad \text{and} \quad G_R = \frac{C\omega_a}{Q_a}. \tag{2.27}$$

It may not be appropriate to call the diagram here shown as equivalent circuit, because some of the circuit elements remain unknown. A complete equivalent circuit will later be obtained for the stationary state.

3. Self-Oscillatory System

Before treating the problem of entrainment phenomenon, we briefly discuss the normal operation of the reflex klystron.

For convenience of calculation, a dimensionless time $\tau=\omega_a t$ is used in this section. Then Eq. (2.23) becomes

$$\frac{d^2v}{d\tau^2}+v+\frac{1}{Q_a}\frac{dv}{d\tau}+\frac{1}{Q_{ext}}\frac{d(Y^{(0)}v)}{d\tau}+\frac{1}{C\omega_a}\frac{di}{d\tau}=0. \tag{3.1}$$

Now putting

$$[F^{(0)}]\equiv-\frac{1}{Q_a}\frac{dv}{d\tau}-\frac{1}{Q_{ext}}\frac{d(Y^{(0)}v)}{d\tau}-\frac{1}{C\omega_a}\frac{di}{d\tau} \tag{3.2}$$

as a function of v, $dv/d\tau$ and τ, Eq. (3.1) can be witten in the simple form:

$$\frac{d^2v}{d\tau^2}+v=[F^{(0)}], \tag{3.3}$$

or more conveniently in the form:

$$\left.\begin{aligned}\frac{dv}{d\tau}&=w\\ \frac{dw}{d\tau}&=-v+[F^{(0)}]\end{aligned}\right\}. \tag{3.4}$$

Since we are considering a system which has nearly harmonic oscillation, we assume the solution of Eq. (3.4) as

$$\left.\begin{aligned}v&=V\cos(\tau-\vartheta)\\ w&=-V\sin(\tau-\vartheta)\end{aligned}\right\}, \tag{3.5}$$

where V and ϑ are taken as slowly varying function of τ. Transforming (3.4) to the differential equation about V and ϑ, we find

$$\left.\begin{aligned}\frac{dV}{d\tau}&=-[F^{(0)}]\sin(\tau-\vartheta)\\ V\frac{d\vartheta}{d\tau}&=[F^{(0)}]\cos(\tau-\vartheta)\end{aligned}\right\}, \tag{3.6}$$

provided that the variables v and w appearing in $[F^{(0)}]$ take the assumed form of Eq. (3.5).

In order to solve this equation, it is necessary to know the concrete form of $[F^{(0)}]$ as a function of v, w and τ. In the first place, i may be taken as the fundamental component of the beam current which appears when the time variation of v is purely harmonic, because the electronic transit time is of the order of several periods and in such a short time interval the cavity oscillation is considered as almost purely harmonic. Thus, from the bunching theory of a reflex klystron, we may use

$$i = -2MI_b J_1(\mathrm{X}) \sin(\tau-\vartheta-\theta_0), \tag{3.7}$$

where I_b is the d-c component of beam current and V_b is the beam voltage, M the gap constant, θ_0 the d-c transit angle in the state of steady oscillation, and X the velocity modulation parameter which is related to V as follows:

$$X = \frac{M\theta_0}{2V_b} V. \tag{3.8}$$

Here it should be noted that in Eq. (3.7) both V (accordingly X) and ϑ are taken as slowly varying functions of time as was formerly assumed. If the oscillator operates at a electronic transit angle ϕ_0 from the mode center value in its steady state of oscillation, that is,

$$\theta_0 = 2\pi\left(n+\frac{3}{4}\right)+\phi_0, \tag{3.9}$$

Eq. (3.7) may be written

$$i = [g^{(0)}-b^{(0)}\tan(\tau-\vartheta)]v, \tag{3.10}$$

where

$$\begin{pmatrix} g^{(0)} \\ b^{(0)} \end{pmatrix} = \frac{M^2\theta_0 I_b}{2V_b}\frac{2J_1(X)}{X}\begin{pmatrix} -\cos\phi_0 \\ \sin\phi_0 \end{pmatrix}. \tag{3.11}$$

We note that $g^{(0)}$ and $b^{(0)}$ coincide respectively with the real and imaginary part of so called electronic admittance when V and ϑ take constant values. Differentiating the both sides of Eq. (3.10), we get

$$\frac{di}{d\tau} = [g^{(0)}-b^{(0)}\tan(\tau-\vartheta)]w-b^{(0)}\sec^2(\tau-\vartheta)\cdot v \tag{3.12}$$

where differential coefficints of $g^{(0)}$, $b^{(0)}$ and ϑ are neglected according to our hypothesis.

In the next place, $Y^{(0)}$ becomes from the assumption of Eq. (3.5)

$$Y^{(0)} = G^{(0)}-B^{(0)}\tan(\tau-\theta), \tag{3.13}$$

since i_g and v_g have the same time variation as v in our approximation. Generally speaking, $G^{(0)}$ and $B^{(0)}$ should be considered as functions of time, because these depend on frequency characteristics of the load, whereas the instantaneous frequency of the cavity oscillation varies slowly. But if the load is fairly insensitive of frequency, or, if the state near steady oscillation is concerned, $G^{(0)}$ and $B^{(0)}$ may be taken as certain constant parameters. In this case these reduce respectively to the ordinary conductance and susceptance when looking at the load from S'.

Using Eqs. (3.12) and (3.13), we can express $[F^{(0)}]$ by

$$[F^{(0)}]=\left(\frac{b^{(0)}}{C\omega_a}+\frac{B^{(0)}}{Q_{axt}}\right)v-\left(\frac{g^{(0)}}{C\omega_a}+\frac{G^{(0)}}{Q_{ext}}+\frac{1}{Q_a}\right)w \tag{3.14}$$

which together with Eq. (3.5) gives the functional form of $[F^{(0)}]$ in terms of V and ϑ.

Now we return to the problem of solving Eq. (3.6). Since V and ϑ are assumed to vary with time much more slowly than $\sin\tau$ or $\cos\tau$, these will remain constant during one period of the latter. Accordingly the right hand side of Eq. (3.6) will be replaced without much error by the time average over one period. Then approximately we have

$$\left.\begin{aligned}\frac{dV}{d\tau}&=-\frac{1}{2\pi}\int_0^{2\pi}[F^{(0)}]\sin\xi\,d\xi\\ \frac{d\vartheta}{d\tau}&=\frac{1}{2\pi V}\int_0^{2\pi}[F^{(0)}]\cos\xi\,d\xi\end{aligned}\right\}, \tag{3.15}$$

where V and ϑ in the integrand are to be regarded as constant, thus transforming the integration variable to $\xi=\tau-\vartheta$. Carrying out the integration, we get

$$\left.\begin{aligned}\frac{dV}{d\tau}&=-\frac{V}{2}\left(\frac{g^{(0)}}{C\omega_a}+\frac{1}{Q_a}+\frac{G^{(0)}}{Q_{ext}}\right)\equiv\Phi(V)\\ \frac{d\vartheta}{d\tau}&=\frac{1}{2}\left(\frac{b^{(0)}}{C\omega_a}+\frac{B^{(0)}}{Q_{ext}}\right)\equiv\Psi(V)\end{aligned}\right\}. \tag{3.16}$$

Stationary state is given by $\Phi(V)=0$, i.e.

$$-\frac{g^{(0)}}{C\omega_a}=\frac{1}{Q_a}+\frac{G^{(0)}}{Q_{ext}}. \tag{3.17}$$

The stationary value of oscillation amplitude then can be obtained from this equation, which we denote by $V=V_0$. Since the stationary value of ϑ becomes $\Psi(V_0)\tau+$cont., the frequency of oscillation settles in the value

$$\omega_0=\omega_a[1-\Psi(V_0)]. \tag{3.18}$$

This may be written in another form by use of Eq. (3.16):

$$-\frac{b^{(0)}}{C\omega_a}=\frac{2(\omega_0-\omega_a)}{\omega_a}+\frac{B^{(0)}}{Q_{ext}}. \tag{3.18$'$}$$

Equations (3.17) and (3.18)′ are well known as the condition of oscillation of a reflex klystron.

We next examine the stability of the stationary solution here obtained. Supposing that V deviated from V_0 by δV, approximately we have

$$\left[\frac{dV}{d\tau}\right]_{V_0+\delta V}=\Phi(V_0+\delta V)\doteqdot\left[\frac{d\Phi}{dV}\right]_{V_0}\delta V,$$

since $\Phi(V_0)=0$. In the case $[d\Phi/dV]_{V_0}<0$, oscillation is stable at $V=V_0$, since the sign of $dV/d\tau$ is always contrary to that of δV in the vicinity of $V=V_0$. In our case, it is easily shown that

$$\left[\frac{d\Phi}{dV}\right]_{V_0} = \frac{M^2\theta_0 I_b}{2V_b}\frac{\cos\phi_0}{C\omega_a}\left[J_0(X)-\frac{2J_1(X_0)}{X_0}\right]<0$$

holds in the necessary range of X.

Further, in relation to later discussion, we wish to evaluate an approximate value of V_0. It was impossible to obtain V_0 analytically from Eq. (3.17) where V is contained as the argument of the first order Bessel's function. The following approximation is found to be convenient:

$$\frac{2J_1(X)}{X} \doteqdot 1-\nu X^2 \tag{3.19}$$

where ν is a constant, the value of which is chosen so that $y=1-\nu X^2$ may satisfactorily fit to $y=2J_1(X)/X$ in the necessary range of X.; for instance, $\nu=0.1058$ when ν is chosen in such a way that two curves coincide at $X=2$. Using this approximation, Eq. (3.17) becomes

$$\left.\begin{aligned} 2\Phi(V) &= \alpha_0 V-\mu_0 c_0 V^3 \\ 2\Psi(V) &= \beta_0-\mu_0 s_0 V^2 \end{aligned}\right\}, \tag{3.20}$$

where

$$\left.\begin{aligned} \begin{pmatrix} c_0 \\ s_0 \end{pmatrix} &= \frac{M^2\theta_0 I_b}{2V_b}\frac{1}{C\omega_a}\begin{pmatrix}\cos\phi_0 \\ \sin\phi_0\end{pmatrix}, \\ \alpha_0 &= c_0-\left(\frac{1}{Q_a}+\frac{G^{(0)}}{Q_{\text{ext}}}\right), \\ \beta_0 &= s_0+\frac{B^{(0)}}{Q_{ext}}, \end{aligned}\right\} \tag{3.21}$$

and

$$\mu_0 = \nu\left(\frac{M\theta_0}{2V_b}\right)^2 .$$

From Eq. (3.20) the approximate value of V_0 is given by

$$V_0 = \sqrt{\frac{\alpha_0}{\mu_0 c_0}} . \tag{3.22}$$

4. Entrained System

Behaviors of a reflex klystron impressed by a sinusoidal external signal of definite frequency p, may be discussed by solving Eq. (2.26) if we put in that equation

$$i_e = I_e \cos pt, \tag{4.1}$$

which we call the "signal current". The signal frequency is assumed to be so close to the resonant frequency of the klystron cavity that the signal may resonate the cavity sufficiently; for it is easilly supposed that the smaller frequency difference causes the larger effect.

By use of the transformation of the independent variable, $\tau=pt$, Eq. (2.26) can be arranged in the same form as Eq. (3.3), that is

$$\frac{d^2v}{d\tau^2}+v=[F] \tag{4.2}$$

with

$$[F]\equiv\frac{\omega_a}{p}\left[h_a v-\frac{1}{Q_a}\frac{dv}{d\tau}-\frac{1}{Q_{ext}}\frac{d(Yv)}{d\tau}-\frac{1}{C\omega_a}\frac{di}{d\tau}-\frac{E_e}{Q_{ext}}\frac{d}{d\tau}[(1+Y\cdot)\cos\tau]\right], \tag{4.3}$$

where

$$h_a=\frac{p^2-\omega_a^2}{p\omega_a}\doteqdot\frac{2(p-\omega_a)}{\omega_a} \tag{4.4}$$

and

$$E_e=I_e/(mY_0)\,. \tag{4.5}$$

The approximate form of h_a in Eq. (4.4) is allowed when p is very close to ω_a. The quantity E_e may be interpreted as the amplitude of induced gap voltage due to application of the signal, because I_e/Y_0 is the amplitude of signal voltage at S' and this is tranformed to the voltage at the position of gap by multiplying the factor $1/m$. If $E_e\lesssim V_0$, that is, the power of incident signal is of the order or below the output power of the oscillator, we can assume that the effect of the term $[F]$ in the right hand side of Eq. (4.2) is so small that nearly harmonic oscillation of the cavity appears as in the case of the self-oscillatory system previously studied. We may then follow the procedure used in the preceding section; that is, we have merely to write down the functional form of $[F]$ with respect to v, w and τ, and then replace $[F^{(0)}]$ by $[F]$ in Eq. (3.15).

The beam current is taken to have the same form as Eq. (3.7), but in this θ_0 should be replaced by

$$\theta=2\pi\left(n+\frac{3}{4}\right)+\phi$$

with

$$\phi=\phi_0+\frac{p-\omega_0}{\omega_0}\theta_0\,, \tag{4.6}$$

provided that the d-c transit time is not changed from the case of normal operation. Then $g^{(0)}$ and $b^{(0)}$ also must be replaced respectively by

$$\begin{pmatrix} g \\ b \end{pmatrix}=\frac{M^2\theta I_b}{2V_b}\frac{2J_1(X)}{X}\begin{pmatrix} -\cos\phi \\ \sin\phi \end{pmatrix} \tag{4.7}$$

in the corresponding expression of i to Eq. (3.10) or Eq. (3.12).

Next, considering that i_g' and v_g' are almost sinusoidal for any short time interval, and that their ratio YY^0 may be identified in nature with the load admittance looking out from S' at an instantaneous frequency of i_g' or v_g', we may put, in the oprator form,

$$Y\cos=G\cos-B\sin\,. \tag{4.8}$$

Here G and B are respectively the load conductance and the load susceptance at that frequency, but are assumed simply to be constant in the frequency region concerned. Thus we shall hereafter equate G and B with $G^{(0)}$ and $B^{(0)}$, respectively.

This assumption will be reasonable as long as the load admittance seen from S' is fairly insensitive to frequency change, and accordingly the load admittance is matched to the characteristic admittance of the waveguide in the frequency region concerned.

Thus $[F]$ may be expressed in the form

$$[F] = \frac{\omega_a}{p}\Big[\Big(h_a+\frac{b}{C\omega_a}+\frac{B}{Q_{ext}}\Big)v-\Big(\frac{g}{C\omega_a}+\frac{1}{Q_a}+\frac{G}{Q_{ext}}\Big)w + \frac{E_e}{Q_{ext}}\{(1+G)\sin\tau+B\cos\tau\}\Big] \tag{4.9}$$

Substituting this expression to the equation which has the same form as Eq. (3.15) except for replacement of $[F^{(0)}]$ by $[F]$, we obtain the following equations which determine the time variation of V and ϑ:

$$\left.\begin{aligned} -2\frac{dV}{d\tau} &= \frac{\omega_a}{p}\Big(\frac{g}{C\omega_a}+\frac{1}{Q_a}+\frac{G}{Q_{ext}}+\frac{A\cos\Theta}{Q_{ext}}\Big)V \\ 2\frac{d\vartheta}{d\tau} &= \frac{\omega_a}{p}\Big(h_a+\frac{b}{C\omega_a}+\frac{B}{Q_{ext}}+\frac{A\sin\Theta}{Q_{ext}}\Big) \end{aligned}\right\}, \tag{4.10}$$

where

$$\Theta = \vartheta+\tan^{-1}[B/(1+G)], \tag{4.11}$$

and

$$A = \eta E_e/V, \quad \eta = \sqrt{(1+G)^2+B^2}. \tag{4.12}$$

The quantity A here defined simply means the proportion of the externally applied voltage included in the total r-f gap voltage. It is clear from Eq. (4.10) that the stationary oscillation of the cavity becomes possible when V and Θ take definite values, that is, when frequency entrainment is achieved. Basing on Eq. (4.10), the equivalent circuit of the entrained system at stationary state may be drawn as in Fig. 3.

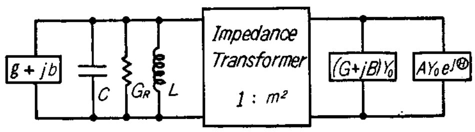

Fig. 3. Equivalent circuit representation of the entrained system at stationary state..

Putting now $\Delta b=b-b^{(0)}$, Eq. (3.18) of the previous section is expressible in the form

$$\frac{b}{C\omega_a}+\frac{B}{Q_{ext}}=-\frac{2(\omega_0-\omega_a)}{\omega_a}+\frac{\Delta b}{C\omega_a}$$

The second equation of (4.10) can then be written

$$2\frac{d\Theta}{d\tau}=\frac{\omega_a}{p}\left\{\frac{2(p-\omega_0)}{\omega_a}+\frac{A\sin\Theta}{Q_{ext}}+\frac{\Delta b}{C\omega_a}\right\}. \tag{4.13}$$

In this expression, the last term in the right hand side is negligible as compared with the first term, if we assume that our klystron operates at the center of electronic mode in the absence of the external signal, that is, $\phi_0 \simeq 0$. The order of magnitude of $\Delta b/(C\omega_a)$, under the assumption above mentioned may be estimated as follows. By use of the relations $b=-g\tan\phi$ and $b^{(0)}=-g^{(0)}\tan\phi_0$ which come directly from Eq. (4.1) and Eq. (3.11) respectively, and of the fact that $g/C\omega_a$ and $g^{(0)}/C\omega_a$ are of the order of

$$\frac{1}{Q_a}+\frac{1}{Q_{ext}}+\frac{A\cos\Theta}{Q_{ext}},$$

we have

$$\frac{\Delta b}{C\omega_a}\simeq\left(\frac{1}{Q_a}+\frac{1}{Q_{ext}}\right)\frac{p-\omega_0}{\omega_0}\theta_0.$$

Thus $\Delta b/C\omega_a$ proves to be a small quantity of lower order compared with $(p-\omega_0)/\omega_0$, assuming Q_a and $Q_{ext}\simeq 500$.

We hereafter limit ourselves to the case in which the two conditions already mentioned are satisfied, that is, (i) $\phi_0\simeq 0$; (ii) $G\simeq 1$ and $B\simeq 0$.
Thus, in the case of our concern, approximately we have

$$\frac{d\Theta}{d\tau}=\frac{\omega_a}{p}\left(\frac{p-\omega_0}{\omega_a}+\frac{A\sin\Theta}{2Q_{ext}}\right), \tag{4.14}$$

or returning to the initial time variable t,

$$\frac{d\Theta}{dt}=p-\omega_0+\frac{\omega_a A\sin\Theta}{2Q_{ext}}. \tag{4.14)'$$

The stationary state is then given by

$$\left.\begin{aligned}&\frac{g}{C\omega_a}+\frac{1}{Q_a}+\frac{G}{Q_{ext}}+\frac{A\cos\Theta}{Q_{ext}}=0\\&\frac{2(p-\omega_0)}{\omega_a}+\frac{A\sin\Theta}{Q_{ext}}=0\end{aligned}\right\}. \tag{4.15}$$

It is easily seen from this equation that the absolute value of electronic conductance changes nearly by $(A\cos\Theta/Q_{ext})C\omega_a$ due to the application of the signal, practically either decreasing or increasing, as is seen later, depending on the values of $p-\omega_0$ and A. The electronic susceptance, however, does not change appreciably within our assumption, so that the effectively added susceptance due to the signal, $C\omega_a A\sin\Theta/Q_{ext}$, is considered to be approximately counterbalanced by the change of susceptance of the cavity due to the frequency shift $p-\omega_0$.

For determining the stationary values of V and Θ, we use the approximate form of $2J_1(X)/X$ introduced in the previous section. Then Eq. (4.15) can be approximately written

$$\left.\begin{aligned}(\mu c V^2-\alpha)V+E\cos\Theta &= 0\\ hV+E\sin\Theta &= 0\end{aligned}\right\}, \tag{4.16}$$

in which we used the notation

$$E = \eta E_e/Q_{ext} \tag{4.17}$$

and

$$h = 2(p-\omega_0)/\omega_a\,; \tag{4.18}$$

μ, c, and α correspond to μ_0, c_0 and α_0 respectively, and are obtained from Eq. (3.21) by replacing ϕ_0 by ϕ, and $G^{(0)}$ by G. Under the given conditions, however, the differences between μ's and μ_0's are quite trivial. Then Eq. (3.17) becomes, using Eq. (3.22),

$$\left.\begin{aligned}\mu c(V^2-V_0^2)\,V+E\cos\Theta &= 0\\ hV+E\cos\Theta &= 0\end{aligned}\right\}. \tag{4.19}$$

This equation is the same in form as the corresponding one in the theory of frequency entrainment of an ordinary vacuum tube oscillator having one resonant circuit. Hence it can be said that the phenomenon of frequency entrainment in a reflex klystron appears to be quite analogous to the same phenomenon in an ordinary oscillator. Thus the subsequent discussion comes straightforwardly because we can make use of the various results obtained by the analysis of the latter problem.

Now the stationary values of V and Θ can be algebraically obtained from Eq. (4.19), both as a function of the fractional frequency deviation, h, and the strength of external force, E. In Fig. 4, we show the sketch of the $h-V^2$ relations for the various values of E. The region of stable state is known to be expressed by

$$\left.\begin{aligned}&V^2\geqq V_0^2/2\\ &9\left(\frac{V^2}{V_0^2}-\frac{2}{3}\right)^2+\frac{3}{\alpha^2}h^2\geqq 1\end{aligned}\right\} \tag{4.20}$$

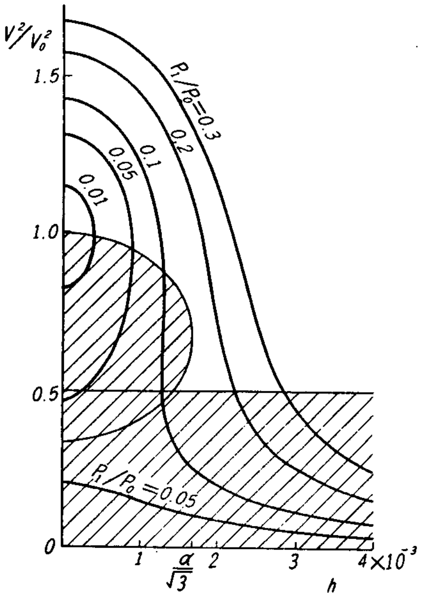

Fig. 4. $h-V^2$ diagram of the entrained system. As a parameter, P_1/P_0 was used instead of E. The hatched region corresponds to unstable state. This diagram was drawn for the assumed values: $Q_a=Q_{ext}=500$, $X=1.84$, $M^2=0.5$, $n=2$, $V_b=300$ and $\nu=0.1058$.

in the $h-V^2$ plane.

From Fig. 4, it can be seen that the stable oscillation at the frequency entrainment appears in a finite band of frequencies when an external force of strength E is impressed. The width of this frequency band will later be called the bandwidth of frequency entrainment.

(i) Estimation of the bandwidth of frequency entrainment.

We are now interested in how much bandwidth will be obtained by a given strength, E, of external force. For relatively small values of E, the frequency band may be determined by the value of h which gives $V^2=V_0^2$ and which we denote by $h_{\max}$, as is seen from Fig. 4. This holds almost exactly when $E\leq \alpha V_0/2$. Even for larger values of E, $h_{\max}$ will give a rough estimate of the bandwidth of frequency entrainment.

Putting $V=V_0$ in the equation

$$(\mu c)^2(V^2-V_0^2)^2V^2+h^2V^2=E^2$$

which comes from Eq. (4.19) by eliminating Θ, we obtain

$$h_{\max}=E/V_0$$

Then, defining the bandwidth of frequency entrainment by $W=\omega_a h_{\max}/2$, we have

$$W=\frac{\omega_a E}{2V_0} \tag{4.21}$$

which shows that the bandwidth is just proportional to the strength of external signal. More conveniently we can express this relation in terms of the ratio of the signal power, P_1, to the output power of the oscillator at its normal operation, P_0. If we put

$$A_{V=V_0}=\eta E_e/V_0\equiv A_0,$$

this quantity is shown to have the following relation with P_1/P_0:

$$A_0=\eta\sqrt{GP_1/P_0} \tag{4.22}$$

By use of this relation, together with $E/V_0=A_0/Q_{ext}$, Eq. (4.21) is rewritten as

$$W=\frac{\eta\omega_a}{2Q_{ext}}\sqrt{\frac{GP_1}{P_0}}\,. \tag{4.23}$$

Now consider the case $Y=1$ for brevity, remembering that the above results are in good approximation only for case $Y\simeq 1$, and considering that we practically operate a reflex klystron under this condition. In this case Eq. (4.23) becomes

$$W=\frac{\omega_a}{Q_{ext}}\sqrt{\frac{P_1}{P_0}} \tag{4.24}$$

Here the factor $1/Q_{ext}$ may be related to the width of the resonance curve of the cavity, W', in such a way that the latter is given by the difference between two frequencies at which $2Q_{ext}(\omega-\omega_a)/\omega_a=\pm 1$. We may then write it as

$$W = W'\sqrt{P_1/P_0}\,. \tag{4.25}$$

Thus the frequency range of entrained oscillation of a reflex klystron is at most of the order of the width of the resonance curve of the klystron cavity, if the power of the applied signal is not larger than the output power of the klystron at its normal operation. In other words, the frequency of the signal of small power must be close to ω_0 as is indicated in Eq. (4.25), in order to be able to entrain a reflex klystron whose cavity appreciablly resonates in the frequency range of width W' centered at ω_a. This means that the signal of such a small power can entrain the klystron only when the klystron cavity is sufficiently tuned to the signal.

(ii) Phase retardation of the cavity oscillation from the signal

Next we investigate the phase relation between the external signal and the oscillation of the cavity at the entrained state. The following equation is obtained for the stationary values of V and Θ from the second of Eq. (4.19):

$$\sin\Theta = -\frac{p-\omega_0}{W}\frac{V}{V_0}\,, \tag{4.26}$$

For determining the stationary value of Θ from this equation, the first of Eq. (4.19) also must be taken into account. This requires that $\cos\Theta$ is either negative or positive, according to whether $V>V_0$ or $V<V_0$, The latter case, $V<V_0$, occurs for a relatively large deviation of p from ω_0, that is, when $p-\omega_0>W$. Then we can write down the explicit forms of Θ and $\cos\Theta$ as follows:

$$\Theta = \begin{cases} \pi+\mathrm{Sin}^{-1}\left(\frac{p-\omega_0}{W}\frac{V}{V_0}\right) & \text{for } p-\omega_0 \leqq W \\ -\mathrm{Sin}\left(\frac{p-\omega_0}{W}\frac{V}{V_0}\right) & \text{for } p-\omega_0 > W \end{cases} \tag{4.27}$$

and

$$\cos\Theta = \mp\sqrt{1-\left(\frac{p-\omega_0}{W}\frac{V}{V_0}\right)^2} \tag{4.28}$$

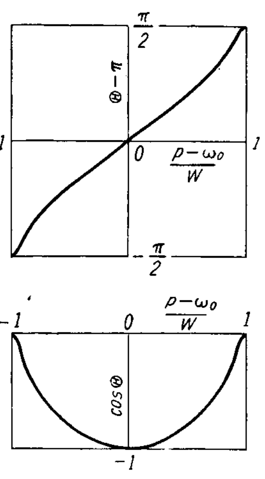

Fig. 5. Graphical representation of Θ and $\cos\Theta$ as functions of $p-\omega_0$ for $P_1/P_0=0.1$,

Since, for the small value of E or P_1, V takes a value almost equal to V_0, approximate values of Θ and $\cos\Theta$ will be obtained by putting $V/V_0=1$ in Eqs. (4.27) and (4.28). Graphical representations of Eqs. (4.27) and (4.28) are given in Fig. 5 for the region $p-\omega_0\leqq W$.

Again in the case $Y=1$, Θ reduces to ϑ and expresses the phase retardation of the voltage oscillation of the cavity from the signal current oscillation and since the phase of the applied signal voltage differs from that of the signal current by π in our assignment of sign to current, the voltage oscillation of the cavity is concluded to have a phase retardation of $\Theta-\pi$ with reference to the signal voltage. It is seen from Fig. 5, that the larger signal or the smaller value of $p-\omega_0$ leads to the smaller retardation of phase. Thus in the case of extremity, $\Theta-\pi=0$, it is supposed that v (accordingly v_g) and v_e will be in phase, whereas i_g and i_e will be out of phase with each other.

Further, the shift of electronic conductance from before to after the application of signal, which is proportional to $A\cos\Theta$, has the larger value for the smaller value of $p-\omega_0$, or for the larger signal power. In this connection, the net value of output power, P_{out}, which flows out of the cavity will be affected both by the value of $p-\omega_0$ and of P_1. The expression of this quantity is given by

$$P_{out}=\frac{C\omega_a}{2Q_{ext}}\left(1+\frac{2V_0}{V}\sqrt{\frac{P_1}{P_0}}\cos\Theta\right)V^2 \tag{4.29}$$

with

$$P_0=\frac{C\omega_a}{2Q_{ext}}V_0^2.$$

from the evaluation of the integral $(1/2\pi)\int_0^{2\pi}v_g i_g d\tau$. Eq. (4.29) can be obtained more directly as $(1/2)\,gV^2$ times the circuit efficiency which is given by

$$\left(\frac{1}{Q_{ext}}+\frac{A\cos\Theta}{Q_{ext}}\right)\Big/\left(\frac{1}{Q_a}+\frac{1}{Q_{ext}}+\frac{A\cos\Theta}{Q_{ext}}\right).$$

Conclusions

In the treatment formerly given by Slater, it was necessary to assume that the signal power is much smaller than the output power of the oscillator at its normal operation, that is, $P_1 \ll P_0$. This restriction has been safely relaxed in our treatment to the extent that P_1 is of the same order of magnitude as P_0, because then the non-linearity of the differential equation, Eq. (4.3), is still of the order of 10^{-2}, provided that both Q_a and Q_{ext} take values of several hundreds. The theory is intended to be verified by further experiments.*

In conclusion, a reflex klystron can theoretically be entrained by a relatively small signal, as was shown in section 4. This phenomenon then may possibly be utilized for the amplification of a signal of fixed frequency. For that purpose, however, further investigations will be required particularly as to the fluctuation of the phase difierence of the entrained oscillation from the signal, which depends on the frequency stability of the klystron oscillator under test.

* to be reported in this Memoirs

References

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Appendix

Validity of Using the Resonance Frequency Expressed by Fq. (2.14)

This expression was first used by Slater, where ω_a' was introduced as a resonant frequency of the damped oscillation of a cavity due to wall loss. Here we may directly assure the validity of using ω_a' in our analyses as follows.

If we go on with analysis without use of the replacement of Eq. (2.14), coefficients of v in the expression of $[F^{(0)}]$ and $[F]$ become

$$\frac{b^{(0)}}{C\omega_a}+\frac{B^{(0)}}{Q_{ext}}+\frac{1}{Q_w} \tag{1a}$$

and

$$\frac{\omega_a}{p}\left(\frac{2(p-\omega_a)}{\omega_a}+\frac{b}{C\omega_a}+\frac{B}{Q_{ext}}+\frac{1}{Q_w}\right) \tag{1b}$$

respectively, while in the text these are given, using the original notation, as

$$\frac{b}{C\omega_a'}+\frac{B^{(0)}}{Q_{ext}} \tag{2a}$$

and

$$\frac{\omega_a'}{p}\left(\frac{2(p-\omega_a')}{\omega_a'}+\frac{b}{C\omega_a'}+\frac{B}{Q_{ext}}\right). \tag{2b}$$

Thus, for the frequency of the self-excited oscillation, we have

$$\omega_0=\omega_a\left[1-\frac{1}{2}\left(\frac{b^{(0)}}{C\omega_a}+\frac{B^{(0)}}{Q_{ext}}+\frac{1}{Q_w}\right)\right]. \tag{3}$$

The corresponding formula, Eq. (3.18) in the text, is

$$\omega_0=\omega_a'\left[1-\frac{1}{2}\left(\frac{b^{(0)}}{C\omega_a'}+\frac{B^{(0)}}{Q_{ext}}\right)\right]. \tag{4}$$

Comparison of (3) and (4) leads in the first order approximation to the relation between ω_a' and ω_a, Eq. (2.14), as long as $(B^{(0)}/Q_{ext})\ll 1/Q_w$.

Eq. (2.14) is shown to be equally the condition for which (lb) and (2b) may be identified. The use of Eq. (4.10) are then justified in the first order approximation.