Exact Periodic Solutions for the Forced Oscillations of a Symmetric Nonlinear System with "Set-up Springs"

By

Yoshikazu SAWARAGI* and Hidekatsu TOKUMARU**

This paper presents the exact periodic solutions for the forced oscillation of a symmetric nonlinear system with "set-up springs". There exist many kinds of solutions, the "harmonic" solution, the subharmonic solution of odd order and the subharmonic solution of even order of equi-interval type and the "harmonic" and the even order subharmonic solution of non-equi-interval type. By "equi-interval" is meant that the time interval in which the solutions take positive or negative values in a cycle is a half of the fundamental period of the solution.

The stability of the periodic solutions is investigated exactly with the aid of the Tsypkin's method used for the stability problems of relay control systems.

1. Introduction

Recently M. Yamaguti[1] showed that

$$m\ddot{x}+c\dot{x}+F(x) = P(t)\,, \tag{1}$$

where

$$\left.\begin{aligned} &F(-x) = -F(x)\,, \\ &P(t+T) = P(t) \\ \text{and}\quad &P(t+T/2) = -P(t)\,, \end{aligned}\right\} \tag{2}$$

have at least one periodic solution of period T satisfying

$$x(t+T/2) = -x(t) \tag{3}$$

under some other weak conditions for $P(t)$ and $F(x)$.

However, there may exist the periodic solution of the period T or the period of integral times T satisfying

$$x(t+T'/2) \neq -x(t) \tag{4}$$

where T' is the period of the solution. In fact, C. Hayashi[2] and C.P. Atkinson[3] showed such oscillations of period T experimetally and A. M. Katz[4] calculated these solutions of Duffing's equation approximately. M. Tsumura investigated the possibility of the existence of the solution of this type theoretically.

* Department of Applied Mathematics and Physics
** Department of Aeronautical Engineering

In this paper, the authors find the exact periodic solutions for the forced oscillation of a symmetric nonlinear system with "set-up springs" to see the existence of the solution of the type of Eq. (4). As the analysis of this type of nonlinearity is simple, other properties of the periodic solutions are also clarified.

2. Equation of motion and its solution

The equation of motion for the forced vibration of symmetric nonlinear system with "set-up springs" is as follows:

$$m\ddot{x}+kx\pm F = P\sin\omega t\,. \qquad (\pm\,;\; x\gtrless 0) \tag{5}$$

For simplicity, the damping effect is not introduced in this equation. If the period of the solution of this equation is $2\pi/\omega$, the solution is a "harmonic" solution, and if the period of the solution is equal to $2\pi\lambda/\omega$, where λ is an integer, the solution is a subhamonic solution.

For the convenience of treatment, let us discuss the solutions of period $2\pi/\omega$ of the following equation:

$$m\ddot{x}+kx\pm F = P\sin\lambda\omega t\,, \qquad (\pm\,;\; x\gtrless 0) \tag{6}$$

where λ is an integer. When $\lambda=1$ the solution of this equation is harmonic and when $\lambda\neq 1$ the solution is subharmonic. Putting

$$\omega_0^2 = k/m\,,\quad u=\omega_0/\omega\,,\quad F_0 = F/m\omega^2\,,\quad P_0 = P/m\omega^2\,, \tag{7}$$

$$\tau = \omega t-\frac{\varphi}{\lambda}\,, \tag{8}$$

Eq, (6) becomes

$$\frac{d^2x}{d\tau^2}+u^2x\pm F_0 = P_0\sin(\lambda\tau+\varphi)\,, \qquad (\pm\,;\; x\gtrless 0) \tag{9}$$

where φ is an unknown constant which satisfies the following conditions:

$$x=0 \quad\text{and}\quad \frac{dx}{d\tau}>0 \quad\text{at}\quad \tau=0\,. \tag{10}$$

Now, let us confine our discussions to the solutions that intersect τ-axis at only one point in the interval $0<\tau<2\pi$. Denoting the instant of this zero by τ_0, this restriction may be described by the following expressions:

$$\begin{aligned} & x(\tau)>0 \quad\text{at}\quad 0<\tau<\tau_0 \\ \text{and}\qquad & x(\tau)<0 \quad\text{at}\quad \tau_0<\tau<2\pi\,. \end{aligned} \tag{11}$$

$\pm F_0$ in Eq. (9) takes the rectangular wave form as shown in Fig. 1. Expanding this wave into Fourier's series, we have

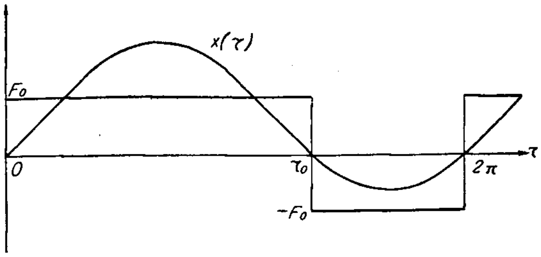

Fig. 1.

$$\pm F_0 = \frac{F_0}{\pi}(\tau_0-\pi)+\frac{2F_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\tau_0}{n}\cos n\tau + \frac{2F_0}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\tau_0}{n}\sin n\tau\,. \tag{12}$$

Putting

$$x = a_0+\sum_{n=1}^{\infty}a_n\cos n\tau+\sum_{n=1}^{\infty}b_n\sin n\tau\,, \tag{13}$$

and substituting Eqs. (12) and (13) into Eq. (9), we have

$$x = \frac{P_0}{u^2-\lambda^2}\sin(\lambda\tau+\varphi)+\frac{F_0}{\pi u^2}(\pi-\tau_0) + \frac{2F_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\tau-\sin n(\tau-\tau_0)}{n(n^2-u^2)}\,, \tag{14}$$

where u must not be an integer.

The unknown quantities τ_0 and φ may be determined by the following conditions:

$$\left.\begin{aligned} x(0) &= 0 \\ \text{and} \qquad x(\tau_0) &= 0\,, \end{aligned}\right\} \tag{15}$$

Substituting Eq. (14) into Eq. (15), we have the following simultaneous equations to determine τ_0 and φ:

$$\left.\begin{aligned} &\sin\varphi = \sin(\lambda\tau_0+\varphi) \\ &\frac{\pi P_0}{2F_0}\sin\varphi = \frac{u^2-\lambda^2}{2u^2}(\tau_0-\pi)-(u^2-\lambda^2)\sum_{n=1}^{\infty}\frac{\sin n\tau_0}{n(n^2-u^2)}\,. \end{aligned}\right\} \tag{16}$$

If the equations are solved, substituting the obtained values of τ_0 and φ in Eq. (14) we can establish the solution of Eq. (9) unless it contradicts the assumptions (11).

3. Solutions of equi-interval type, the case of $\tau_0=\pi$

The equations (16) are certainly satisfied by

$$\tau_0 = \pi\,, \quad \varphi = 0 \tag{17}$$

and
$$\tau_0 = \pi\,, \quad \varphi = \pi\,. \tag{18}$$

Using Eq. (17) and denoting $x(\tau)$ for this case by $x_P(\tau)$, we have

$$x_P(\tau) = \frac{P_0}{u^2-\lambda^2}\sin\lambda\tau + f(\tau)\,. \tag{19}$$

Similarly, for the case of Eq. (18) we have

$$x_Q(\tau) = -\frac{P_0}{u^2-\lambda^2}\sin\lambda\tau + f(\tau)\,. \tag{20}$$

$f(\tau)$ in these two equations is given as follows:

$$f(\tau) = \frac{4F_0}{\pi}\sum_{i=1}^{\infty}\frac{\sin(2i-1)\,\tau}{(2i-1)\{(2i-1)^2-u^2\}}\,. \tag{21}$$

Synthesizing the Fourier's series in Eq. (21), we obtain

$$f(\tau) = \frac{F_0}{u^2}\left\{\frac{\cos u\left(\dfrac{\pi}{2}-\tau\right)}{\cos\dfrac{u\pi}{2}}-1\right\}; \qquad 0\leq\tau\leq\pi\,. \tag{22}$$

From Fq. (21) it is clear that

$$f(\tau) = -f(-\tau) \quad \text{and} \quad f(\tau) = -f(\tau+\pi)\,. \tag{23}$$

Therefore, when λ is an odd integer, both $x_P(\tau)$ and $x_Q(\tau)$ have the property of Eq. (3). However, when λ is an even integer, neither $x_P(\tau)$ nor $x_Q(\tau)$ follow Eq. (3) but they satisfy Eq. (4). When $u<1$, $f(\tau)$ is nothing but the solution of free oscillation of the system.

Now, we are considering only the case of $\varphi=0$ and $\varphi=\pi$, but $\varphi=2i\pi$ and $\varphi=\pi+2i\pi$ (i: integer), when $\tau_0=\pi$, also satisfy Eq. (16). Retransforming back τ to t in Eq. (6) we obtain

$$x_{P,Q}(t) = \pm\frac{P_0}{u^2-\lambda^2}\sin(\lambda\omega t-\varphi) + f\left(\omega t-\frac{\varphi}{\lambda}\right). \tag{24}$$

Therefore, all the cases of $\varphi=0, 2\pi, \cdots, 2(\lambda-1)\,\pi$ correspond to different solutions respectively. Similarly, so do the cases of $\varphi=\pi, 3\pi, \cdots, \pi+2(\lambda-1)\,\pi$. That is, there are λ solutions belonging to x_P series and λ solutions belonging to x_Q series. However, these λ solutions are the same in the τ domain. Therefore, the only cases of $\varphi=0$ and $\varphi=\pi$ will be discussed later.

(1) Odd order subharmonics

Let us discuss first the case that λ is an odd integer other than unity. Now, if Eq. (11) is not satisfied by x_P or x_Q, x_P or x_Q obtained formally as (19) or (20) is not a solution.

When $u>1$, it can be verified that both $x_P(\tau)$ and $x_Q(\tau)$ always take negative values somewhere in the interval $0<\tau<\pi$ (see Appendix I). Therefore, neither $x_P(\tau)$ nor $x_Q(\tau)$ is a solution when $u>1$.

When $u<1$, $f(\tau)$ is always positive in the interval $0<\tau<\pi$. Calculating $f(\tau)/f(\pi/2)$ at the limit of $u\to 0$ and $u\to 1-0$, we have

$$\lim_{u\to 0} f(\tau)/f(\pi/2) = \frac{4}{\pi^2}\tau(\pi-\tau),$$
$$\lim_{u\to 1-0} f(\tau)/f(\pi/2) = \sin\tau.$$

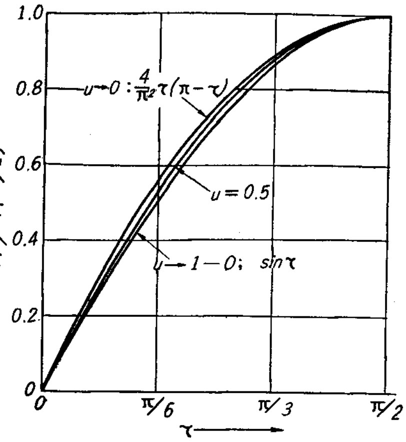

Fig. 2. Wave form of $f(\tau)$ for $u<1$.

Plotting the curves of $f(\tau)/f(\pi/2)$ for several values of u, we have Fig. 2. From this figure it is seen that $f(\tau)$ is so well behaved that a curve for any u is similar to the sine curve. Then the conditions (10) and (11) for $x_P(\tau)$ may be replaced in good approximation by $x_P'(0)>0$, where the prime means $d/d\tau$. Thus we have the following inequality giving the region in which $x_P(\tau)$ is a solution:

$$u<1 \quad \text{and} \quad \frac{P_0}{F_0}<\frac{\lambda^2-u^2}{\lambda u}\tan\frac{u\pi}{2}. \tag{25}$$

Let us adopt $x_Q(3\pi/2\lambda)>0$ as the approximate condition for the condition (11). Then we obtain the approximated region of existence of $x_Q(\tau)$ as follows:

$$u<1 \quad \text{and} \quad \frac{P_0}{F_0}<\frac{\lambda^2-u^2}{u^2}\left[\frac{\cos\left\{\frac{u\pi}{2}\left(1-\frac{3}{\lambda}\right)\right\}}{\cos\frac{u\pi}{2}}-1\right]. \tag{26}$$

As a wave form of $f(\tau)$ for $0<u<1$ is not so distorted from sine form as mentioned above, the biharmonic approximation for the subharmonic oscillation of odd order may retain considerably high accuracy. The nonlinearity of this system treated here is fairly remarkable, accordingly a system having usual continuous nonlinearity may be analysed with good accuracy by the biharmonic approximation.

The first terms of $x_P(\tau)$ and $x_Q(\tau)$ have the same amplitude but opposite phase with each other. On the other hand the second terms of $x_P(\tau)$ and $x_Q(\tau)$ are the same and equal to the free oscillation. In the biharmonic solution of Duffing's equation without damping similar circumstances are seen at a large amplitude.

Putting

$$\lambda = 2s+1$$

when s is an odd integer, the first term and the second term of $x_P(\tau)$ are in phase, and $x_P(\pi/2)$ gives the amplitude. But, when s is an even integer these two terms are not in phase and $x_P(\pi/2)$ is not the amplitude. In the case of $x_Q(\tau)$, the situation is just contrary.

(2) Subharmonic oscillation of even order

Let us consider the case that λ is an even integer. In this case the relation $x(\tau+\pi)=-x(\tau)$ does not hold. Hitherto, for a symmetric system the subharmonic oscillation of even order has not been discussed. However, there may exist this type of oscillation as will be discussed later.

It is easily shown that neither $x_P(\tau)$ nor $x_Q(\tau)$ is a solution for $u<1$. When $u<1$, $x'_{P,Q}(0)>0$ may be adopted as a substitute of the condition (11) for $x_{P,Q}(\tau)$. Then, we have

$$u<1 \quad \text{and} \quad \frac{P_0}{F_0}<\frac{\lambda^2-u^2}{\lambda u}\tan\frac{u\pi}{2} \tag{27}$$

as the inequality giving the region where $x_{P,Q}(\tau)$ is a solution. In the case of even λ the regions of existence of $x_P(\tau)$ and $x_Q(\tau)$ are the same although in the case of odd λ these are different. $x_P(\pi/2)$ and $x_Q(\pi/2)$ are not the extreme value in this case.

(3) "Harmonic" oscillation

Let us discuss the case of $\lambda=1$. When $u<1$, the first and the second term of $x_Q(\tau)$ are in phase. $x_Q(\tau)$ then always satisfies the condition (11). On the other hand, $x_P(\tau)$ can be a solution under the conditions $x_P(\pi/2)>0$, because the first and the second term of $x_P(\tau)$ are out of phase with each other. The inequality $x_P(\pi/2)>0$ may be written as follows:

$$\frac{P_0}{F_0}<\frac{1-u^2}{u^2}\left\{\frac{1}{\cos\frac{u\pi}{2}}-1\right\}, \qquad (u<1). \tag{28}$$

When $u>1$, it is easily proved that $x_Q(\tau)$ is not a solution. However, $x_P(\tau)$ may be a solution under some conditions at $u>1$. When $u>1$, $f(\tau)$ is more oscillatory than $\sin\tau$. When $2i-1<u<2i$ or $2i<u<2i+1$ $(i=1, 2, \cdots)$, $f(\tau)$ takes the minimum value

$$f_{\text{min.}} = -\frac{F_0}{u^2}\left\{\frac{1}{\left|\cos\frac{u\pi}{2}\right|}+1\right\}, \tag{29}$$

$$\text{at} \qquad \tau = \tau_1,\ \tau_2, \cdots, \tau_i, \tag{30}$$

where

$$\tau_j = \frac{\pi}{2} - \frac{(i-2j+1)}{u}\pi. \tag{31}$$

When $2i-1<u<i$, $f'(0)<0$ always, and when $2i<u<2i+1$, $f'(0)>0$ always. When i is an odd integer $f(\pi/2)$ is the minimum value, and when i is an even integer $f(\pi/2)$ is the maximum value in either case when u is in $2i-1<u<2i$ or in $2i<u<2i+1$. The maximum value of $f(\tau)$ in the interval $0<\tau<\pi$ is

$$f_{\max.} = \frac{F_0}{u^2}\left\{\frac{1}{\left|\cos\frac{u\pi}{2}\right|}-1\right\}. \tag{32}$$

When $u>4$, $f(\tau)$ in the interval $0<\tau<\pi$ is the periodic function of the period $2\pi/u$ as shown in Fig. 3.

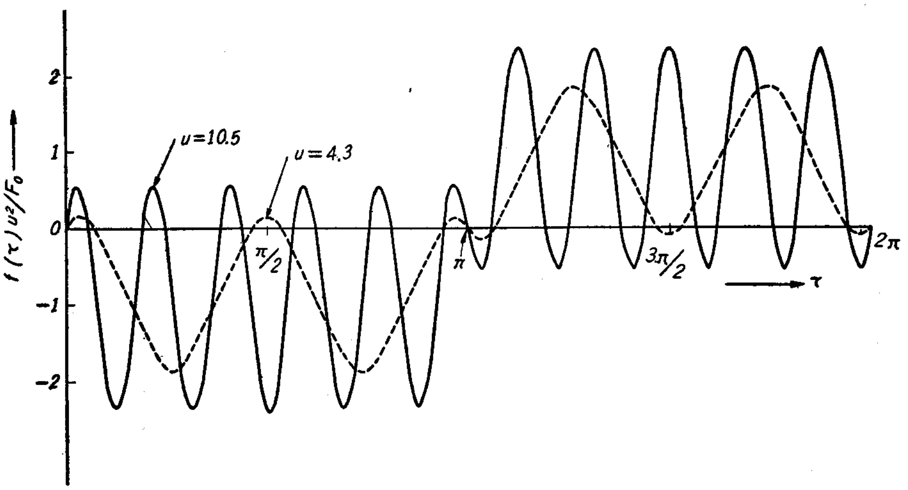

Fig. 3. Wave form of $f(\tau)$ for $u>1$.

Considering the fact mentioned above, we may adopt the following inequalities as the approximate conditions of Eq. (11):

$$x_P'(0)>0 \quad\text{and}\quad x_P\left\{\frac{\pi}{2}-\frac{(i-1)}{u}\pi\right\}>0 \qquad \text{for}\quad 2i-1<u<2i \tag{33}$$

and

$$x_P\left\{\frac{\pi}{2}-\frac{(i-1)}{u}\pi\right\}>0 \qquad \text{for}\quad 2i<u<2i+1 \tag{34}$$

Fig. 4 shows the regions of the existence of the solutions $x_P(\tau)$ and $x_Q(\tau)$ when $\lambda=1$.

If $u \doteqdot 2i-1$,

$$f_{\max.} \doteqdot |f_{\min.}|, \tag{35}$$

and if $u \doteqdot 2i$,

$$f_{\max.} \ll |f_{\min.}| \,. \tag{36}$$

Then if $u \doteqdot 2i-1$ the wave form of $f(\tau)$ is nearly sinusoidal, but if $u \doteqdot 2i$ the wave form of $f(\tau)$ is far from the sinusoidal form as shown in some examples illustrated in Fig. 3. Accordingly, in the case of $u \not\doteqdot 2j-1$, we can not grasp the feature of the solution accurately by the biharmonic approximation.

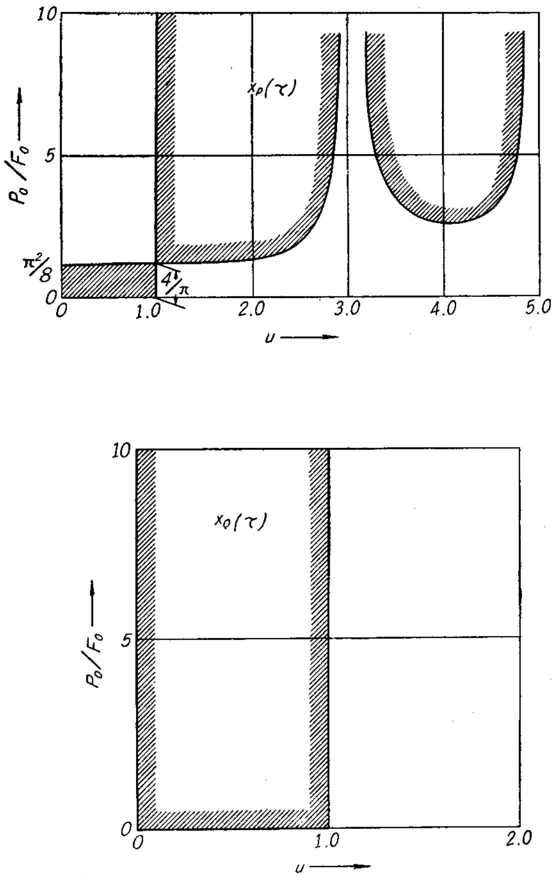

Fig. 4. Regions of existence of $x_P(\tau)$ and $x_Q(\tau)$ when $\lambda=1$.

As $x_P(\tau)$ and $x_Q(\tau)$ are nearly sinusoidal when $u<1$, the terminology as "harmonic" will be allowable. However, when $u>1$, $x_P(\tau)$ is far from sinusoidal. In this region of u the terminology "superharmonic" seems to be rather appropriate. But, in this paper the word "harmonic solution" is employed to mean the solution having the same period as forcing function.

4. Solution of non-equi-interval type (the case of $\tau_0 \neq \pi$)

In this case, the following holds always.

$$x(\tau) \neq -x(\tau+\pi).$$

If a combination, τ_0 and φ, satisfies the simultaneous equations (16), then the combination, $2\pi-\tau_0$ and $2\pi-\varphi$, also satisfies the same simultaneous equations. Thus, if $x(\tau)$ is a solution, then $-x(-\tau)$ is also a solution, thereby it is sufficient to consider only the case $\tau_0<\pi$.

Though $x(\tau)$ and $-x(-\tau)$ are the same solution in the case of $\tau_0=\pi$, in the case of $\tau_0 \neq \pi$ these are not the same. From the first relation in Eq. (16) we have

$$\tau_0 = \frac{2i\pi}{\lambda} \quad \text{or} \quad \varphi = \frac{(2i+1)\pi-\lambda\tau_0}{2}.$$

It may be shown that there is no solution satisfying (11) for $\tau_0=\frac{2i\pi}{\lambda}$. Then Eq. (16) may be rewritten as follows:

$$\left.\begin{aligned} (-1)^i \frac{P_0}{F_0}\cos\frac{\lambda\tau_0}{2} &= \frac{\lambda^2-u^2}{u^2}\cdot\frac{\sin u(\pi-\tau_0)}{\sin u\pi}, \\ \varphi &= \frac{(2i+1)\pi-\lambda\tau_0}{2}, \end{aligned}\right\} \tag{37}$$

where i is the integer which gives a value of $0\sim 2\pi$ to φ.

Synthesizing the Fourier's series of Eq. (14), we obtain

$$\left.\begin{aligned} x(\tau) &= (-1)^i \frac{P_0}{u^2-\lambda^2}\cos\lambda\left(\tau-\frac{\tau_0}{2}\right)+g(\tau), \\ g(\tau) &= \frac{F_0}{u^2}\left\{\frac{\sin u(\pi-\tau)+\sin u(\pi-\tau_0+\tau)}{\sin u\pi}-1\right\};\ 0<\tau<\tau_0 \\ g(\tau) &= \frac{F_0}{u^2}\left\{\frac{\sin u(\pi-\tau)-\sin u(\pi+\tau_0-\tau)}{\sin u\pi}+1\right\};\ \tau_0<\tau<2\pi. \end{aligned}\right\} \tag{38}$$

The wave form of $x(\tau)$ is symmetric as to $\tau=\tau_0/2$ and $\tau=\pi+\tau_0/2$ regardless of λ, while the wave form of the even order subharmonic oscillations in the preceding section has not such symmetricity.

(1) Subharmonic oscillation

We can prove that there is no solution satisfying the condition (11) when $\lambda>1$ and $u>1$. When λ is odd there is no solution for $u<1$ either. Namely, if a subharmonic solution of non-equi-interval type exists, it is always of even order. Investigating numerically whether the inequality (11) holds or not, we have the regions of existence of the subharmonic solutions of even order of non-equi-interval type as shown in Fig. 5. From this figure it is seen that the region of existence is confined to the vicinity of $\tau_0=\pi$ when λ is a large number. Letting τ_0 tend to π, Eq. (37) yields

$$\frac{P_0}{F_0} \to 0 \quad \text{and} \quad \varphi \to \frac{\pi}{2}$$

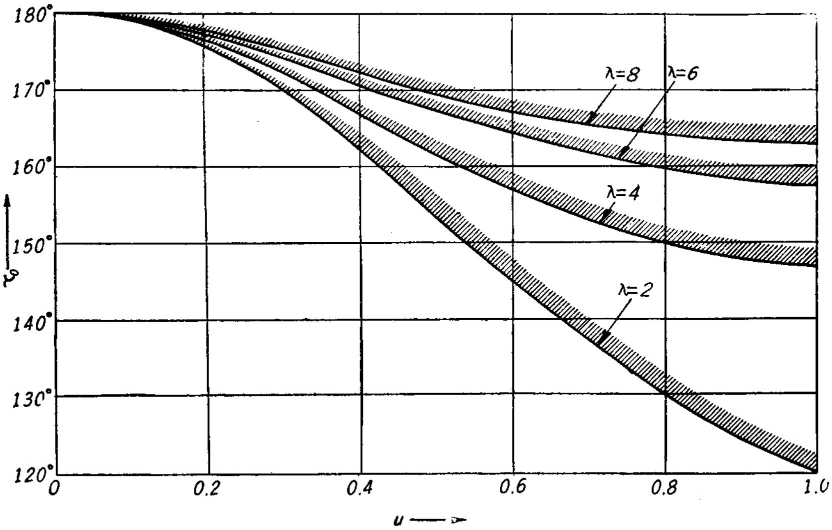

Fig. 5. Regions of existence of subharmonic solutions of even order of non-equi-interval type.

As $\frac{P_0}{F_0} \rightarrow 0$ means that the nonlinearity tends to ∞, it is seen that a subharmonic solution of higher order of non-equi-interval type appears only in the case of striking nonlinearity.

Some examples of wave form of $x(\tau)$ are shown in Fig. 6.

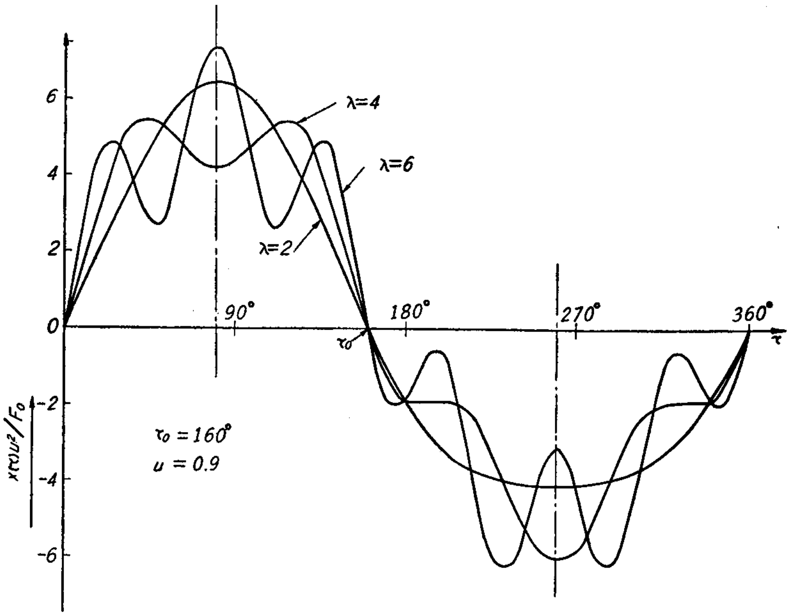

Fig. 6. Examples of wave form of subharmonic solution of even order of non-equi-interval type.

(2) Harmonic oscillation

Let us discuss the case of $\lambda=1$. In this case, there is no solution for $u<1$, but for $u>1$ there is a case that a solution exists.

$\frac{P_0}{F_0}$ -u diagram given in Fig. 7 shows the regions where a solution exists, but

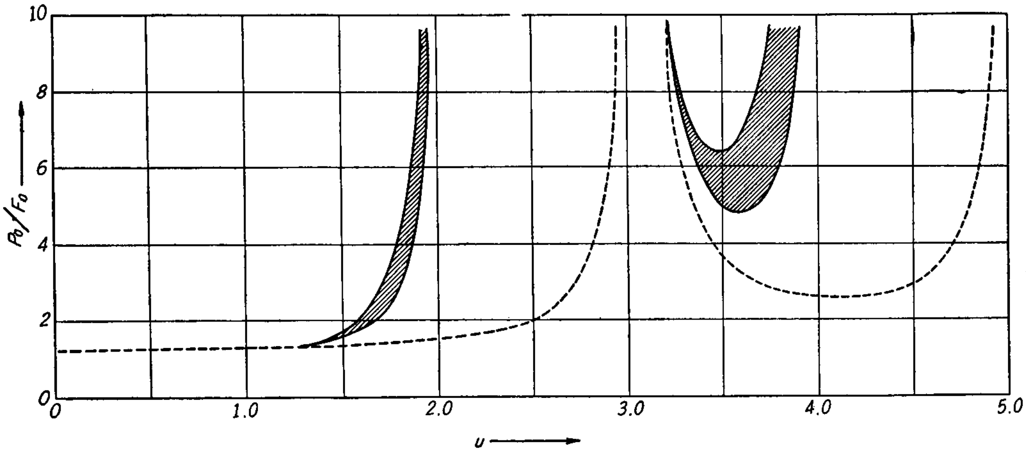

Fig. 7. Region of existence of "harmonic" solution of non-equi-interval type.

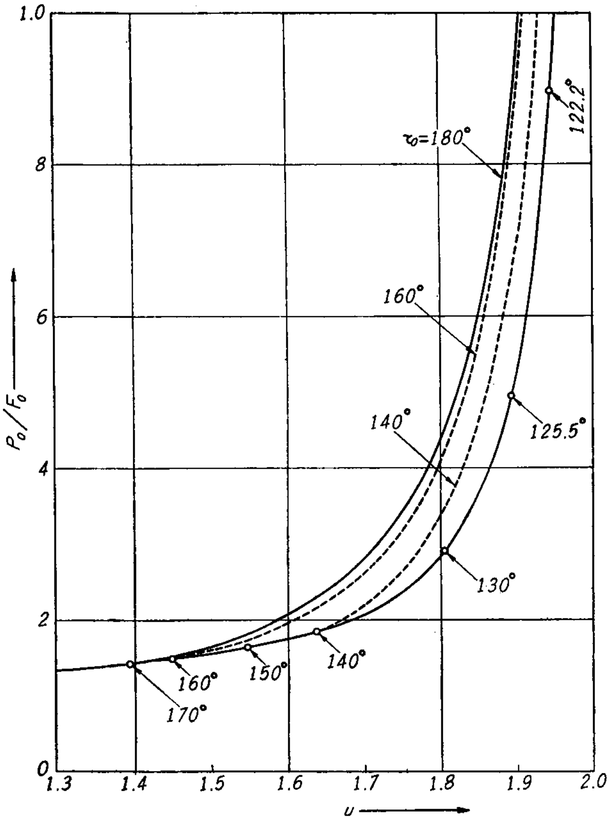

Fig. 8 (a). Detailed figure of the region shown in Fig. 7.

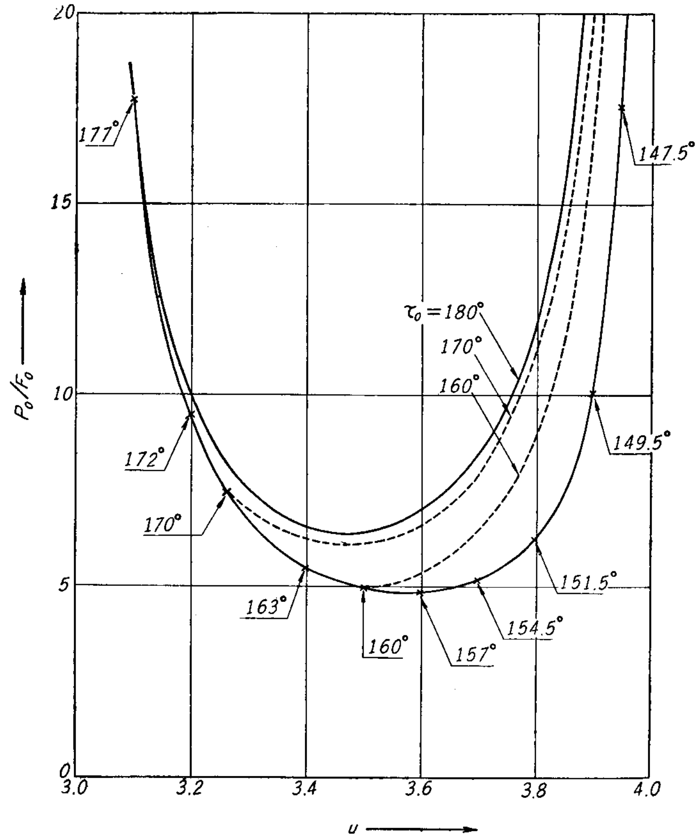

Fig. 8 (b). Detailed figure of the region shown in Fig. 7.

only in a case of $u<4$. The border lines of upper limit of the regions are given as follows:

$$\frac{P_0}{F_0} = 2\frac{1-u^2}{u}\operatorname{cosec} u\pi. \quad (39)$$

The border lines of lower limit of the regions are determined by numerical calculations. From the diagram, we shall find this region very small and narrow. The brocken line in this figure is the limiting line of the region shown in Fig. 4. It is seen that the region shown in Fig. 4 contains the region of this case.

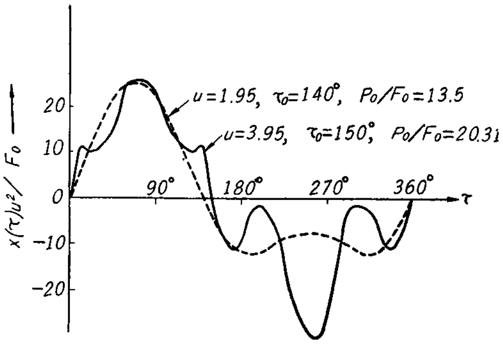

Fig. 9. Examples of wave form of "harmonic" solution of non-equi-interval type.

The more detailed figures of the region are shown in Fig. 8. In these figures parameter τ_0 is scaled on. The region for a solution of $\tau_0>\pi$ is naturally the same as that indicated in Fig. 8.

Several examples of the wave form of $x(\tau)$ are shown in Fig. 9.

5. Stability of the periodic solutions

Let us investigate the stability of the solutions with the aid of the Tsypkin's method[5] for the stability problems of relay control systems.

The Leplace transform of Eq. (8) is given as follows:

$$L[x(\tau)] = L[P(\tau)]W(s) - L[\Phi\{x(\tau)\}]W(s)\,,$$

where (40)

$$\Phi(x) \equiv \pm F_0 \qquad (\pm\,;\; x \gtrless 0)\,,$$

$$P(\tau) \equiv P_0 \sin(\lambda\tau+\varphi)\,,\quad W(s) = \frac{1}{s^2+u^2}\,.$$

If $x(\tau)$ becomes $x(\tau)+\varepsilon(\tau)$ because of the superposition of small disturbance $\varDelta(\tau)$ on $P(\tau)$, the Laplace transform of equation of motion becomes

$$L[x(\tau)+\varepsilon(\tau)] = L[P(\tau)+\varDelta(\tau)]W(s) - L[\Phi\{x(\tau)+\varepsilon(\tau)\}]W(s)\,. \tag{41}$$

Subtraction of Eq. (40) from Eq. (41) yields the following equation.

$$E(s) = D(s)W(s) - L[\Phi'\{x(\tau)\}\varepsilon(\tau)]W(s) \tag{42}$$

where

$$\left.\begin{aligned} &\Phi'\{x(\tau)\} \equiv \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\,[\Phi(x+\varepsilon)-\Phi(x)] = 2F_0\delta\{x(\tau)\}\,,\\ &E(s) \equiv L[\varepsilon(\tau)]\,,\quad D(s) \equiv L[\varDelta(\tau)]\,, \end{aligned}\right\} \tag{43}$$

and δ is the symbol of Dirac's delta function.

Now, there exists the following relation[5]:

$$\delta\{x(\tau)\} = \sum_{\nu} \frac{\delta(\tau-\tau_\nu)}{\left|\dfrac{dx}{d\tau}\right|_{\tau=\tau_\nu}} \tag{44}$$

where τ_ν is the time of zeros of $x(\tau)$. From Eq. (3), $\tau_\nu=2\pi k$ and $\tau_\nu=\tau_0+2\pi k$ $(k=0, 1, 2, \cdots)$. Thus we have

$$\Phi'\{x(\tau)\} = 2F_0\sum_{k=0}^{\infty}\left[\frac{\delta(\tau-2\pi k)}{|\dot{x}(0)|}+\frac{\delta\{\tau-(\tau_0+2\pi k)\}}{|\dot{x}(\tau_0)|}\right], \tag{45}$$

where

$$\dot{x}(0) \equiv \left.\frac{dx}{d\tau}\right|_{\tau=0} \quad\text{and}\quad \dot{x}(\tau_0) \equiv \left.\frac{dx}{d\tau}\right|_{\tau=\tau_0}.$$

Substituting Eq. (45) into Eq. (42), we have

$$E(s) = W(s)\left[D(s) - \frac{2F_0}{|\dot{x}(0)|}E^*(s) - \frac{2F_0 e^{-\tau_0 s}}{|\dot{x}(\tau_0)|}E^*(s, \tau_0)\right] \tag{46}$$

where

$$E^*(s) \equiv \sum_{k=0}^{\infty} e^{-2\pi ks}\varepsilon(2\pi k), \quad E^*(s, \tau_0) \equiv \sum_{k=0}^{\infty} e^{-2\pi ks}\varepsilon(2\pi k+\tau_0)$$

Now, Eq. (46) precisely corresponds to the sampling system shown in Fig. 10 in which the two samplers act with the same sampling period 2π but are non-synchronous. The bottom sampler operates τ_0 after the upper sampler.

Taking the z transform and modified z transform of Eq. (46), we have

$$E(z) = WD(z) - \frac{2F_0}{|\dot{x}(0)|}W(z)E(z) - \frac{2F_0}{|\dot{x}(\tau_0)|}W(z, -\tau_0)E(z, \tau_0) \tag{47}$$

$$E(z, \tau_0) = WD(z, \tau_0) - \frac{2F_0}{|\dot{x}(0)|}W(z, \tau_0)E(z) - \frac{2F_0}{|\dot{x}(\tau_0)|}W(z)E(z, \tau_0) \tag{48}$$

where

$$E(z) = Z[E(s)], \quad E(z, \tau_0) = Z[E(s)e^{\tau_0 s}], \quad W(z) = Z[W(s)],$$
$$W(z, \tau_0) = Z[W(s)e^{\tau_0 s}], \quad W(z, -\tau_0) = Z[W(s)e^{-\tau_0 s}],$$
$$WD(z) = Z[W(s)D(s)], \quad WD(z, \tau_0) = Z[W(s)D(s)e^{\tau_0 s}].$$

Solving for $E(z)$ and $E(z, \tau_0)$, we have

$$E(z) = \frac{1}{\psi(z)}\left[\left\{1 + \frac{2F_0}{|\dot{x}(\tau_0)|}W(z)\right\}WD(z) - \frac{2F_0}{|\dot{x}(\tau_0)|}W(z, -\tau_0)WD(z, \tau_0)\right], \tag{49}$$

$$E(z, \tau_0) = \frac{1}{\psi(z)}\left[-\frac{2F_0}{|\dot{x}(0)|}W(z, \tau_0)WD(z) + \left\{1 + \frac{2F_0}{|\dot{x}(0)|}W(z)\right\}WD(z, \tau_0)\right], \tag{50}$$

where

$$\psi(z) = 1 + 2F_0\left\{\frac{1}{|\dot{x}(0)|} + \frac{1}{|\dot{x}(\tau_0)|}\right\}W(z) + \frac{4F_0^2}{|\dot{x}(0)|\cdot|\dot{x}(\tau_0)|}\{W^2(z) - W(z, \tau_0)W(z, -\tau_0)\}. \tag{51}$$

Then, the characteristic equation of the sampling system of Fig. 10 is

$$\psi(z) = 0. \tag{52}$$

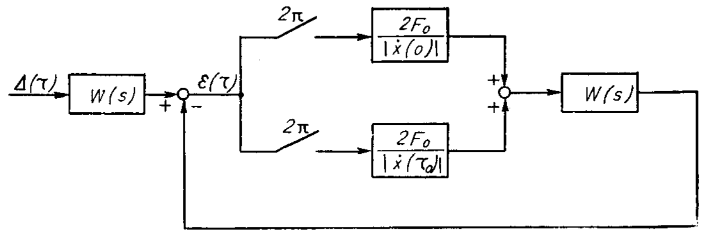

Fig. 10. Sampling system corresponding to stability of periodic solutions.

Calculating $W(z)$, $W(z, \tau_0)$ and $W(z, -\tau_0)$ and substituting these into Eq. (51), we obtain

$$\psi(z) = \frac{z^2-2Bz+1}{z^2-2z\cos 2\pi u+1}, \tag{53}$$

where

$$B = \cos 2\pi u - \frac{F_0 \sin 2\pi u}{u}\left(\frac{1}{|\dot{x}(0)|}+\frac{1}{|\dot{x}(\tau_0)|}\right)+\frac{2F_0^2 \sin \tau_0 u \sin u(2\pi-\tau_0)}{u^2|\dot{x}(0)|\cdot|\dot{x}(\tau_0)|}. \tag{54}$$

Accordingly the characteristic equation becomes

$$z^2-2Bz+1=0. \tag{55}$$

When $|B|>1$, either of the two roots of this equation always lies outside of the unit circle of the z plane. Then the system is unstable.

On the other hand, in the case of $|B|<1$, both roots of Eq. (55) always lie on the unit circle of the z plane. Therefore, the time sequences $\varepsilon(2\pi k)$ and $\varepsilon(\tau_0+2\pi k)$ neither diverge nor converge to zero. Cosidering the fact that there is no damping effect in the system, it may be concluded that the system is stable though $\varepsilon(t)$ does not converge to zero. However, it must be noticed that there is a possibility of hidden divergent oscillation. In practice, this hidden oscillation condition arises so rarely that it remains undetected. When $|B|=1$, two roots of Eq. (55) lie on the unit circle of z plane but they are double roots. So, the system is unstable generally.

6. Stability of the solution of equi-interval type

The stability condition of the solutions of this type is

$$0<\left(\cos \pi u-\frac{F_0 \sin \pi u}{u|\dot{x}(0)|}\right)\left(\cos \pi u-\frac{F_0 \sin \pi u}{u|\dot{x}(\pi)|}\right)<1. \tag{56}$$

(i) Subharmonic solutions of odd order

In this case, there exists the relation

$$\dot{x}(0)=-\dot{x}(\pi)>0. \tag{57}$$

Therefore the inequality of Eq. (56) resolves to

$$-1<\cos \pi u-\frac{F_0 \sin \pi u}{u\dot{x}(0)}<1 \tag{58}$$

$$\text{and} \qquad \dot{x}(0) \neq F_0 \tan u\pi. \tag{59}$$

When $\dot{x}(0)=F_0 \tan u\pi$, $|B|=1$. But, in this case (59) is unnecessary for the stability of the solution, because the factors $(1+z)$ which yield in terms of $W(z)$ and $W(z,\pm\tau_0)$ in the case of $\dot{x}(0)=F_0 \tan u\pi$ cancel the same factor in $\psi(z)$. Thus, it is sufficient to check only the inequality (58) for the investigation of the stability. Inequality (58) becomes

$$\left.\begin{aligned}&(u^2-\lambda^2)\frac{P_0}{F_0}>0\,,\\ \text{and}\quad &\frac{\lambda u}{u^2-\lambda^2}\cdot\frac{P_0}{F_0}>-2\,\mathrm{cosec}\,u\pi\,,\end{aligned}\right\}\tag{60}$$

for $x_P(\tau)$, and

$$\left.\begin{aligned}&(u^2-\lambda^2)\frac{P_0}{F_0}<0\,,\\ \text{and}\quad &\frac{\lambda u}{u^2-\lambda^2}\cdot\frac{P_0}{F_0}<2\,\mathrm{cosec}\,u\pi\,,\end{aligned}\right\}\tag{61}$$

for $x_Q(\tau)$. Eq. (60) always contradicts Eq. (25), and on the contrary (61) always implies Eq. (26). That is, $x_P(\tau)$ is always unstable and $x_Q(\tau)$ is always stable.

(ii) Subharmonic solution of even order

Evaluating $\dot{x}_P(0)$ and $\ddot{x}_P(\pi)$ and substituting these into (56), we have

$$(P_0/F_0)^2<0 \tag{62}$$

for $x_P(\tau)$. Similarly, we have the same inequality as (62) for $x_Q(\tau)$. From these inequalities it is clear that both $x_P(\tau)$ and $x_Q(\tau)$ are unstable.

(iii) Harmonic solution

The stability condition of this case is the same as (60) or (61) for $x_P(\tau)$ or $x_Q(\tau)$, respectively. Let us discuss the problem, dividing the total range of u into three parts as $0<u<1$, $2i-1<u<2i$ and $2i<u<2i+1$ (i: integer).

a) $0<u<1$. In this case the same discussions as in the case of odd order subharmonic oscillations hold. Thus $x_P(\tau)$ is unstable and $x_Q(\tau)$ is stable.

b) $2i-1<u<2i$. In this range of u, $x_Q(\tau)$ does not exist. Rearranging the inequality of (60), we have

$$\frac{P_0}{F_0}>2\frac{1-u^2}{u}\,\mathrm{cosec}\,u\pi\,. \tag{63}$$

As the equation $\frac{P_0}{F_0}=2\frac{1-u^2}{u}\,\mathrm{cosec}\,u\pi$ corresponds to the border line of upper limit of the region of existence of the harmonic solution of non-equi-interval type as mentioned in Eq. (39), $x_P(\tau)$ in this region is found to be unstable. Namely, the harmonic solutions of non-equi-interval type and of equi-interval type can not coexist experimentally.

c) $2i<u<2i+1$. As in the case of b) $x_Q(\tau)$ does not exist. In this case the inequalisy of (60) is always satisfied so long as $P_0/F_0>0$. That is, $x_P(\tau)$ always stable.

7. Stability of the solution of non-equi-interval type

The value of B in Eq. (54) for $x(\tau)$ is equal to the one for $-x(-\tau)$. So, the stability of $-x(-\tau)$ is precisely the same as that of $x(\tau)$. That is, it is sufficient to discuss only the case of $\tau_0=\pi$. Substituting Eq. (38) into (54), we have

$$\left\{\dot{x}(0)-\frac{2F_0}{u}\frac{\sin u\left(\pi-\frac{\tau_0}{2}\right)\cos\frac{u\tau_0}{2}}{\cos u\pi}\right\}\left\{\dot{x}(0)-\frac{2F_0}{u}\frac{\cos u\left(\pi-\frac{\tau_0}{2}\right)\sin\frac{u\tau_0}{2}}{\cos u\pi}\right\}>0 \tag{64}$$

$$\left\{\dot{x}(0)+\frac{2F_0}{u}\frac{\cos u\left(\pi-\frac{\tau_0}{2}\right)\cos\frac{u\tau_0}{2}}{\sin u\pi}\right\}\left\{\dot{x}(0)-\frac{2F_0}{u}\frac{\sin u\left(\pi-\frac{\tau_0}{2}\right)\sin\frac{u\tau_0}{2}}{\sin u\pi}\right\}>0. \tag{65}$$

(i) Subharmonic solution of even order

It was found from the numerial investigation that the solution is stable in the region of existence of solution shown in Fig. 5.

(ii) Harmonic solution

As is shown in Fig. 7 the solution exists only in the interval $2i-1<u<2i$. In this interval the inequalities (64) and (65) becomes

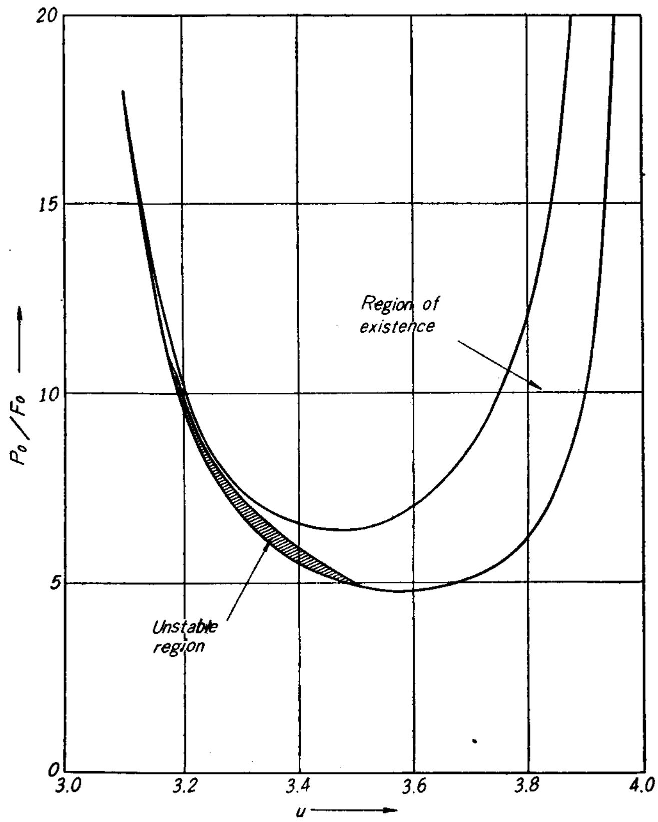

Fig. 11. Unstable region of harmonic solution of non-equi-interval type.

$$\left\{\frac{\sin\frac{\tau_0}{2}}{u^2-1}\cdot\frac{P_0}{F_0}-\frac{4}{u}\operatorname{cosec}2u\pi\sin^2 u\left(\pi-\frac{\tau_0}{2}\right)\right\}$$

$$\times\left\{\frac{\sin\frac{\tau_0}{2}}{u^2-1}\cdot\frac{P_0}{F_0}-\frac{4}{u}\operatorname{cosec}2u\pi\sin^2\frac{u\tau_0}{2}\right\}>0\,, \tag{66}$$

and
$$\frac{P_0}{F_0}>-\frac{2(u^2-1)}{u}\operatorname{cosec}u\pi\operatorname{cosec}\frac{\tau_0}{2}\cos u(\pi-\tau_0)\,. \tag{67}$$

The inequality of (67) always holds in the region of existence of solution according to the numerical investigation. The inequanlity of (66) does not hold in the shaded region shown in Fig. 11. Therefore, the solutions $x(\tau)$ is unstable in this region.

8. Conclusions

It is concluded that:

(i) It is exemplified that there can exist the solution $x(t)$ such as

$$x(t)\neq -x\left(t+\frac{T}{2}\right)$$

and also there can exist the subharmonic solution of even order, even if the restoring force have the characteristic

$$F(x) = -F(-x)$$

and the periodic forcing function $P(t)$ of period T have the property,

$$P\left(t+\frac{T}{2}\right) = -P(t)\,.$$

(ii) There exist two kinds of solution, the equi-interval type and the non-equi-interval type.

(iii) The subharmonic solution of odd order of the non-equi-interval type does not exist.

(iv) The harmonic solution of non-equi-interval type appears only in the range of $2i-1<u<2i$ (i: positive integer).

(v) The solutions for $u<1$ or for $u\fallingdotseq 2i-1$ may be well approximated by harmonic or biharmonic approximation.

(vi) If the subharmonic oscillation of even order occurs experimentally, it is always that of non-equi-interval type.

(vii) It is never the case that the stable solution of equi-interval type and the stable solution of non-equi-interval type coexist.

(viii) The harmonic solution of non-equi-interval type is apt to occur at the vicinity of $u=2i$ (i: positive integer).

(ix) It seems that the behavior of the C. P. Atkinson's solution by an analog computation for the superharmonic oscillation of Duffing's equation qualitatively agrees with what is mentioned above.

Appendix I

When $u>1$, $f(\tau)$ always takes negative value somewhere in the interval $0<\tau<\pi$. Both the value of $f(0)$ and $f(\pi)$ are equal to zero. $f(\tau)$ is symmetric with respect to $\tau=\pi/2$. Denoting the length of subinterval where $f(\tau)>0$ by L_{f+} and also denoting the length of subinterval where $f(\tau)<0$ by L_{f-}, it is easily seen that

$$L_{f+} < L_{f-}.$$

The first term of $x_P(\tau)$ or $x_Q(\tau)$, $\pm\frac{P_0}{u^2-\lambda^2}\sin\lambda\tau$, is symmetric with respect to $\tau=\pi/2$ and is equal to zero at $\tau=0$ and $\tau=\pi$. When $\lambda\geqq 3$, this term always takes negative value somewhere in the interval $0<\tau<\pi$. Defining $L_{\lambda+}$ and $L_{\lambda-}$ for this term similarly as L_{f+} and L_{f-}, it is clear that

$$L_{\lambda+} = L_{\lambda-}.$$

Using the above mentioned facts, it is easily verified that $\pm\frac{P_0}{u^2-\lambda^2}\sin\lambda\tau+f(\tau)$ always takes negative value somewhere in the interval $0<\tau<\pi$ when $u>1$ and $\lambda\geqq 3$.

Appendix II

Let us suppose that $x(t)$ is a periodic solution of period T of Eq. (1) under the conditions (2). If $x(t)$ is so, $-x\left(t+\frac{T}{2}\right)$ is also a solution.

Putting

$$\left.\begin{aligned} -x\left(t+\frac{T}{2}\right) &\equiv y(t)\,, \\ x(t)-y(t) &\equiv \zeta(t)\,, \end{aligned}\right\} \qquad (a)$$

and substituting Eq. (a) into Eq. (1), we have

$$m\ddot{\zeta}+c\dot{\zeta}+F(x)-F(y) = 0. \qquad (b)$$

Here, we may write as follows:

$$F(x)-F(y) = \zeta g(x, y). \qquad (c)$$

As $g(x, y)$ satisfies

$$\left.\begin{aligned} g(x, y) &= g(y, x)\,, \\ g(x, y) &= g(-x, -y)\,, \end{aligned}\right\} \qquad (d)$$

$g(x, y)$ is the periodic function of period $T/2$.

Putting

$$F(x) = kx + \mu F_1(x)\,, \tag{e}$$

where $F_1(x)$ consists of higher order of x, we may represent $g(x, y)$ as follows:

$$g(x, y) = k + \mu g_1(x, y)\,. \tag{f}$$

Expanding $g_1(x, y)$ in Fourier's series and denoting its Fourier's coefficient by a_n, b_n and a_0 ($n=1, 2, 3, \cdots$), Eq. (b) becomes

$$m\ddot{\zeta} + c\dot{\zeta} + \left\{(k+\mu a_0) + \mu\left(\sum_{n=1}^{\infty} a_n \cos\frac{4n\pi}{T}t + \sum_{n=1}^{\infty} b_n \cos\frac{4n\pi}{T}t\right)\right\}\zeta = 0\,. \tag{g}$$

Putting

$$\left.\begin{aligned} &\tau = 4\pi t/T\,,\quad k/m = \omega_0^2\,,\quad 2\pi/T = \omega\,,\quad \omega_0/\omega = u\,,\\ &2\sqrt{mk} = c_c\,,\quad c\omega/c_c\omega_0 = \beta\,,\quad \mu/4m\omega^2 = \varepsilon\,. \end{aligned}\right\} \tag{h}$$

Eq. (g) becomes

$$\frac{d^2\zeta}{d\tau^2} + \beta\frac{d\zeta}{d\tau} + \left\{\left(\frac{u^2}{4} + \varepsilon a_0\right) + \varepsilon\left(\sum_{n=1}^{\infty} a_n \cos n\tau + \sum_{n=1}^{\infty} b_n \sin n\tau\right)\right\}\zeta = 0\,. \tag{i}$$

This is an equation of parametric excitation.

If a_n, b_n and a_0 are given, we can say that corresponding to transition values of $\left\{\frac{u^2}{4} + \varepsilon a_0\right\}$ and ε from stability to instability there must exist at least one periodic solution of Eq. (i) with the period 2π or the period 4π. When $\beta=0$, the transition value of $\left\{\frac{u^2}{4} + \varepsilon a_0\right\}$ corresponding to the periodic solution of the period 2π tends to i^2 (i: positive integer) provided $\varepsilon \to 0$. Then, the pair of $\left\{\frac{u^2}{4} + \varepsilon a_0\right\}$ and ε corresponding to the periodic solution $\zeta(\tau)$ of period 2π will be something like as shown in Fig. a.

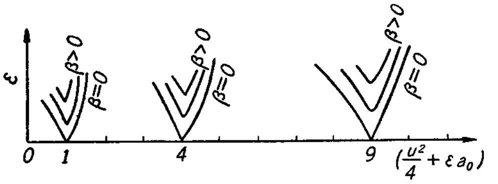

Fig. a.

On the other hand, from the definition of $\zeta(t)$ it must be a periodic solution of the period $T/2$. Therefore, $\zeta(\tau)$ must have the period 2π.

Thus, when $\beta > 0$ (i.e. $c > 0$) there cannot exist the periodic solution $\zeta(\tau)$ of priod 2π other than $\zeta(\tau)\equiv 0$ provided ε is sufficiently small. Also in the case of $\beta=0$, if $u \not\doteq 2i$ (i: positive integer), there cannot exist the periodic solution of the period 2π other than $\zeta(\tau)\equiv 0$ provided ε is small.

When $\zeta(\tau)\equiv 0$, the periodic solution $x(t)$ of the period T of Eq. (1) have the property

$$x(t) = -x(t+T/2)\,.$$

So, if the periodic solution of the period T of $x(t) \neq -x(t+T/2)$ appears when ε is small, it must be restricted to the case of $u \doteq 2i$.

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