Accuracy Considerations of the Equivalent Linearization Technique for the Analysis of a Non-Linear Control System with a Gaussian Random Input

By

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In this paper, the accuracy of the results which can be obtained by the linearization technique of a non-linear control system with a Gaussian random input is investigated. Presence of non-linearity in the control system destroys the Gaussian nature of the response signals. In the linearization technique, therefore, assumption of the Gaussian input for the non-linear element is approximate. The accuracy analysis is restricted to consideration of the cumulants of the second and the fourth order of the input to the non-linear element. This results from the consideration that a linear element with a memory tends to make non-Gaussian signals more nearly Gaussian. A simple control system with a Gaussian input is analyzed as an example. It is concluded that the equivalent linearization technique is accurate enough for engineering purposes.

1. Introduction

In dealing with a non-linear control system subjected to a stationary Gaussian random input, we have so far assumed that the input to the non-linear element in the system also belongs to a Gaussian random process, in order to analyze the non-linear control system by the equivalent linear technique[1]. However, if rigorously considered, the assumption is not valid because of the presence of the non-linearity in the system. Therefore, the results obtained by the equivalent linear technique will be only approximate, although from the point of view of practical applications such solutions will be satisfactory. At times one is interested in checking or improving the accuracy of analytical results by considering the effects of the signal portions which can not be described by the equivalent linear technique. In the equivalent linear technique, it is assumed that the distortion from the Gaussian properties of the input to the non-linear element are small. As a matter of fact, this assumption is reasonable if the non-linearity is small and the low-pass characteristics in the linear parts are dominant. In this

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paper, we will treat a random signal which does not belong to a Gaussian random process but is nearly Gaussian from the above mentioned point of view. Before considering the closed loop system we must develop a general approximate description of a random process in such a way that it will afford many mathematical advantages for the later discussions.

2. Moment-Generating Functions and Cumulant Functions

T. N. Thiele[2] originally introduced the notion of the cumulants or the semi-invariants instead of the moments as a set of statistical parameters for a random process $\{y(t)\}$. Now, the moment-generating function $M(s_1)$ of a random variable $y_1=y(t_1)$, when it exists, is given by

$$M(s_1) = E[\exp s_1 y_1] \tag{1}$$

where s_1 is an auxiliary variable. This function always exists and is, moreover, an uniformly continuous function of s_1 when s_1 is imaginary, say jw_1. It is then a function $\varphi(w_1)$ of the real variable w_1, and is known as the characteristic function. The moment-generating function gives the moments of a distribution.

If we expand $M(s_1)$ in the MacLaurin series, we have

$$M(s_1) = M(0)+M'(0)\frac{s_1}{1!}+M''(0)\frac{s_1^2}{2!}+\cdots \tag{2}$$

where

$$\left.\begin{aligned} M(0) &= 1 \\ M'(0) &= E[y_1] \\ M''(0) &= E[y_1^2] \\ &\vdots \\ M^{(n)}(0) &= E[y_1^n] \end{aligned}\right\} \tag{3}$$

Thus Eq. (2) is expressed as

$$M(s_1) = 1+\sum_{n=1}^{\infty}\frac{E[y_1^n]}{n!}s_1^n \tag{4}$$

Eq. (4) gives the moments of the distribution in terms of the coefficients of the expansion for $M(s_1)$.

The logarithm of $M(s_1)$ defines the cumulant or semi-invariant function $K(s_1)$, that is,

$$K(s_1) = \log M(s_1) \tag{5}$$

and whose n-th derivative (when it exists) at $s_1=0$ is the n-th cumulant or semi-invariant, that is,

$$\lambda_n = K^{(n)}(0) \tag{0}$$

Therefore, by expanding $K(s_1)$ in a MacLaurin series, we have

$$K(s_1) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n!} s_1^n \tag{7}$$

Then by expanding $\exp K(s_1)$ in a power series in s_1, and equating the coefficients to those of the corresponding powers in Eq. (4), the following relations between the moments and the cumulants are obtained.

$$\left.\begin{aligned}
\lambda_1 &= E[y_1] \\
\lambda_2 &= E[y_1^2] - \{E[y_1]\}^2 \\
\lambda_3 &= E[y_1^3] - 3E[y_1^2]\cdot E[y_1] + 2\{E[y_1]\}^3 \\
\lambda_4 &= E[y_1^4] - 4E[y_1^3]\cdot E[y_1] - 3E[y_1^2] + 12E[y_1^2]\{E[y_1]\}^2 \\
&\quad -6\{E[y_1]\}^4 \\
&\text{etc.}
\end{aligned}\right\} \tag{8}$$

These relations mentioned above can be extended to the case of the n random variables, $y_1=y(t_1)$, $y_2=y(t_2)$, $\cdots$ and $y_n=y(t_n)$. Then the cumulant function $K(s_1, s_2, \cdots, s_n)$ is defined by

$$\begin{aligned}
K(s_1, s_2, \cdots, s_n) &= \log M(s_1, s_2, \cdots, s_n) \\
&= \sum_{i=1}^{n} \lambda(t_i) s_i + \frac{1}{2!} \sum_{i,j}^{n} \lambda(t_i, t_j) s_i s_j + \frac{1}{3!} \sum_{i,j,k}^{n} \lambda(t_i, t_j, t_k) s_i s_j s_k + \cdots
\end{aligned} \tag{9}$$

where the function $M(s_1, s_2, \cdots, s_n)$ is the moment generating function of the n random variables, and the coefficients λ's of the expansion are the cumulants and serve as an interpretation of the distribution. In a similar way as before, the relations between the moments and the cumulants are obtained as follows,

$$\left.\begin{aligned}
\lambda(t_i) &= E[y_i] \\
\lambda(t_i, t_j) &= E[y_i\cdot y_j] - \lambda(t_i)\cdot\lambda(t_j) \\
\lambda(t_i, t_j, t_k) &= E[y_i\cdot y_j\cdot y_k] - \lambda(t_i, t_j)\lambda(t_k) - \lambda(t_i, t_k)\lambda(t_j) \\
&\quad - \lambda(t_k, t_j)\lambda(t_i) - \lambda(t_i)\lambda(t_j)\lambda(t_k) \\
\lambda(t_i, t_j, t_k, t_l) &= E[y_i\cdot y_j\cdot y_k\cdot y_l] - \lambda(t_i, t_j, t_k)\lambda(t_l) - \lambda(t_i, t_j, t_l)\lambda(t_k) \\
&\quad - \lambda(t_i, t_k, t_l)\lambda(t_j) - \lambda(t_j, t_k, t_l)\lambda(t_i) - \lambda(t_i, t_j)\lambda(t_k, t_l) \\
&\quad - \lambda(t_i, t_k)\lambda(t_j, t_l) - \lambda(t_i, t_l)\lambda(t_j, t_k) - \lambda(t_i, t_j)\lambda(t_k)\lambda(t_j) \\
&\quad - \lambda(t_i, t_k)\lambda(t_j)\lambda(t_l) - \lambda(t_i, t_l)\lambda(t_j)\lambda(t_k) - \lambda(t_j, t_k)\lambda(t_i)\lambda(t_l) \\
&\quad - \lambda(t_i, t_l)\lambda(t_j)\lambda(t_k) - \lambda(t_k, t_l)\lambda(t_i)\lambda(t_j) \\
&\quad - \lambda(t_i)\lambda(t_j)\lambda(t_k)\lambda(t_l) \\
&\quad \text{etc.}
\end{aligned}\right\} \tag{19}$$

A familiar and typical model for this manner of description of a distribution is readily provided by the shot effect, which affords mathematical description of the voltage fluctuation that might be expected in the output of a vacuum-tube circuit. The shot effect consists of a collection of impulses with random magnitudes at randomly occurring times. It is known that the joint cumulant function for $y_1, y_2, \cdots, y_n$ is written as

$$K(s_1, s_2, \cdots, s_n) = v\bar{a}\sum_i^n s_i + \frac{1}{2!}\sum_{i,j}^n v\overline{a^2}\int_{-\infty}^{\infty} \delta(t_i-\tau)\,\delta(t_j-\tau)\,d\tau\, s_i s_j$$
$$+\frac{1}{3!}\sum_{i,j,k}^n v\overline{a^3}\int_{-\infty}^{\infty} \delta(t_i-\tau)\,\delta(t_j-\tau)\,\delta(t_k-\tau)\,d\tau\, s_i s_j s_k + \cdots \tag{11}$$

where v represents the constant average density with which the times of occurrence of the impulses are distributed at random and independently over the time axis, and $\overline{a^n}$ is the n-th moment of the magnitude of the impulses.

In general, a random process $\{y(t)\}$ can be completely characterized by an infinite sequence of functions $p_1, p_2, \cdots, p_n$, where the function p_n is the joint probability density function of the n random variables $y_1, y_2, \cdots$ and y_n. It is expected that one of the satisfactory approximations for the probability distribution function of an actual random process would be obtained by employing the cumulants of the first few orders. In fact if we take up only the first and second cumulant, $\lambda(t_i)$ and $\lambda(t_i, t_j)$, the corresponding approximate distribution becomes a well-known n-dimensional normal one. On the other hand, except for the first and second cumulants, all the others vanish if a random process is Gaussian.

3. The Relations between the Cumulants of the Input and the Output of a Linear System

Let $W(t,\tau)$ denote the weighting function of a linear system, then the output $x(t)$ is expressed by

$$x(t) = \int_{-\infty}^{t} W(t, \tau)\, y(\tau)\, d\tau \tag{12}$$

where $y(t)$ is the input to the linear system. We assume that $W(t, \tau)$ is essentially zero when the absolute difference between t and τ exceeds some fixed value T_1, that is,

$$W(t, \tau) = 0 \qquad \text{for } |t-\tau| > T_1 \tag{13}$$

We fix upon a large but finite interval $(-T \leqq t \leqq T)$, where $T \gg T_1$, and divide the interval at the points $t_1, t_2, \cdots, t_{n+1}$ $(-T = t_1 < t_2 < \cdots < t_{n+1} = T)$ into n parts. If t is now constrained to lie within a sub-interval,

$$-T + T_1 < t < T - T_1$$

and if the number of the points of sub-intervals can be sufficiently small, then we can write Eq. (12) in the following form

$$x(t) = \sum_{i=1}^{\infty} W(t, t_i) y_i \Delta t_i \tag{14}$$

where

$$\Delta t_i = t_{i+1} - t_i$$

By definition we have the cumlant function of the m random variables $x_1 = x(t_1')$, $x_2 = x(t_2')$, $\cdots$, $x_m = x(t_m')$ and then by using Eq. (14) it is rewritten as

$$\begin{aligned} K_x^m(u_1, u_2, \cdots, u_m) &= \log E\Big[\exp \sum_j^m x_j u_j\Big] \\ &= \log E\Big[\exp \sum_i^n y_i \big(\sum_j^m W(t_j', t_i) \Delta t_i u_j\big)\Big] \end{aligned} \tag{15}$$

where the u's are auxiliary variables. It is found that the last expression represents the cumlant function of the n random variables $y_1, y_2, \cdots, y_n$ of the input with the arguments $s_i = \sum_j^m W(t_j', t_i) t_i u_j$ $(i = 1, 2, \cdots, n)$, which has been given by Eq. (9). Therefore, we have formally

$$\begin{aligned} &K_x^m(u_1, u_2, \cdots, u_m) \\ &= K_y^n\Big\{\sum_j^m W(t_j', t_1) \Delta t_1 u_j, \sum_j^m W(t_j', t_2) \Delta t_2 u_j, \cdots, \sum_j^m W(t_j', t_n) \Delta t_n u_j\Big\} \end{aligned} \tag{16}$$

Each of the cumlants of the output can be obtained as a sum of the products of the input cumlant by the weighting function over all the sampled points. when the number of the sub-division n tends to infinity such that $max(\Delta t_i) \to 0$, the cumlant function K_x^m can be expressed by the cumlants which are functions of the input cumlants and the weighting function, that is

$$K_x^m = \sum_i^m \lambda_{x(t_i)} u_i + \frac{1}{2!} \sum_{ij}^m \lambda_{x(t_i, t_j)} u_i u_j + \frac{1}{3!} \sum_{i,j,k}^m \lambda_{x(t_i, t_j, t_k)} u_i u_j u_k + \cdots \tag{17}$$

where

$$\left.\begin{aligned} \lambda_x(t_i) &= \int_{-\infty}^{\infty} W(t_i, t_0) \lambda_y(t_0) dt_0 \\ \lambda_x(t_i, t_j) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t_i, t_0) W(t_j, t_p) \lambda_y(t_0, t_p) dt_0 dt_p \\ \lambda_x(t_i, t_j, t_k) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t_i, t_0) W(t_j, t_p) W(t_k, t_q) \lambda_y(t_0, t_p, t_q) dt_0 dt_p dt_q \\ \lambda_x(t_i, t_j, t_k, t_l) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(t_i, t_0) W(t_j, t_p) W(t_k, t_q) W(t_l, t_r) \times \\ &\quad \times \lambda_y(t_0, t_p, t_q, t_r) dt_0 dt_p dt_q dt_r \end{aligned}\right\} \tag{18}$$

When $W(t, \tau)$ cannot be considered zero as the absolute difference between t and τ becomes large, say in such systems that have integral characteristics or are unstable we can also obtain the same relations of cumlants as Eq. (18). These are regarded as the limiting cases when T_1 tends to infinity.

In particular, if the input $y(t)$ is stationary and the linear system is time-invariant, all the functions considered above are invariable for the shift of the time origin. Therefore, these functions depend on t and τ only through the difference $(t-\tau)$.

$$\left.\begin{aligned} &W(t, \tau) = W(t-\tau) \\ &\lambda_y(t_i) = m_y \\ &\lambda_y(t_i, t_j) = \lambda_y(\tau_1) \\ &\lambda_y(t_i, t_j, t_k) = \lambda_y(\tau_1, \tau_2) \\ &\lambda_y(t_i, t_j, t_k, t_l) = \lambda_y(\tau_1, \tau_2, \tau_3) \end{aligned}\right\} \tag{19}$$

where

$$\tau_1 = t_j - t_i, \quad \tau_2 = t_k - t_i, \quad \tau_3 = t_l - t_i, \cdots$$

Then Eq. (18) can be expressed as follows,

$$\left.\begin{aligned} m_x &= \int_{-\infty}^{\infty} W(\sigma)\, d\sigma \cdot m_y \\ \lambda_x(\tau_1) &= \iint_{-\infty}^{\infty} W(\sigma_1) W(\sigma_2) \lambda_y(\tau_1 - \sigma_2 + \sigma_1)\, d\sigma_1 d\sigma_2 \\ \lambda_x(\tau_1, \tau_2) &= \iiint_{-\infty}^{\infty} W(\sigma_1) W(\sigma_2) W(\sigma_3) \lambda_y(\tau_1 + \sigma_1 - \sigma_2, \tau_2 - \sigma_3 + \sigma_1)\, d\sigma_1 d\sigma_2 d\sigma_3 \\ \lambda_x(\tau_1, \tau_2, \tau_3) &= \iiiint_{-\infty}^{\infty} W(\sigma_1) W(\sigma_2) W(\sigma_3) W(\sigma_4) \lambda_y(\tau_1 - \sigma_2 + \sigma_1, \tau_2 - \sigma_3 + \sigma_1, \\ &\qquad \tau_3 - \sigma_4 + \sigma_1)\, d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_4 \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{20}$$

If we take the Fourier transform of Eq. (20), we have

$$\left.\begin{aligned} &m_x \delta(\omega) = G(0)\, \delta(\omega)\, m_y \\ &\mu_x(j\omega_1) = G(-j\omega_1)\, G(j\omega_1)\, \mu_y(j\omega_1) \\ &\mu_x(j\omega_1, j\omega_2) = G(-j\omega_1 - j\omega_2)\, G(j\omega_1)\, G(j\omega_2)\, \mu_y(j\omega_1, j\omega_2) \\ &\mu_x(j\omega_1, j\omega_2, j\omega_3) = G(j\omega_1 - j\omega_2 - j\omega_3)\, G(j\omega_1)\, G(j\omega_2)\, G(j\omega_3)\, \mu_y(j\omega_1, j\omega_2, j\omega_3) \\ &\qquad\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{21}$$

where

$$\int_{-\infty}^{\infty} W(\sigma) \exp(-j\omega\sigma)\, d\sigma = G(j\omega)$$

$$\mu(j\omega_1, j\omega_2, \cdots, j\omega_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda(\tau_1, \tau_2, \cdots, \tau_n) \exp(-j\omega_1\tau_1 - \cdots - j\omega_n\tau_n)\, d\tau_1 \cdots d\tau_n \tag{22}$$

Now, if the mean value of the input m_y is zero, the mean value of the output m_x becomes zero, and $\lambda_y(\tau_1)$ and $\lambda_x(\tau_1)$ correspond to the auto-correlation functions $R_y(\tau_1)$ and $R_x(\tau_1)$ of the input and the output respectively. Therefore, $\mu_y(j\omega_1)$ and $\mu_x(j\omega_1)$ correspond to the spectral densities, $S_y(\omega_1)$ and $S_x(\omega_1)$, of the input and the output respectively.

4. The Distribution of the Output Signal Passing Through Linear Filters

If a linear system has a Gaussian input, every signal appearing in the system is also Gaussian. However, the presence of a non-linear element in the system destroys the Gaussian nature of the response signals. It is said that the low-pass nature of a linear element usually tends to make non-Gaussian signals more nearly Gaussian. This statement is equivalent to saying that when the memory of a linear filter becomes so long as to hold as the same order of the variance

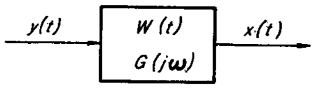

Fig. 1.

of the output as before, the higher the order of the cumulants of interest is, the more rapidly the corresponding cumulants of the output decrease. To see this fact, consider the stable linear system as shown in Fig. 1, where $W(t)$ and $G(j\omega)$ are the weighting function and the frequency-response function of the linear filter respectively. Then, let us write the weighting function of the linear filter transformed by the time scale as follows,

$$H(T)W(t/T) \qquad \text{for} \quad T>1 \tag{23}$$

where $H(T)$ is to be determined as a normalizing factor under the above condition, because by this transformation the memory of the linear filter will become long but the magnitude of the output of the transformed linear filter will generally increase.

Now, if we assume that the input $y(t)$ is a non-Gaussian signal with the mean zero, we have

$$\mu_y(j\omega) = S_y(\omega)\,, \quad \mu_x(j\omega) = S_x(\omega)$$

where $S_y(\omega)$ and $S_x(\omega)$ are the spectral densities of the input $y(t)$ and the output $x(t)$ respectively. Therfore, the spectral density $S_x(\omega, T)$ of the output from the transformed linear filter can be obtained from Eq. (21) as follows,

$$S_x(\omega,T) = T^2H^2|G(jT\omega)|^2S_y(\omega) \tag{24}$$

and the variance $\sigma_x(T)$ is given by

$$\sigma_x(T) = \frac{1}{2\pi}\int_0^\infty S_x(\omega, T)\,d\omega$$
$$= TH^2\int_{-\infty}^{\infty} |G(j\omega)|^2 \mu_y(j\omega/T)\,d\omega \tag{25}$$

In particular, if the input $y(t)$is a shot noise with the mean zero, we can write the Fourier transforms of its cumulants from Eq. (11) as follows,

$$\left.\begin{aligned}\mu_y(j\omega_1) &= \frac{1}{2\pi}\frac{v\overline{a^2}}{2!}\\ \mu_y(j\omega_1,\ j\omega_2) &= \frac{1}{(2\pi)^2}\frac{v\overline{a^3}}{3!}\\ \mu_y(j\omega_1,\ j\omega_2,\ j\omega_3) &= \frac{1}{(2\pi)^3}\frac{v\overline{a^4}}{4!}\\ \text{etc.}\end{aligned}\right\} \tag{26}$$

From Eq. (25) we have

$$\sigma_x(T) = TH^2\frac{v\overline{a^2}}{2!}\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 d\omega$$
$$= TH^2\sigma_x(1)$$

where $\sigma_x(1)$ is the variance of the output from the original linear filter. Therefore, in order that $\sigma_x(T)$ be of the same order as $\sigma_x(1)$ when $T\to\infty$, the normalizing factor $H(T)$ must be chosen as

$$H(T) = T^{-1/2} \tag{27}$$

Hence, the Fourier transforms of the cumulants of the output from tne transformed filter become

$$\left.\begin{aligned}\mu_x(j\omega_1,\ j\omega_2\,;\ T) &= T^{3/2}G(-jT\omega_1-jT\omega_2)G(jT\omega_1)G(jT\omega_2)\mu_y(j\omega_1,\ j\omega_2)\\ \mu_x(j\omega_1,\ j\omega_2,\ j\omega_3\,;\ T) &= T^2G(-jT\omega_1-jT\omega_2-jT\omega_3)G(jT\omega_1)G(jT\omega_2)\times\\ &\quad\times G(jT\omega_3)\mu_y(j\omega_1,\ j\omega_2,\ j\omega_3)\end{aligned}\right\} \tag{28}$$

etc.

and the corresponding cumulants can be obtained from Eq. (21) as follows,

$$\lambda_x(\tau_1,\ \tau_2\,;\ T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu_x(j\omega_1,\ j\omega_2\,;\ T)\exp\,(j\omega_1\tau_1+j\omega_2\tau_2)\,d\omega_1 d\omega_2$$
$$= T^{-1/2}\frac{v\overline{a^3}}{3!}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(-j\omega_1-j\omega_2)G(j\omega_1)\times$$
$$\times G(j\omega_2)\exp\,(j\omega_1\tau_1+j\omega_2\tau_2)\,d\omega_1 d\omega_2$$

$$= T^{-1/2}\frac{v\overline{a^3}}{3!}\int_{-\infty}^{\infty} W(t)\,W(t+\tau_1)\,W(t+\tau_2)\,dt$$

$$= T^{-1/2}\lambda_x(\tau_1,\ \tau_2\,;1) \tag{29}$$

Similarly we have

$$\lambda_x(\tau_1,\ \tau_2,\ \tau_3\,;T) = T^{-1}\lambda_x(\tau_1,\ \tau_2,\ \tau_3,\ 1) \tag{30}$$

In general, the n-th cumulant $\lambda_x(\tau_1,\ \tau_2, \cdots, \tau_{n-1}\,; T)$ of the output is of the order $T^{-(n/2-1)}$. Therefore, if the value of T becomes large, the output tends closely to a member of a Gaussian random process.

Next we consider the general cases where the input $y(t)$ is not a shot noise but satisfies the following conditions,

$$\left.\begin{aligned} &m_y = 0 \\ &\mu_y(0) = 0 \\ &|\mu_y(j\omega_1,\ j\omega_2, \cdots, j\omega_n)| < M_{yn} \qquad \text{for} \quad n = 1, 2, \cdots \quad \text{and} \quad -\infty < \omega < \infty \end{aligned}\right\} \tag{31}$$

where the M_y's are finite numbers. Let us assume that the linear filter satisfies the following conditions

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |G(-j\omega_1,\ j\omega_2, \cdots, j\omega_n)\,G(j\omega_1)\cdots G(j\omega_n)|\,d\omega_1\cdots d\omega_n < M_{gn} \qquad \text{for} \quad (n = 1, 2, \cdots) \tag{32}$$

where the M_g's are finite numbers. Under these conditions, we have from Eq. (25)

$$\lim_{T\to\infty} \sigma_x(T) = \mu_y(0)\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega = \text{const} \tag{33}$$

if the normalizing factor $H(T)$ is equal to $T^{-1/2}$. That is, in order that $\sigma_x(T)$ be of the same order as $\sigma_x(1)$ even if T tends to infinity, the normalizing factor must be chosen in the same form as before. Then we have the following inequalities,

$$\left.\begin{aligned} |\lambda_x(\tau_1,\ \tau_2\,;T)| &\leq \int_{-\infty}^{\infty} |\mu_x(j\omega_2,\ j\omega_2\,;T)|\,d\omega_1 d\omega_2 \\ &< T^{-1/2} M_{y2} M_{g2} \\ |\lambda_x(\tau_1,\ \tau_2,\ \tau_3 : T)| &> T^{-1} M_{y3} M_{g3} \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{34}$$

From these results, it is found that the absolute value of the n-th cumulant $|\lambda_x(\tau_1,\ \tau_2, \cdots, \tau_{n-1},\ T)|$ is of the order $T^{-(n/2-1)}$.

In the case where a linear filter has purely integral characteristics, condition (32) does not satisfy. The transfer function of such a filter can be expressed by

$$G(s) = G'(s)/s \tag{35}$$

where $G'(s)$ has neither pole nor zero-point at the origin of the s-plane. Lest the cumulants of the output from the linear filter should become infinite, the Fourier transforms of the cumulants of the input must be of the following form

$$\mu_y(j\omega_1, \cdots, j\omega_n) = (-j\omega_1 - \cdots - j\omega_n)(j\omega_1)\cdots(j\omega_n)\mu'_y(j\omega_1, \cdots, j\omega_n) \tag{36}$$

where $\mu'_y(j\omega_1, \cdots, j\omega_n)$ has neither pole nor zero-point at the origin of the n-dimensional space $(\omega_1, \cdots, \omega_n)$. This description is confirmed by the fact that, when a control system containing a purely integral element in the forward path is excited by random noise, the output has finite variance. The variance of the output from the transformed filter can be expressed from Eqs. (25), (35), and (36) by

$$\sigma_x = TH^2 \int_{-\infty}^{\infty} |G(j\omega)|^2 \mu_y(j\omega/T)\, d\omega$$

$$= H^2/T \int_{-\infty}^{\infty} |G'(j\omega)|^2 \mu'_y(j\omega/T)\, d\omega$$

In the limiting case of $T \to \infty$, we have the same result as Eq. (33), if the normalizing factor is chosen to be of the form $H(T) = T^{1/2}$. Therefore, the Fourier transforms of the cumulants of the output are expressed as

$$\left.\begin{aligned} \mu_x(j\omega_1, j\omega_2 : T) &= T^{9/2} G(-jT\omega_1 - jT\omega_2) G(jT\omega_1) G(jT\omega_2) \mu_y(j\omega_1, j\omega_2) \\ &= T^{3/2} G'(-jT\omega_1 - jT\omega_2) G'(jT\omega_1) G'(jT\omega_2) \mu'_y(j\omega_1, j\omega_2) \\ \mu_x(j\omega_1, j\omega_2, j\omega_3 : T) &= T^2 G'(-jT\omega_1 - jT\omega_2 - jT\omega_3) G'(jT\omega_1) \times \\ &\quad \times G'(jT\omega_2) G'(jT\omega_3) \mu'_y(j\omega_1, j\omega_2, j\omega_3) \end{aligned}\right\} \tag{37}$$

etc.

Eq. (37) coincides with Eq. (28). Therefore it is found that the same results are obtained as before.

5. The Cumulant Function of the Input to the Non-Linear Element in Control Systems

Let us consider the system as shown in Fig. 2, where the non-linear element, N. L., is symmetric and the transfer characteristic is expressed by

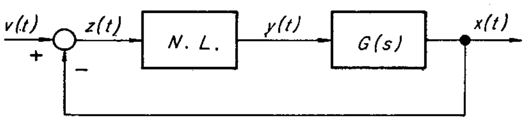

Fig. 2.

$$y = f(z)$$
$$= z + \frac{\gamma}{3!} z^3 \tag{38}$$

where $z(t)$ and $y(t)$ are the input and the output respectively. The coefficient of the third power, γ, is now interpreted as the measure of non-linearity and is assumed to be a small quantity. Then, the two-sided Laplace transform of Eq. (38) is

$$F(s) = \int_{-\infty}^{\infty} f(z) \exp(-sz)\, dz$$
$$= s^{-2} + \gamma s^{-4} \tag{39}$$

The output $y(t)$ is expressed by

$$y(t) = \frac{1}{2\pi} \oint F(jw) \exp(jwz)\, dw$$
$$= \frac{1}{2\pi} \oint (-w^{-2} + w^{-4}) \exp(jwz)\, dw \tag{40}$$

where the symbol $\oint$ is an integral path taken counter clock-wise along a small circle with a center at the origin, and $s = jw$.

As the input to the control system $v(t)$ is assumed to be a random signal belonging to a stationary Gaussian random process with mean zero, the cumulant function of the m-th dimension is expressed by

$$K_v(s_1, \cdots, s_m) = \frac{1}{2} \sum_{i,j}^{m} \lambda_v(t_j - t_i)\, s_i s_j$$
$$= \frac{1}{2} \sum_{i,j}^{m} R_v(t_j - t_i)\, s_i s_j \tag{41}$$

where $R_v(\tau)$ is the auto-correlation function of $v(t)$.

As the control system considered here is symmetric and the cumulant function of $z(t)$ also become symmetric about the origin of the auxiliary variables, all the cumulants of odd order can be assumed to be zero. Therefore, the cumulant function of $z(t)$ can be expressed as

$$K_z(s_1, s_2, \cdots, s_m) = \frac{1}{2} \sum_{i,j}^{m} \lambda_z(t_i, t_j)\, s_i s_j$$
$$+ \frac{1}{4!} \sum_{i,j,k,l}^{m} \lambda_z(t_i, t_j, t_k, t_l)\, s_i s_j s_k s_l$$
$$+ \cdots \tag{42}$$

where

$$\left.\begin{aligned}&\lambda_z(t_i,\ t_j) = E[z_i z_j]\\&\lambda_z(t_i,\ t_j,\ t_k,\ t_l) = E[z_i z_j z_k z_l]-\lambda_z(t_i,\ t_j)\lambda_z(t_k,\ t_l)\\&\qquad -\lambda_z(t_i,\ t_k)\lambda_z(t_j,\ t_1)-\lambda_z(t_i,\ t_1)\lambda_z(t_i,\ t_k)\\&\cdots\cdots\cdots\end{aligned}\right\} \tag{43}$$

Now, let us assume that the input to the non-linear element characterized by Eq. (38) belongs to a Gaussian random process with the mean zero and the variance unity, then the cumulants of the output become as follows,

$$\left.\begin{aligned}&\lambda_1 = \lambda_3 = \lambda_5 = \cdots = 0\\&\lambda_2 = 1+\gamma+\cdots\\&\lambda_4 = 4\gamma+\cdots\\&\lambda_6 = 12\gamma^2+\cdots\\&\cdots\cdots\end{aligned}\right\} \tag{44}$$

If γ is small, all the cumulants higher than the sixth are negligibly small. Therefore, in Eq. (42) we can safely neglect all the cumulants higher than the sixth.

6. Analysis of the Closed System

Let us again consider the non-linear control system as shown in Fig. 2. We have the following relation from Fig. 2.

$$v(t) = z(t)+x(t) \tag{45}$$

If the cumulant function of $z(t)+x(t)$ is expressed as

$$\begin{aligned}K_{zx}(s_1,\ s_2,\cdots,s_m) &= \log E\Big[\exp\Big\{\sum_i^m (z_i+x_i)\,s_i\Big\}\Big]\\&= \frac{1}{2}\sum_{i,j}^m\Big\{\lambda_{zz}(t_i,\ t_j)+\lambda_{zx}(t_i,\ t_j)+\lambda_{xz}(t_i,\ t_j)+\lambda_{xx}(t_i,\ t_j)\,s_i s_j\\&\quad+\frac{1}{4!}\sum_{i,j,k,l}^m\Big\{\lambda_{zzzz}(t_i,\ t_j,\ t_k,\ t_l)+\lambda_{zzzx}(t_i,\ t_j,\ t_k,\ t_l)+\cdots\\&\quad+\cdots\cdots\cdots\\&\quad+\cdots+\lambda_{xxxx}(t_i,\ t_j,\ t_k,\ t_l)\Big\}s_i s_j s_k s_l\end{aligned} \tag{46}$$

Therefore we have

$$K_v(s_1,\ s_2,\cdots,s_m) = K_{zx}(s_1,\ s_2,\cdots,s_m) \tag{47}$$

If on both sides of Eq. (46) we put the coefficient of each of the auxiliary variables equal to the corresponding one on the other side and if we use the relations of Eq. (18), we have the following two relations. One relation is for the 2nd cumulants,

$$\lambda_{vv}(t_i, t_j) = \lambda_{zz}(t_i, t_j) + \int_{-\infty}^{\infty} W(t_j, t_0)\lambda_{zy}(t_i, t_0)dt_0$$
$$+ \int_{-\infty}^{\infty} W(t_i, t_p)\lambda_{yz}(t_p, t_j)dt_p + \int_{-\infty}^{\infty}\int_{\infty}^{\infty} W(t_i, t_0)W(t_j, t_p)\lambda_{yy}(t_0, t_p)dt_0 dt_p \quad (48)$$

and the other is for the 4th cumulants,

$$0 = \lambda_{zzzz}(t_i, t_j, t_k, t_l) + \int_{-\infty}^{\infty} W(t_l, t_r)\lambda_{zzzy}(t_i, t_j, t_k, t_r)dt_r$$
$$+ \int_{-\infty}^{-\infty} W(t_k, t_q)\lambda_{zzyz}(t_i, t_j, t_q, t_l)dt_q + \int_{-\infty}^{\infty} W(t_j, t_p)\lambda_{zyzz}(t_i, t_p, t_k, t_l)dt_p$$
$$+ \int_{-\infty}^{\infty} W(t_i, t_0)\lambda_{yzzz}(t_0, t_j, t_k, t_l)dt_0$$
$$+ \iint_{-\infty}^{\infty} W(t_k, t_q)W(t_l, t_r)\lambda_{zzyy}(t_i, t_j, t_q, t_r)dt_q dt_r + \cdots$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \cdots$$
$$+ \iiiint_{-\infty}^{\infty} W(t_i, t_0)W(t_j, t_p)W(t_k, t_q)W(t_l, t_r)\lambda_{yyyy}(t_0, t_p, t_q, t_r)dt_0 dt_p dt_q dt_r \quad (49)$$

If the control system is time-invariant and the input to the system is stationary, and if we take the Fourier transforms of Eqs. (48) and (49), we have

$$\mu_{vv}(j\omega) = \mu_{zz}(j\omega) + G(j\omega)\mu_{zy}(j\omega) + G(-j\omega)\mu_{yz}(j\omega)$$
$$+ G(-j\omega)G(j\omega)\mu_{yy}(j\omega) \quad (50)$$

and

$$0 = \mu_{zzzz} + G(j\omega_3)\mu_{zzzy} + G(j\omega_2)\mu_{zzyz} + G(j\omega_1)\mu_{zyzz}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)\mu_{yzzz} + G(j\omega_2)G(j\omega_3)\mu_{zzyy}$$
$$+ G(j\omega_3)G(j\omega_1)\mu_{zyzy} + G(j\omega_1)G(j\omega_2)\mu_{zyyz}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_3)\mu_{yzzy} + G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_2)\mu_{yzyz}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_1)\mu_{yyzz} + G(j\omega_1)G(j\omega_2)G(j\omega_3)\mu_{zyyy}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_2)G(j\omega_3)\mu_{yzyy}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_3)G(j\omega_1)\mu_{yyzy}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_1)G(j\omega_2)\mu_{yyyz}$$
$$+ G(-j\omega_1 - j\omega_2 - j\omega_3)G(j\omega_1)G(j\omega_2)G(j\omega_3)\mu_{yyyy} \quad (51)$$

where we denote the expression $\mu(j\omega_1, j\omega_2, j\omega_3)$ by the abbreviated form μ.

Now, by using expression (40) and by changing the order of integration and average, the cross-correlation function between z_i and y_j, $\lambda_{zy}(t_i, t_j)$, becomes as follows,

$$\lambda_{zy}(t_i,\ t_j) = E[z_i \cdot y_j] = \frac{1}{(2\pi)^2} \oint (jw_i)^{-2} dw_i \int_C F(jw_j)\, E[\exp(jw_i z_i + jw_j z_j)]\, dw_j \tag{52}$$

The expression $E[\exp(jw_i z_i + jw_j z_j)]$ is the characteristic function of x_i and z_j, and by using Eq. (42) and by putting $s_i = jw_i$ it can be expressed as follows,

$$\begin{aligned} &E[\exp(jw_i z_i + jw_j z_j)] \\ &= \exp\Big[-\frac{1}{2}\Big\{\lambda_z(t_i,\ t_i) w_i^2 + 2\lambda_z(t_i,\ t_j) w_i w_j + \lambda_z(t_i,\ t_j) w_j^2\Big\} \\ &+ \frac{1}{4!}\Big\{\lambda_z(t_i,\ t_i,\ t_i,\ t_i) w_i^4 + 4\lambda_z(t_i,\ t_i,\ t_i,\ t_j) w_i^3 w_j \\ &+ 6\lambda_z(t_i,\ t_i,\ t_j,\ t_j) w_i^2 w_j^2 + 4\lambda_z(t_i,\ t_j,\ t_j,\ t_j) w_i w_j^3 \\ &+ \lambda_z(t_j,\ t_j,\ t_j,\ t_j) w_j^4\Big\}\Big] \end{aligned} \tag{53}$$

If we put

$$\begin{aligned} \beta_n = \frac{1}{2\pi} \int_C (jw)^n F(jw) \exp\Big\{-\frac{1}{2}\lambda_z(t_i,\ t_i) w^2 \\ + \frac{1}{4!}\lambda_z(t_i,\ t_i,\ t_i,\ t_i) w^4\Big\} dw \end{aligned} \tag{54}$$

Eq. (52) becomes

$$\lambda_{zy}(t_i,\ t_j) = \beta_1 \lambda_z(t_i,\ t_j) + \frac{\beta_3}{3!}\lambda_z(t_i,\ t_j,\ t_j,\ t_j) \tag{55}$$

In a stationary case, the above equation is written as

$$\lambda_{zy}(\tau) = \beta_1 \lambda_z(\tau) + \frac{\beta_3}{3!}\lambda_z(0,\ 0,\ -\tau) \tag{56}$$

In the similar way, we have

$$\lambda_{yz}(\tau) = \beta_1 \lambda_z(\tau) + \frac{\beta_3}{3!}\lambda_z(0,\ 0,\ \tau) \tag{57}$$

and

$$\begin{aligned} \lambda_y(\tau) = \beta_1^2 \lambda_z(\tau) + \frac{\beta_1 \beta_3}{3!}\Big\{\lambda_z(0,\ 0,\ \tau) + \lambda_z(0,\ 0,\ -\tau)\Big\} \\ + \frac{\beta_3^2}{3!}\lambda_z^3(\tau) + O(\beta_3^3) \end{aligned} \tag{58}$$

By substituting the Fourier transforms of Eqs. (56), (57) and (58) into Eq. (50), we have

$$\begin{aligned} \mu_v(j\omega) = \Big\{1 + \beta_1 G(j\omega)\Big\}\Big\{1 + \beta_1 G(-j\omega)\Big\}\mu_z(j\omega) \\ + \frac{\beta_3}{3!}\Big[G(j\omega) A_2(-j\omega) + G(-j\omega) A_z(j\omega) \\ + \beta_1 G(j\omega) G(-j\omega)\Big\{A_z(-j\omega) + A_z(j\omega)\Big\}\Big] \\ + \frac{\beta_3^2}{3!} G(j\omega) G(-j\omega)\, \mu_z^{(3)}(j\omega) + O(\beta_3^3) \end{aligned} \tag{59}$$

where

$$A_z(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mu_z(0, 0, \tau)\exp(-j\omega\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu_z(j\omega_1,\ j\omega_2,\ j\omega)\,d\omega_1 d\omega_2 \tag{60}$$

and

$$\mu_z^{(3)}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda_z^3(\tau)\exp(-j\omega\tau)\,d\tau \tag{61}$$

In a similar way, we have the following relation from Eq. (51)

$$\mu_z(j\omega_1,\ j\omega_2, j\omega_3) = -\beta_3\Big\{G_c(j\omega_3)\,{}_3\mu_z + G_c(j\omega_2)\,{}_2\mu_z$$
$$+ G_c(j\omega_1)\,{}_1\mu_z + G_c(-j\omega_1 - j\omega_2 - j\omega_3)\,{}_0\mu_z\Big\} \tag{62}$$

where

$$G_c(j\omega) = \int_{-\infty}^{\infty}W_c(t)\exp(-j\omega t)\,dt$$
$$= \frac{G(j\omega)}{1+\beta_1 G(j\omega)} \tag{63}$$

and

$$\left.\begin{aligned}
{}_3\mu_z &= \mu_z(j\omega_1+j\omega_2+j\omega_3)\,\mu_z(j\omega_1)\,\mu_z(j\omega_2)\\
{}_2\mu_z &= \mu_z(j\omega_1+j\omega_2+j\omega_3)\,\mu_z(j\omega_3)\,\mu_z(j\omega_1)\\
{}_1\mu_z &= \mu_z(j\omega_1+j\omega_2+j\omega_3)\,\mu_z(j\omega_2)\,\mu_z(j\omega_3)\\
{}_0\mu_z &= \mu_z(j\omega_1)\,\mu_z(j\omega_2)\,\mu_z(j\omega_3)
\end{aligned}\right\} \tag{64}$$

Therefore, we have from the inverse Fourier Transform of Fq. (62).

$$\lambda_z(\tau_1,\ \tau_2,\ \tau_3) = -\beta_3\int_{-\infty}^{\infty}\Big\{\lambda_z(\sigma)\,\lambda_z(\tau_1\sigma)\,\lambda_z(\tau_2-\sigma)\,W_c(\tau_3-\sigma)$$
$$+\lambda_z(\sigma)\,\lambda_z(\tau_1-\sigma)\,W_c(\tau_2-\sigma)\,\lambda_z(\tau_3-\sigma)$$
$$+\lambda_z(\sigma)\,W_c(\tau_1-\sigma)\,\lambda_z(\tau_2-\sigma)\,\lambda_z(\tau_3-\sigma)$$
$$+W_c(\sigma)\,\lambda_z(\tau_1-\sigma)\,\lambda_z(\tau_2-\sigma)\,\lambda_z(\tau_3-\sigma)\Big\}d\sigma \tag{65}$$

The required quantity $\lambda_z(0, 0, \tau)$ becomes

$$\lambda_z(0, 0, \tau) = -\beta_3\Big\{\int_{-\infty}^{\infty}W_c(\sigma)\,\lambda_z^3(\tau-\sigma)\,d\sigma$$
$$+3\int_{-\infty}^{\infty}W_c(\sigma)\,\lambda_z^2(\sigma)\,\lambda_z(\tau-o)\,d\sigma\Big\} \tag{66}$$

and the 4th cumulant of $z(t)$ is given by

$$\lambda_4^z = \lambda_z(0, 0, 0) = -4\beta_3 \int_{-\infty}^{\infty} W_c(t)\, \lambda_z^3(t)\, dt \tag{67}$$

As Eq. (60) is equal to the Fourier transform of Eq. (66), we have

$$A_z(j\omega) = -\beta_3 \Big\{ G_c(j\omega)\, \mu_z^{(3)}(j\omega) + 3B_z(j\omega)\, \mu_z(j\omega) \Big\} \tag{68}$$

where

$$B_z(j\omega) = \int_{-\infty}^{\infty} W_c(\tau)\, \lambda_z^2(\tau) \exp(-j\omega\tau)\, d\tau \tag{69}$$

By substituting Eq. (68) into Eq. (59) and by neglecting the small quantities of higher order than that of β_3^2, we have

$$\begin{aligned} \mu_v(j\omega) = {} & |1+\beta_1 G(j\omega)|^2 \mu_z(j\omega) \\ & -\beta_3^2 \Big[\mathrm{Real}\Big\{ G(-j\omega)(1+\beta_1 G(j\omega))\, B_z(j\omega) \Big\} \mu_z(j\omega) \\ & + \frac{1}{6} |G(j\omega)|^2 \mu_z^{(3)}(j\omega) \Big] \end{aligned} \tag{70}$$

It is difficult to find the expression for $\mu_z(j\omega)$ from the above equation, so that we intend to solve it by the perturbation method. As β_3^2 generally is a small quantity, we assume that $\mu_z(j\omega)$ can be expanded in series of powers of β_3^2 as follows,

$$\mu_z(j\omega) = \mu_{z0}(j\omega) + \beta_3^2 \mu_{z1}(j\omega) + O(\beta_3^4) \tag{71}$$

The first approximation $\mu_{z0}(j\omega)$ can be found by substituting Eq. (71) into Eq. (70) and by putting β_3^2 equal to zero, that is,

$$\mu_{z0}(j\omega) = \left| \frac{1}{1+\beta_1 G(j\omega)} \right|^2 \mu_v(j\omega) \tag{72}$$

The additional term $\mu_{z1}(j\omega)$ is found by differentiating Eq. (70) with respect to β_3^2 and putting β_3^2 equal to zero, that is,

$$\begin{aligned} \mu_{z1}(j\omega) = {} & \Big[\mathrm{Real}\Big\{ G_c(-j\omega)\, B_{z0}(j\omega) \Big\} \mu_{z0}(j\omega) \\ & + \frac{1}{6} |G_c(j\omega)|^2 \mu_{z0}^{(3)}(j\omega) \Big] \end{aligned} \tag{73}$$

where

$$\mu_{z0}^{(3)}(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda_{z0}^3(\tau) \exp(-j\omega\tau)\, d\tau \tag{74}$$

and

$$B_{z0}(j\omega) = \int_{-\infty}^{\infty} W_c(\tau)\lambda_{z0}^{2}(\tau)\exp(-j\omega\tau)d\tau \tag{75}$$

Hence, the approximate value of the 2nd cumulant, the variance, of $z(t)$ is found by

$$\lambda_2^z = \int_{-\infty}^{\infty}\mu_{z0}(j\omega)d\omega + \beta_3^2\int_{-\infty}^{\infty}\mu_{z1}(j\omega)d\omega \tag{76}$$

As for the 4th cumulant of $z(t)$,, we have from Eq. (62)

$$\begin{aligned}\mu_z(j\omega_1,\ j\omega_2,\ j\omega_3) = -\beta_3\Big\{&\mu_{z0}(j\omega_1+j\omega_2+j\omega_3)\mu_{z0}(j\omega_1)\mu_{z0}(j\omega_2)G_c(j\omega_3)\\ &+\mu_{z0}(j\omega_1+j\omega_2+j\omega_3)\mu_{z0}(j\omega_1)G_c(j\omega_2)\mu_{z0}(j\omega_3)\\ &+\mu_{z0}(j\omega_1+j\omega_2+j\omega_3)G_c(j\omega_2)\mu_{z0}(j\omega_2)\mu_{z0}(j\omega_3)\\ &+G_c(-j\omega_1-j\omega_2-j\omega_3)\mu_{z0}(j\omega_1)\mu_{z0}(j\omega_2)\mu_{z0}(j\omega_3)\Big\}\end{aligned} \tag{77}$$

Therefore, the value of the 4th cumulant of $z(t)$ can be obtained from Eq. (77) or Eq. (67) as

$$\begin{aligned}\lambda_4^z &= \iiint_{-\infty}^{\infty}\mu_z(j\omega_1,\ j\omega_2,\ j\omega_3)d\omega_1 d\omega_2 d\omega_3\\ &= -4\beta_3\int_{-\infty}^{\infty}W_c(\tau)\lambda_{z0}^{3}(\tau)d\tau\end{aligned} \tag{78}$$

But Eqs. (76) and (78) contain the unknown parameters, β_1 and β_3. From Eq. (57) β_1 and β_3 can be calculatated as the functions of λ_2^z and λ_4^z, that is,

$$\begin{aligned}\beta_1 &= \frac{1}{2\pi}\int_C (jw)F(jw)\exp\left\{-\frac{1}{2}\lambda_2^z w^2 + \frac{1}{4!}\lambda_4^z w^4\right\}dw\\ &\doteqdot \frac{1}{2\pi}\int_C (jw)F(jw)\left(1+\frac{1}{24}\lambda_4^z w^4\right)\exp\left(-\frac{1}{2}\lambda_2^z w^2\right)dw\end{aligned} \tag{79}$$

and

$$\beta_3 \doteqdot \frac{1}{2\pi}\int_C (jw)^4 F(jw)\left(1+\frac{1}{24}\lambda_4^z w^4\right)\exp\left(-\frac{1}{2}\lambda_2^z w^2\right)dw \tag{80}$$

Therefore we can find the required values of λ_2^z and λ_4^z from solution of the simultaneous equations (76), (78), (79) and (80).

7. Numerical Example

Let us consider a simple non-linear conrol system as shown in Fig. 3, where N. L. is a non-linear element of zero-memory type and the transfer characteristics are expressed by Eq. (38). We assume that the system input $v_0(t)$ is a white noise and is normally distributed. Fig. 3 is Equivalent to Fig. 4 where

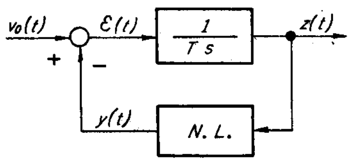

Fig. 3.

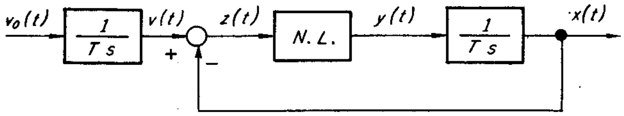

Fig. 4.

$$G(j\omega) = \frac{1}{jT\omega}$$

$$\mu_v(j\omega) = |G(j\omega)|^2 \frac{N^2}{2\pi}$$

We have from Eqs. (79) and (80)

$$\left.\begin{aligned} \beta_1 &= 1+\frac{\gamma}{2}\sigma_z \\ \beta_3 &= \gamma \end{aligned}\right\} \tag{81}$$

where σ_z is the variance of $z(t)$, that is $\sigma_z=\lambda_2^z$

We have from Eq. (63)

$$G_c(j\omega) = \frac{1}{jT\omega+\beta_1} \tag{82}$$

and

$$\left.\begin{aligned} W_c(t) &= \frac{1}{T}\exp\left(-\frac{\beta_1}{T}t\right) & t>0 \\ W_c(t) &= 0 & t<0 \end{aligned}\right\} \tag{83}$$

The first approximation of the spectral density of $z(t)$ is given by

$$\mu_{z0}(j\omega) = \frac{1}{2\pi}\frac{N^2}{T^2\omega^2+\beta_1^2} \tag{84}$$

In this case, as β_1 is equal to the equivalent gain of the non-linear element, Eq. (84) is coincident with the result obtained by the equivalent linear technique.

We have from Eq. (84)

$$\begin{aligned} \lambda_{z0}(\tau) &= \int_{-\infty}^{\infty} \mu_{z0}(j\omega)\exp(j\omega\tau)d\tau \\ &= \frac{N^2}{2\beta_1 T}\exp\left(-\frac{\beta_1}{T}|\tau|\right) \end{aligned} \tag{85}$$

and

$$\lambda_{z0}^{3}(\tau) = \frac{N^6}{8\beta_1^3 T^3} \exp\left(-\frac{3\beta_1}{T}|\tau|\right) \tag{86}$$

Eq. (74) becomes

$$\mu_{z0}^{(3)}(j\omega) = \frac{1}{2\pi} \frac{3N^6}{4\beta_1^2 T^2} \frac{1}{T^2\omega^2 + 9\beta_1^2} \tag{87}$$

As we have the reletion,

$$\left.\begin{aligned} W_c(\tau)\lambda_{z0}^{z}(\tau) &= \frac{N^4}{4\beta_1^2 T^3} \exp\left(-\frac{3\beta_1}{T}\tau\right) & \tau > 0 \\ &= 0 & \tau < 0 \end{aligned}\right\} \tag{88}$$

Eq. (75) becomes

$$B_{z0}(j\omega) = \frac{N^4}{4\beta_1^2 T^2} \frac{1}{j\omega T + 3\beta_1} \tag{89}$$

Therefore, we have

$$\begin{aligned} &\text{Real}\left[G_c(-j\omega) B_{z0}(j\omega)\right] \\ &= \frac{N^4}{4\beta_1^2 T^2} \frac{T^2\omega^2 + 3\beta_1^2}{(T^2\omega^2 + 9\beta_1^2)(T^2\omega^2 + \beta_1^2)} \end{aligned} \tag{90}$$

So that we have from Eq. (73)

$$\mu_{z1}(j\omega) = \frac{1}{2\pi} \frac{N^6}{8\beta_1^2 T^2} \frac{3T^2\omega^2 + 7\beta_1^2}{(T^2\omega^2 + \beta_1^2)(T^2\omega^2 + 9\beta_1^2)} \tag{91}$$

Hence the variance of $z(t)$ can be found from Eq. (76), that is,

$$\begin{aligned} \lambda_2^z &= \sigma_z \\ &= \frac{N^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{T^2\omega^2 + \beta_1^2} d\omega + \beta_3^2 \frac{N^6}{8\beta_1^2 T^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3T^2\omega^2 + 7\beta_1^2}{(T^2\omega^2 + \beta_1^2)(T^2\omega^2 + 9\beta_1^2)} d\omega \\ &= \frac{N^2}{2T\beta_1} + \beta_3^2 \frac{N^6}{12T^3\beta_1^5} \end{aligned} \tag{92}$$

And the 4th cumulant of $z(t)$ is given by Eq. (79) as

$$\begin{aligned} \lambda_4^z &= -4\beta_3 \frac{N^6}{\beta_1^3 T^4} \int_0^{\infty} \exp\left(-\frac{4\beta_1}{T}\tau\right) d\tau \\ &= -\frac{\beta_3 N^6}{8\beta_1^4 T^3} \end{aligned} \tag{93}$$

From Eqs. (81) and (92), the variance of $z(t)$ can be found graphically. For $N=1$, these numerical results are shown in Fig. 5. It is found from Eq. (92) that the variance of $z(t)$ is almost equal to the one obtained by the equivalent linear technique if γ is a small quantity, that is,

$$\lambda_2^z \doteqdot \frac{N^2}{2T\beta_1} = \sigma_{z0} \tag{94}$$

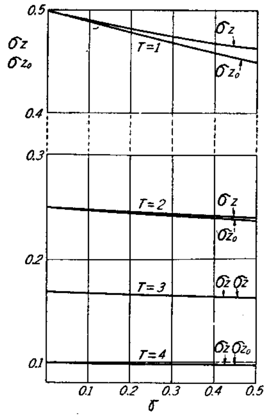

Fig. 5.

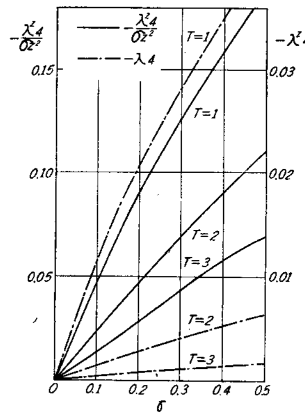

Fig. 6.

Fig. 6 shows the values of the 4th cumulant of $z(t)$ and the peakedness of the one-dimensional distribution of $z(t)$, λ_4^z/σ_z^2.

8. Conclusions

By assuming the cumulant function of the input to a non-linear element in a closed system, we obtain the expressions for the 2nd cumulant and the 4th cumulant of the input to the non-linear element. But these equations do not explicitly give quantitative results. Therefore, we considered a first-order system with a non-linear element characterized by a cubic curve as an example. The variance of the input to the non-linear element is expressed by a sum of the first approximation and the corrective term. The corrective term is of the order of γ^2, where γ is the coefficient of the cubic term of the non-linear element. The 4th cumulant is of the order of γ and the magnitude is inversely proportional to the time constant of the system. The peakedness, which is a measure of the discrepancy from the Gaussian distribution, is positive for $\gamma<0$ and is negative for $\gamma>0$.

Thus it is concluded that, from the viewpoint of engineering application, the equivalent linear technique may be employed to obtain a rapid evaluation of a system and to provide results for engineering purposes.

References

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