Sampled-data Control System Design Using Reverse Element

By

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In this paper, the authors deal with one method for the control of sampled-data control systems. In contrast with ordinary sampled-data control systems, the polarity of the control signal is reversed at several instants in every sampling period. By deciding the instants properly, the finite settling time response can be obtained. Especially, for a step input, it is shown that the system error can be reduced to zero in one sampling period, irrespective of the order of the controlled element. Furthermore, the compensator is simpler than that for ordinary sampled-data control systems, because it consists only of a sampler, a hold element, and a reverse element which reverses the polarity of the control signal. Even for a controlled element with a symmetric saturation characteristic, the settling time has a finite value. Moreover, it is shown that the reverse element can be used along with an ordinary compensator containing delay elements, and that the settling time can be made shorter. The response for a random input is also analyzed for a typical sampled-data control system with a reverse element.

I. Introduction

Sampled-data control systems with a compensator containing delay elements have been treated by many authors, and the mathematical techniques and the theory developed for them are now well known.

In this paper, the authors deal with sampled-data control systems of a different kind. In contrast with ordinary sampled-data systems, the polarity of the control signal is reversed at several instants in every sampling period. The instants and the number of reversals are determined by the sampling period and the characteristic of the controlled element. In the following discussion, a compensator which reverses the polarity of the control signal is called a reverse element.

II. System Design for a Second Order Controlled Element

(1) Design Procedure for a Step Input

In this section, a design procedure is discussed for the control system shown in Fig. 1. The polarity of the control signal is reversed when $t=T_1$ (measured

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from the beginning of every sampling period) as shown in Fig. 2.

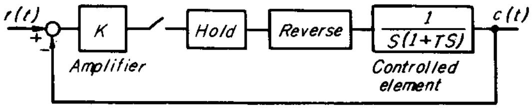


Fig. 1. Block diagram of a sampled-data control system with a second order controlled element.

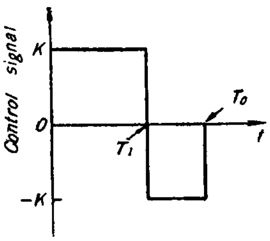


Fig. 2. Reversal of the control signal in a sampling period.

The instant T_1 is chosen so that the system error becomes identically zero after one sampling period.

The system output $c(t)$ for a step input is

$$c(t) = K\left\{g(t)-2g(t-T_1)\,H(t-T_1)\right\} \tag{1}$$

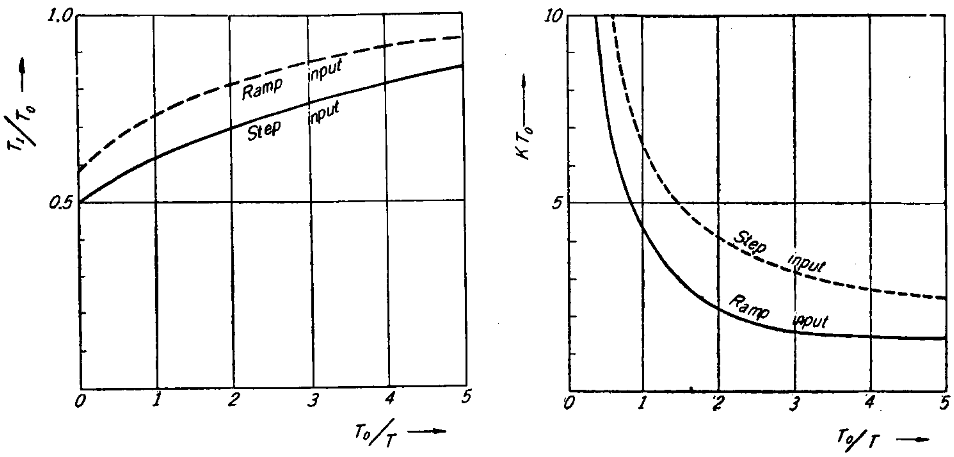
where $H(t)$ is a unit step function, and $g(t)$ is the indicial response of the controlled element, which is written as

$$g(t) = t-T(1-e^{-t/T}) \tag{2}$$

According to the design condition mentioned above, we obtain

$$\left.\begin{aligned} c'(T_0) &= 0 \\ c\,(T_0) &= 1 \end{aligned}\right\} \tag{3}$$

where T_0 is the sampling period. Substitution of Eq. (1) into Eq. (3) yields



(a) (b)

Fig. 3. Conditions for finite setting time response.

$$\left.\begin{aligned} e^{-am} &= 0.5(1+e^{-a}) \\ \frac{1}{KT_0} &= 1-2m \end{aligned}\right\} \quad (4)$$

where

$$m \equiv 1 - T_1/T_0 \quad (5)$$

$$a \equiv T_0/T \quad (6)$$

The relations given by Eq. (4) are plotted in Figs. 3(a) and 3(b) with solid lines. Figure 4 gives the step response of the system output $c(t)$ for $a=1$. The polarity of the control signal is reversed when $T_1/T_0=0.62$, and $KT_0=4.2$.

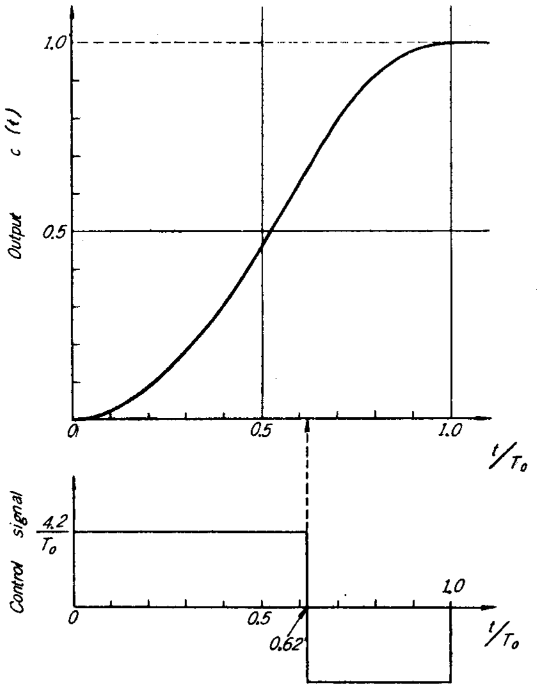


Fig. 4. Step response of the control system in Fig. 1. ($a=1.0$)

(2) Design Procedure for a Ramp Input

The discussion in the preceding section indicates that the simple control system shown in Fig. 1 has a finite settling time for a step input. But the system with a finite settling time for a ramp input should have a compensator containing delay elements.

Consider the control system shown in Fig. 5, and let the time darivative of the system input be $1/T_0$. The design conditions corresponding to Eq. (3) are given by

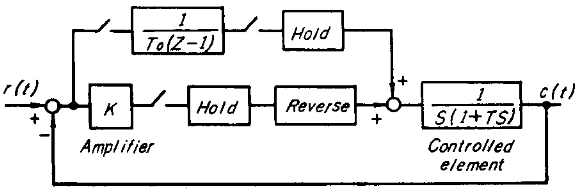


Fig. 5. Block diagram of a sampled-data control system which is designed for a ramp input.

$$\left.\begin{aligned} c'(T_0) &= 1/T_0 \\ c\,(T_0) &= 2 \end{aligned}\right\} \quad . \tag{7}$$

Substitution of Eq. (1) into Eq. (7) yields

$$\left.\begin{aligned} e^{-am} &= 0.5\left(1+e^{-a}+\frac{1-2m}{2+1/a}\right) \\ \frac{1}{KT_0} &= \frac{1-2m}{2+1/a} \end{aligned}\right\} \tag{8}$$

The relations given by Eq. (8) are plotted in Figs. 3(a) and 3(b) with dotted lines. Figure 6 gives the step response of the system output $c(t)$ for $a=1$. The polarity of the control signal is reversed when $T_1/T_0=0.73$ and $KT_0=6.6$.

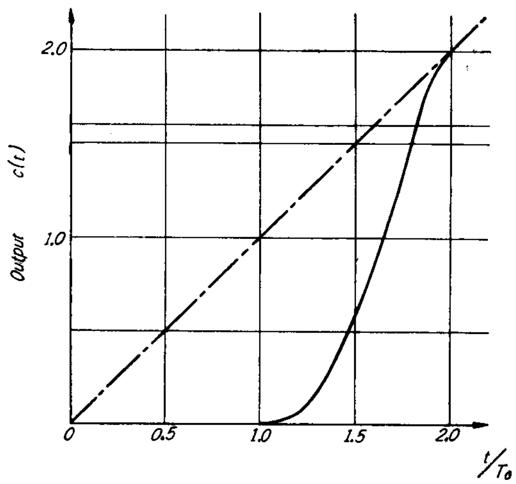


Fig. 6. Velocity response of the control system shown in Fig. 5. ($a=1.0$).

III. System Design for a Third Order Controlled Element

Suppose that the transfer function of the controlled element $G(s)$ is given as

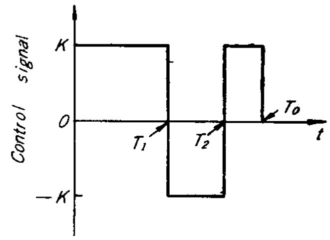


Fig. 7. Reversal of the control signal in a sampling period.

$$G(s) = \frac{1}{s(1+Ts)(1+T's)} \tag{9}$$

In this case, the polarity of the control signal is reversed twice in every sampling period as shown in Fig. 7. Similar to the system design discussed in the previous sections for a second order controlled element, the following conditions can be obtained.

$$\left.\begin{aligned} e^{-am_2}-e^{-am_1} &= 0.5(1-e^{-a}) \\ e^{-bm_2}-e^{-bm_1} &= 0.5(1-e^{-b}) \end{aligned}\right\} \tag{10}$$

$$\frac{1}{KT_0} = 1-2(m_1-m_2) \tag{11}$$

where

$$a \equiv T_0/T \tag{12}$$

$$b \equiv T_0/T' \tag{13}$$

$$m_1 \equiv 1-T_1/T_0 \tag{14}$$

$$m_2 \equiv 1-T_2/T_0 \tag{15}$$

The instants T_1 and T_2 are found from Eq. (10) by solving for m_1 and m_2, and using Eqs. (14) and (15). Figure 8 indicates m_1 and m_2 as a function of T_0/T and T_0/T'. This figure is composed of two regions separated by a straight

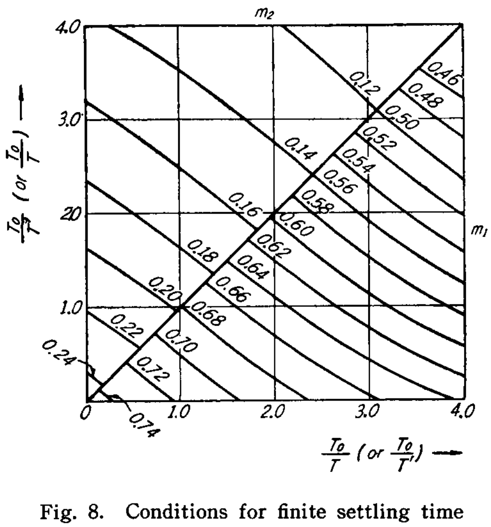


Fig. 8. Conditions for finite settling time response.

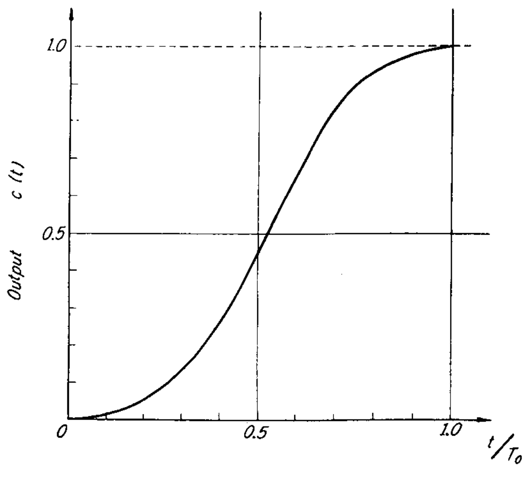


Fig. 9. Step response of a control system with a controlled element of third degree given by Eq. (9). (a=2.0, b=1.0).

line passing through the origin. The value of m_1 is shown in the right-hand region, and that of m_2 is shown in the upper region. Figure 9 gives the step response of the system output $c(t)$ for $a=2.0$ and $b=1.0$.

For a ramp input, a similar method can also be applied to design a control system with a finite settling time.

Now, it will be shown that the reverse element can be used along with an ordinary compensator containing delay elements. In this case the control element can be made simpler, and a shorter settling time can be obtained.

Consider the control system shown in Fig. 10, including both a reverse element and a compensator $D(z)$ containing delay elements. It is assumed that

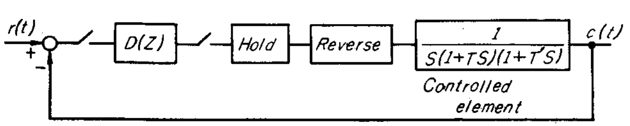


Fig. 10. Block diagram of a sampled-data control system when a reverse element is used along with a compensator $D(z)$ containing delay elements.

the polarity of the control signal is reversed when $t=T_1$ in every sampling period, and that the system error is reduced to zero for $t \geqq 2T_0$.

First, the function $\phi(t)$ is defined by

$$\phi(t) \equiv g(t)-2g(t-T_1)H(t-T_1)+g(t-T_0)H(t-T_0) \tag{16}$$

where $g(t)$ is the indicial response of the controlled element. The system output $c(t)$ is written as

$$c(t) = K\phi(t)+Kh\phi(t-T_0)H(t-T_0) \tag{17}$$

where K and h are unknown constants. Design conditions are given by

$$\left.\begin{aligned} c'(2T_0) &= 0 \\ c''(2T_0) &= 0 \\ c(2T_0) &= 1 \end{aligned}\right\} \tag{18}$$

Substituting Eq. (17) into Eq. (18) yields

$$\left.\begin{aligned} \phi'(2T_0)\phi''(T_0) &= \phi''(2T_0)\phi'(T_0) \\ D(z) &= K\frac{1-\phi'(2T_0)z^{-1}/\phi'(T_0)}{1+\{1-\phi(T_0)\}z^{-1}} \\ \frac{1}{KT_0} &= \phi(2T_0)-\phi(T_0)\phi'(2T_0)/\phi'(T_0) \end{aligned}\right\} \tag{19}$$

Figure 11 gives the step response of the system output $c(t)$ for $a=2.0$ and $b=1.0$, and shows that two solutions exist.

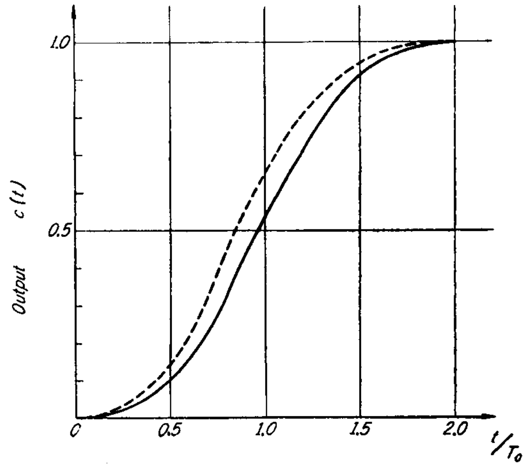


Fig. 11. Step response of the control system shown in Fig. 10. ($a=2.0$, $b=1.0$).

For the solution shown with a solid line,

$$\left.\begin{aligned} T_1 &= 0.72 \\ KT_0 &= 3.65 \\ D(z) &= \frac{3.65}{T_0}\cdot\frac{1-0.368z^{-1}}{1+0.432z^{-1}} \end{aligned}\right\} \tag{20}$$

For the solution shown with a dotted line,

$$\left.\begin{aligned} T_1 &= 0.62 \\ KT_0 &= 4.80 \\ D(z) &= \frac{4.80}{T_0}\cdot\frac{1-0.138z^{-1}}{1+0.325z^{-1}} \end{aligned}\right\} \tag{21}$$

IV. Extension to a General Order Controlled Element

Assume the controlled element $G(s)$ given as

$$\begin{aligned} G(s) &= \frac{1}{s(1+\tau_1 s)\cdots(1+\tau_{n-1}s)} \\ &\equiv \frac{1}{s}+\sum_{i=1}^{n-1}\frac{A_i T_0}{1+\tau_i s} \end{aligned} \tag{22}$$

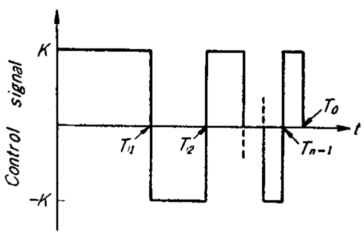


Fig. 12. Reversal of the control signal in a sampling period.

The polarity of the control signal is reversed as shown in Fig. 12. The design conditions are written for a step input as

$$\left.\begin{aligned} & e^{-a_k m_{n-1}} - e^{-a_k m_{n-2}} + \cdots + (-1)^{n-2} e^{-a_k m_1} \\ & \qquad = 0.5\left\{1+(-1)^n e^{-a_k}\right\} \qquad (k = 1, 2, \cdots, n-1) \\ & \frac{1}{KT_0} = 1-2m_1+2m_2-\cdots+2(-1)^{n-1} m_{n-1} \end{aligned}\right\} \tag{23}$$

where

$$a_i \equiv T_0/\tau_i \qquad (i = 1, 2, \cdots, n-1) \tag{24}$$

$$m_i \equiv 1-T_i/T_0 \qquad (i = 1, 2, \cdots, n-1) \tag{25}$$

Similar conditions can also be obtained for a ramp input.

V. The Effect of Saturation of the Controlled Element

It was indicated in previous sections that the system error can be reduced to zero in one sampling period for either a step or ramp input. For a controlled element with a symmetric saturation characteristic, it is clear that the settling

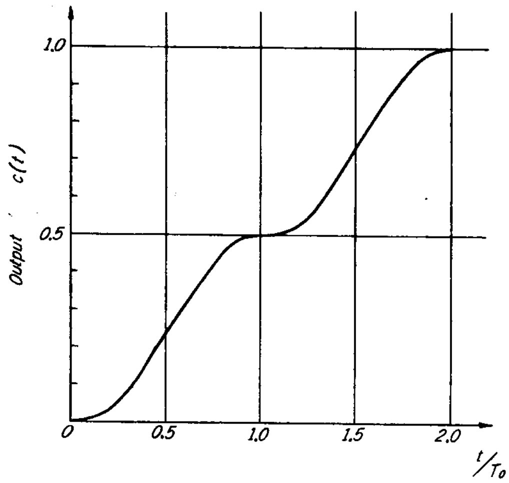


Fig. 13. Step response of a control system with the controlled element given by Eq. (26) when the response speed is limited with saturation.

time becomes longer with an increase in the input level. But when the working point of the controlled element enters a linear region, the system error is reduced to zero in one sampling period. Figure 13 shows the step response of the system output $c(t)$ for a controlled element $G(s)$ given by

$$G(s) = \frac{1}{s(s+1)} \tag{26}$$

VI. Response for a Random Input

In this section, the response for a random input is analysed for the control system shown in Fig. 1. The variance σ_e^2 of the system error at sampling instants is given by[1]

$$\sigma_e^2 = \frac{1}{2\pi j} \oint_{\text{unit circle}} \left| \frac{1}{1+G(z)} \right|^2 \Phi_{rr}(z) \frac{dz}{z} \tag{27}$$

where $G(z)$ is the open loop pulse transfer function, and $\Phi_{rr}(z)$ is the autocorrelation series of the system input.

The polarity of the control signal is reversed when $t=T_1$. Thus $G(z)$ is represented as

$$G(z) = Z\left[\left\{\frac{(1-e^{-T_1 s})}{s} - \frac{(e^{-T_1 s}-e^{-T_0 s})}{s}\right\} \frac{K}{s(1+Ts)}\right] \tag{28}$$

Using tables of the z-transform and the modified z-transform, we obtain $G(z)$ as

$$G(z) = \frac{KT}{(z-1)(z-e^{-a})}\left\{A(m)z+B(m)\right\} \tag{29}$$

where

$$m \equiv 1-T_1/T_0 \tag{30}$$

$$a \equiv T_0/T \tag{31}$$

and

$$\left.\begin{aligned} A(m) &\equiv e^{-a}-2e^{-ma}+(a-2ma+1) \\ B(m) &\equiv 2e^{-ma}-1-e^{-a}(a-2ma+1) \end{aligned}\right\} \tag{32}$$

therefore,

$$\frac{1}{1+G(z)} = \frac{(z-e^{-a})(z-1)}{z^2+C(m)z+D(m)} \tag{33}$$

where

$$\left.\begin{aligned} C(m) &\equiv KTA(m)-(1+e^{-a}) \\ D(m) &\equiv KTB(m)+e^{-a} \end{aligned}\right\} \tag{34}$$

Assume that $\Phi_{rr}(z)$ is given by

$$\Phi_{rr}(z) = \frac{(1-q^2)z}{(z-q)(1-qz)} \tag{35}$$

Substituting Eqs. (33) and (35) into Eq. (27) yields

$$\sigma_e^2 = 4(1-q^2)\cdot\frac{c_1^2 d_3 + c_2^2 d_1}{d_3(d_1 d_2 - d_0 d_3)} \tag{36}$$

where

$$
\left.\begin{aligned}
c_1 &= 1-e^{-a}\\
c_2 &= 1+e^{-a}\\
d_0 &= (1-q)\left\{1+C(m)+D(m)\right\}\\
d_1 &= (3-q)+(1+q)C(m)-(1-3q)D(m)\\
d_2 &= (3+q)-(1-q)C(m)-(1+3q)D(m)\\
d_3 &= (1+q)\left\{1-C(m)+D(m)\right\}
\end{aligned}\right\} \tag{37}
$$

Figure 14 shows the variance σ_e^2 of the system error as a function of KT_0 and m when $a=1$ and $q=0.9$. If the control system discussed above is controlled with

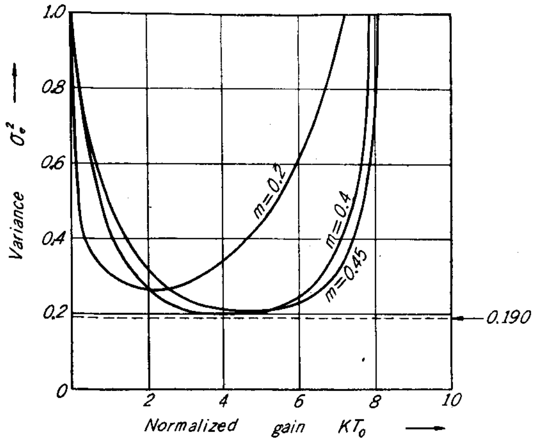


Fig. 14. Variance of the system error as a function of KT_0 and m. ($a=1.0$).

an ordinary compensator containing delay elements, the optimum over-all pulse transfer function $K(z)$ is given by[1)]

$$K(z) = qz^{-1} \tag{38}$$

Therefore, the variance of the system error is equal to $(1-q^2)$ ($=0.19$ when $q=0.9$), which is shown with a dotted line in Fig. 14. Inspection shows that the optimum values of KT_0 and m are approximately 4 and 0.4 respectively which agree well with the results obtained in section II.

Reference

1) S. S. L. Chang: Trans. AIEE, **76**, pt. III. 702 (1957) (Jan. 1958 section).