An Approximate Method for Calculating Turbulent Boundary Layer in Incompressible Fluids

By

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In this paper, the problem of the turbulent boundary layer with pressure gradient is treated. While the velocity profile in the boundary layer is influenced by the pressure gradient, the temperature profile is not greatly influenced by the pressure gradient, and hence a fictitious or virtual convective heat transfer is considered and an auxiliary equation which determines the velocity profile is obtained. The results of this method of calculation are compared with the experimental results.

1. Introduction

The turbulent boundary layer was first treated by v. Kármán and Prandtl and later approximate methods of calculation were developed by Buri and Gruschwitz for the case accompanied with pressure gradient. Almost every approximate method applies the momentum integral equation, but to obtain the change in the velocity profile, another parameter or auxiliary equation is needed to determine the shape of the velocity profile, and Gruschwitz[1], v. Doenhoff-Tetervin[2], and others have proposed empirical equations for the velocity shape parameter. The author has also published an equation in a previous paper[3].

In this paper, however, a virtual convective heat transfer in the boundary layer is introduced to obtain an auxiliary equation.

Let x and y be rectangular coordinates, u and v be the velocity components along x and y respectively, p be the pressure, ρ be the density of fluid, c be the specific heat of fluid, T be the temperature of fluid, τ be the shearing stress and q be the heat flow in the y-direction per unit area per unit time. Then the equation of motion of the boundary layer is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cdot\frac{dp}{dx}+\frac{1}{\rho}\cdot\frac{\partial\tau}{\partial y}$$

and the equation of heat transfer is

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$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=-\frac{1}{\rho gc}\cdot\frac{\partial q}{\partial y}$$

The equation of motion contains a pressure gradient term, but the equation of heat transfer does not, and while the velocity outside the boundary layer varies with the pressure gradient, the difference of temperature between the main stream and the wall surface does not change in the x-direction if the temperature of the wall surface is constant. So the problem of convective heat transfer is similar to the problem of flow along a flat plate in uniform flow. That is why virtual heat transfer is considered and, how an auxiliary equation which determines the shape of the velocity profile is obtained.

2. Velocity Distribution in the Boundary Layer

The velocity distribution in the boundary layer is assumed to be as follows,

$$\frac{u}{u_0}=1+\frac{\zeta}{\kappa}\log\eta-a\left(1-\frac{4}{3}\eta+\frac{1}{3}\eta^4\right) \tag{1}$$

where u is the velocity at distance y from the wall surface, u_0 is the velocity outside the boundary layer, a is a parameter, $\kappa=0.4$, δ is the thickness of the boundary layer and $\eta=y/\delta$. The shearing stress τ_0 acting along the surface is expressed by $\tau_0=c_f\rho u_0^2/2$ or $\tau_0=\zeta^2\rho u_0^2$, where c_f is the coefficient of local skin friction and $\zeta=\sqrt{c_f/2}$. So the friction velocity becomes $u^*=\zeta u_0$.

In the neighbourhood of the wall surface, it is well-known that $u/u^*=(1/\kappa)\log\eta+\text{const.}$, and it can be shown theoretically that the velocity distribution contains a term proportional to η when a pressure gradient exists. The term of η^4 is rather arbitrary and is so determined that the velocity profile of eq. (1) is similar to the velocity profile described in the previous paper[4].

In the case of an incompressible fluid the displacement thickness δ^* and the momentum thickness θ become as follows,

$$\frac{\delta^*}{\delta}=\frac{5}{2}\zeta+\frac{2}{5}a \tag{2}$$

$$\frac{\theta}{\delta}=\frac{5}{2}\zeta+\frac{2}{5}a-\frac{25}{2}\zeta^2-\frac{51}{15}a\zeta-\frac{104}{405}a^2 \tag{3}$$

Now, only the case of a smooth surface will be treated in this paper and for a smooth surface the following well-known relation holds in the neighbourhood of the wall surface even when the flow is accompanied by a pressure gradient, viz.,

$$\frac{u}{u^*}=5.5+\frac{1}{\kappa}\log\eta+\frac{1}{\kappa}\log\frac{u^*\delta}{\nu}$$

where ν is the coefficient of kinematic viscosity.

From eq. (1)

$$\frac{u}{u^*} = \frac{1}{\zeta} + \frac{1}{\kappa}\log\eta - \frac{a}{\zeta} + \cdots$$

and by comparing the above two equations,

$$\log\frac{u^*\delta}{\nu} = \kappa\left(\frac{1-a}{\zeta} - 5.5\right) \tag{4}$$

hence

$$\frac{u_0\theta}{\nu} = \frac{1}{\zeta}\left(\frac{5}{2}\zeta + \frac{2}{5}a - \frac{25}{2}\zeta^2 - \frac{51}{15}a\zeta - \frac{104}{405}a^2\right)\exp\left(\frac{0.4(1-a)}{\zeta} - 2.2\right) \tag{5}$$

Equation (5) gives the relation between Reynolds number $R_\theta = u_0\theta/\nu$ and the coefficient of local skin friction and is shown in Fig. 1.

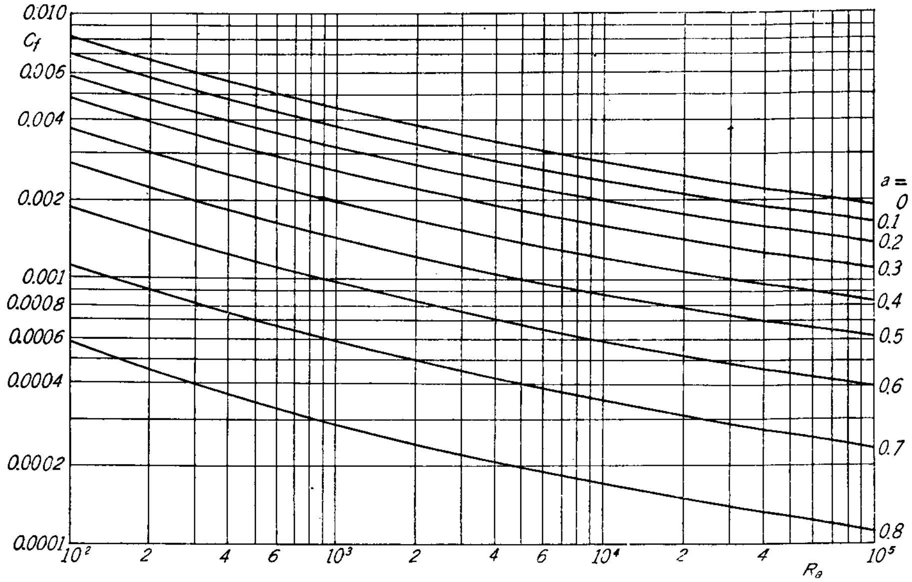

Fig. 1. Coefficient of local skin friction of smooth surface.

The coefficient of local skin friction when $a=0$, denoted hereafter by c_{f_0}, can be expressed approximately by the following relations,

$$\left.\begin{aligned} c_{f_0} &= 0.0172\,R_\theta^{-1/5}, \quad 10^2 < R_\theta < 10^4 \\ c_{f_0} &= 0.0102\,R_\theta^{-1/7}, \quad 10^4 < R_\theta < 10^6 \end{aligned}\right\} \tag{6}$$

and the ratio c_f/c_{f_0} at the same Reynolds number can be expressed approximately by the following relation.

$$\frac{c_f}{c_{f_0}} = 1 - 1.38\,a + 0.527\,a^5, \quad 0 < a < 0.8 \tag{7}$$

3. Heat Transfer by Convection and Auxiliary Equation

Let the temperature of the fluid outside the boundary layer be T_0, the temperature at y be T, the temperature of the wall surface be T_w, the friction temperature be T^*, the thermal conductivity of the fluid be λ, the specific heat of the fluid be c, the coefficient of heat transfer be α and the flow of heat from unit area of the wall surface per unit time be q_w. Then by the definitions of α and T^*,

$$q_w = \alpha(T_w - T_0) \tag{8}$$

$$q_w = \rho g c u^* T^* \tag{9}$$

Assume that the Prandtl number of the fluid i.e. $\nu\rho gc/\lambda$ is unity and let the Nusselt number be $N_\theta = \alpha\theta/\lambda$, then from eqs. (8) and (9),

$$\frac{T_w - T_0}{T^*} = \frac{\zeta R_\theta}{N_\theta} \tag{10}$$

and

$$q_w = \lambda(T_w - T_0)\frac{N_\theta}{\theta} \tag{11}$$

The heat-flow equation of the boundary layer is as follows,

$$\frac{d}{dx}\int_0^\delta \rho g c u(T - T_0)\,dy = q_w \tag{12}$$

So let

$$\frac{\theta_T}{\delta} = \int_0^1 \frac{u}{u_0}\cdot\frac{T - T_0}{T^*}\,d\eta \tag{13}$$

then

$$\int_0^\delta \rho g c u(T - T_0)\,dy = \lambda(T_w - T_0)\frac{N_\theta \theta_T}{\zeta\theta}$$

and by eq. (11), eq. (12) becomes as follows,

$$\frac{d}{dx}\left(\frac{N_\theta \theta_T}{\zeta\theta}\right) = \frac{N_\theta}{\theta} \tag{14}$$

This is the auxiliary equation of the present method.

Strictly speaking, the temperature profile in the boundary layer is affected by the pressure gradient and the temperature profile becomes flatter as the adverse pressure gradient becomes larger, but the effect is not so large as in the case of the velocity profile, and so in this approximate calculation it is assumed that

$$\frac{T - T_0}{T^*} = -\frac{1}{\kappa}\log\eta \tag{15}$$

Then from eqs. (13) and (15)

$$\frac{\theta_T}{\delta} = 2.5-1.7\,a-12.5\,\zeta \tag{16}$$

In the neighbourhood of the wall surface, it is known that

$$\frac{T_w-T}{T^*} = 5.5+\frac{1}{\kappa}\log\frac{u^*y}{\nu}$$

when the Prandtl number is unity, and hence from eqs. (4) and (15)

$$\frac{T_w-T_0}{T^*} = \frac{1-a}{\zeta} \tag{17}$$

From eqs. (10) and (17),

$$N_\theta = \frac{\zeta^2 R_\theta}{1-a} = \frac{c_f R_\theta}{2(1-a)} \tag{18}$$

Let

$$k = \frac{c_f}{c_{f_0}(1-a)} \tag{19}$$

and

$$A = \frac{k\theta_T}{\zeta\theta} \tag{20}$$

then using relations of eqs. (17), (19) and (20), eq. (14) becomes as follows.

$$\frac{d}{dx}\left(c_{f_0}R_\theta A\right) = \frac{kc_{f_0}R_\theta}{\theta} \tag{21}$$

In the above equation, A is a function of a and R_θ as shown in Fig. 2 and by eq. (7)

$$k = \frac{1-1.38\,a+0.527\,a^5}{1-a}$$

in the case of the velocity profile assumed in paragraph 2. Or if eq. (21) is integrated between x_1 and x_2, then

$$\left(c_{f_0}R_\theta A\right)_{x=x_2} = \left(c_{f_0}R_\theta A\right)_{x=x_1}+\int_{x_1}^{x_2}\frac{kc_{f_0}R_\theta}{\theta}\,dx \tag{22}$$

When $u_0=$const., the value of a is zero, so $k=1$, $\theta=\delta\zeta(2.5-12.5\,\zeta)$, $\theta_T=\delta(2.5-12.5\,\zeta)$ and $\zeta=\sqrt{c_{f_0}/2}$, hence eq. (21) becomes as follows,

$$\frac{d\theta}{dx} = \frac{c_{f_0}}{2}$$

which coincides with the momentum integral equation in this case.

4. Practical Method of Calculation

Integrating the momentum integral equation approximately,

$$\left.\begin{aligned}\left(\frac{u_0\theta}{\nu}\right)^{1/5}\theta &= \frac{1}{u_0^4}\left\{0.01032\int_0^x u_0^4 dx + \text{const.}\right\}, \quad R_\theta < 10^4\\ \left(\frac{u_0\theta}{\nu}\right)^{1/7}\theta &= \frac{1}{u_0^4}\left\{0.00583\int_0^x u_0^4 dx + \text{const.}\right\}, \quad R_\theta > 10^4\end{aligned}\right\} \tag{23}$$

Using these equations θ, R_θ and c_{f_0} can be easily calculated.

Equation (21) can be transformed into the following equations if the approximate eqs. (6) are applied, viz.,

$$\left.\begin{aligned}\frac{d}{dx}(R_\theta^{4/5}A) &= \frac{kR_\theta^{4/5}}{\theta}, \quad R_\theta < 10^4\\ \frac{d}{dx}(R_\theta^{6/7}A) &= \frac{kR_\theta^{6/7}}{\theta}, \quad R_\theta > 10^4\end{aligned}\right\} \tag{24}$$

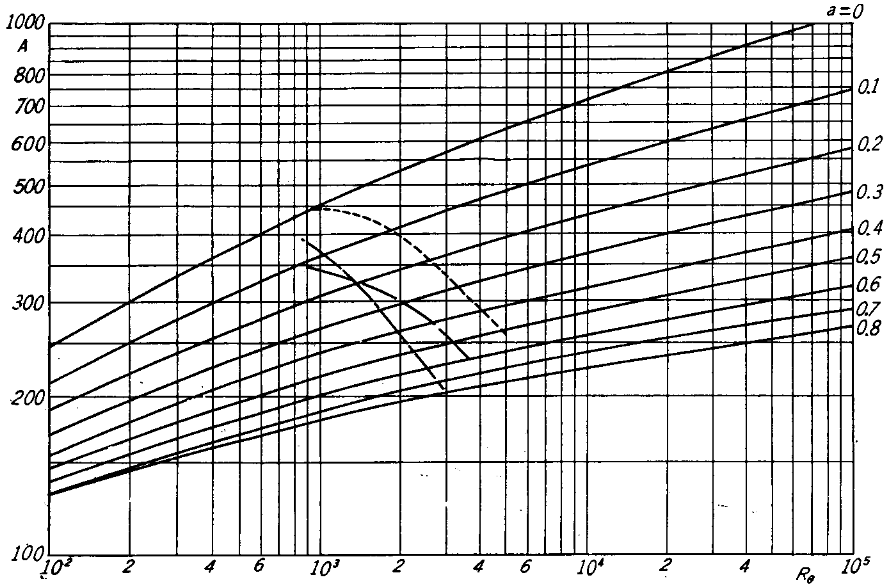

Fig. 2. Relation between auxiliary function A and Reynolds number R_θ.

If, at some point of x say x_1, a is known or assumed then the value of A at this point is determined by eq. (20) or Fig. 2. Therefore if the value of k is assumed properly, then the value of A at any other point say x_2 can be found from eq. (21) or (24) since the values of all quantities in the equation except the value of A at x_2 are known. Hence the values of a and c_f at x_2 are determined.

5. Numerical Examples

Figure 3 shows an example of the change of velocity profile due to the pressure gradient. The flow is assumed to be such that $u_0 = U_0 + U_1 x/l$ where U_0 is the value of u_0 at $x=0$, and it is also assumed that $R_\theta = 10^3$ at $x=0$ and that

$R=U_0l/\nu=10^5$. This figure shows the relation between the value of a at $x=l$ and U_1/U_0. The three cases of $a=0.1$, 0.3 and 0.5 at $x=0$ are shown. The dotted line shows the value of U_1/U_0 which gives no change in velocity profile between $x=0$ and l.

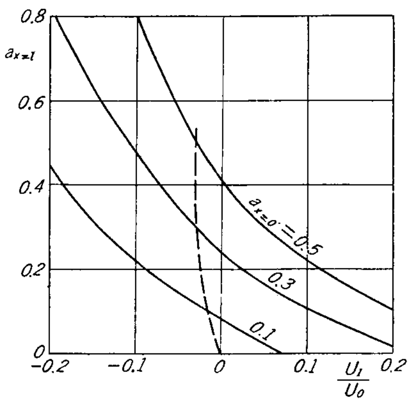

Fig. 3. Change of velocity profile parameter a when $u_0=U_0+U_1x/l$.

Figures 4, 5 and 6 show the results of calculations compared with the experimental results of Gruschwitz[5]. Values of θ and a at the initial point are assumed so that the velocity profile fits well with the experimental results, and then the velocity profiles on the downstream side are calculated. The lines show the velocity profiles calculated by this method. In Fig. 2 the loci of the function A in these three examples are shown, the broken line corresponds to the case shown in Fig. 4, the chain line to Fig. 5 and the dotted line to Fig. 6. The calculated results coincide comparatively well with the experimental results.

As to the separation of the boundary layer, it seems that the value of a or $H=\delta^*/\theta$ at the separation point depends on R_θ, and the condition of separation can not be determined by the present theory. Hence, in this calculation, it is assumed that the separation takes place when a becomes larger than about 0.8. In the present examples, separation has already occurred at $x=56.72$ cm in the

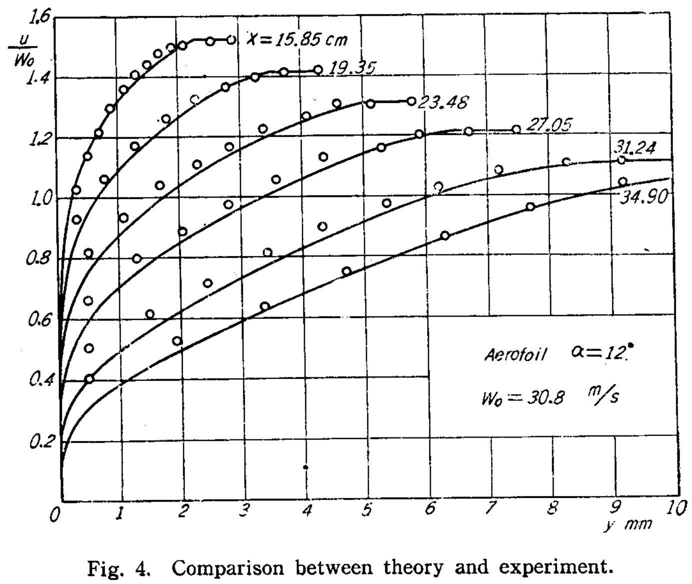

Fig. 4. Comparison between theory and experiment.

case shown in Fig. 5.

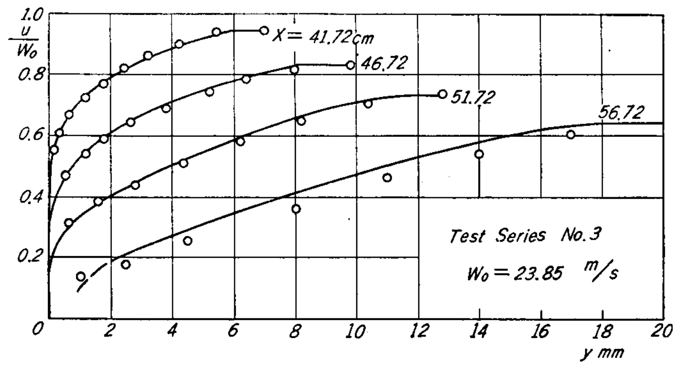

Fig. 5. Comparison between theory and experiment.

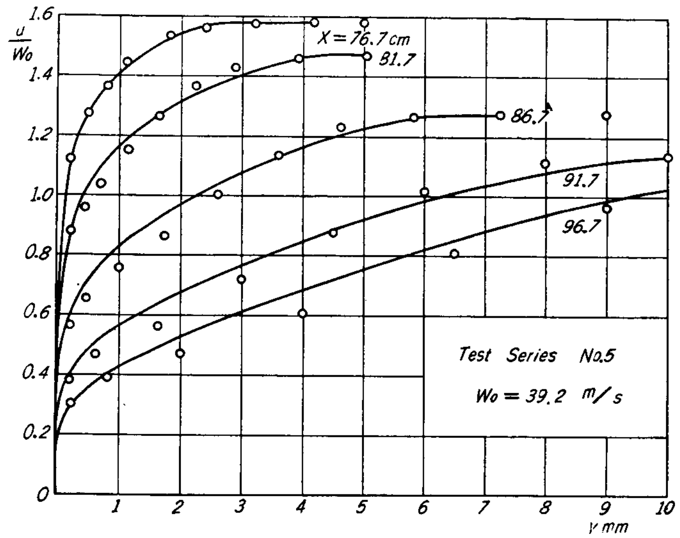

Fig. 6. Comparison between theory and experiment.

6. Conclusion

In this method of calculation, an auxiliary equation which determines the velocity profile is obtained. From this equation, an equation for shape factor H can be obtained but has a complicated form which requires numerical calculation. The auxiliary equation contains a number k which is a function of a, and this makes numerical calculation somewhat cumbersome. It is hoped to find a function A which will make k constant and also that it will be possible to express this function in a simple analytical form.

References

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