Uniqueness Criterion of Deformation of Voigt Type Viscoelastic Body

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The paper deals with the criterion of uniqueness between the stress state and the deformation of the *Voigt* type viscoelastic body, of which stress-strain relations are the *Lagrang*ian form. If the deformed state is given, the forces acting in the body and on its surface are uniquely specified. Conversely, the uniqueness criterion of the infinitesimal displacements for the given infinitesimal variations of body force, surface force and surface displacement, is that a function of the elastic potential and the dissipation function is positive definite.

1. Introduction. The *Lagrang*ian form of the stress-strain relations of the *Voigt* type body was formulated by *Biot*[1] and the author[2]. In the author's paper, generalized *Hamilton*'s principle was applied to the equation of energy conservation, which denotes that the sum of the elastic and the dissipative energy of the body is equal to the work done by the body and the surface force on it. On the condition that the elastic potential is a function only of strain, and the dissipation function is a function of rate of strain and of strain, it was reduced necessarily that such a body is the *Voigt* type one.

The uniqueness between the stress state and the deformation of the hyperelastic media was discussed by *Prager*[3]. He investigated the uniqueness of the infinitesimal displacements with respect to the infinitesimal variations of body force, surface force and surface displacement.

In this paper we will discuss the uniqueness of the stress-strain correspondence of the *Voigt* type body according to *Prager.*

2. General Stress-Strain Relations. A *Voigt* type viscoelastic body is in homogeneous undeformed initial state at time $t=0$, which is bounded by the simply connected finite region V with the sufficiently regular surface S. The body is subjected to the body force $\boldsymbol{F}$ in V and its boundary S is subjected to the surface

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force $\boldsymbol{f}$ on the portion S_f of S and to the displacement $\boldsymbol{u}$ on the portion S_u of S; then it deforms viscoelastically to time t.

The body in the deformed state is in a state of equilibrium. If the elastic potential Φ as a function of $x^{*i}_{,j}$ and the dissipation function Ψ as a function of $x^{*i}_{,j}$ and $\dot{x}^{*i}_{,j}$ are given, the stress is given in the following[2]

$$\tau^{ij} = \frac{\partial \Phi}{\partial \varepsilon_{ij}} + \frac{\partial \Psi}{\partial \dot{\varepsilon}_{ij}}, \qquad (i, j = 1, 2, 3), \tag{1}$$

where $x^{*i} = x^i + u^i(x^1, x^2, x^3; t)$ is the point in the deformed state which is at a point x^i in the undeformed state; the *Lagrang*ian strain component ε_{ij} is defined by

$$\varepsilon_{ij} = \frac{1}{2}(x^{*k}_{,i}\, x^{*k}_{,j} - \delta_{ij}); \tag{2}$$

and τ^{ij} is the *Lagrang*ian stress component.

By means of the definition of the generalized strain and the generalized rate of strain, we have

$$\frac{\partial \Phi}{\partial (x^{*i}_{,j})} = \frac{\partial \Phi}{\partial \varepsilon_{pj}} x^{*i}_{,p} \quad \text{and} \quad \frac{\partial \Psi}{\partial (\dot{x}^{*i}_{,j})} = \frac{\partial \Psi}{\partial \dot{\varepsilon}_{pj}} x^{*i}_{,p}; \tag{3}$$

then we can express the stress-strain relations (1) in the following

$$T^{ij} = \frac{\partial \Phi}{\partial (x^{*j}_{,i})} + \frac{\partial \Psi}{\partial (\dot{x}^{*j}_{,i})}, \tag{4}$$

where

$$T^{ij} = x^{*j}_{,p} \tau^{ip} \quad \text{or} \quad \tau^{ij} = x^{j}_{,*p} T^{ip}. \tag{5}$$

(*Prager*[3] designates τ^{ij} as the *Kirchhoff*ian stress and T^{ij} as the *Lagrang*ian stress.)

The viscoelastic conditions

$$\frac{d}{dt}\left(\frac{\partial \Psi}{\partial \dot{\varepsilon}_{ij}}\right) - \frac{\partial \Psi}{\partial \varepsilon_{ij}} = 0, \qquad (i, j = 1, 2, 3) \tag{6}$$

hold for the dissipation function[2].

3. Uniqueness Criterion. The equilibrium equations are expressed as

$$T^{ij}_{,i} + F^j = 0, \qquad (j = 1, 2, 3), \tag{7}$$

where F^j $(j=1, 2, 3)$ is the body force per unit volume before deformation.

A surface element, which has the area dS and the exterior normal λ_i in the undeformed state, is subjected to the force

$$\lambda_i T^{ij} dS = \lambda_i \left(\frac{\partial \Phi}{\partial (x^{*j}_{,i})} + \frac{\partial \Psi}{\partial (\dot{x}^{*j}_{,i})}\right) dS \tag{8}$$

in the deformed state by (4) and a volume element, which has the volume dV in the undeformed state, is subjected to the force

$$F^j dV = -\left(\frac{\partial \Phi}{\partial(x^{*j}_{,i})} + \frac{\partial \Psi}{\partial(\dot{x}^{*j}_{,i})}\right)_{,i} dV \tag{9}$$

in the deformed state by (7) and (4).
Therefore, when the deformed state $(x^{*i}_{,j})$ is given, then the forces acting on the surface and in the body are uniquely determined.

Conversely, we will find the condition of the uniqueness of the equilibrium position x^{*i} for given body force and given boundary conditions.

When the equilibrium position x^{*i} is given with the specified body force $\boldsymbol{F}$, surface force $\boldsymbol{f}$ and surface displacement $\boldsymbol{u}$, the uniqueness of the infinitesimal displacements δx^{*i} will be discussed for the given infinitesimal variation δF^i of the specified body force and for the given infinitesimal variations δf^i and δx^{*i} of the specified surface force and surface displacement.

The displacement variations $\delta\bar{x}^{*i}$ and $\delta\bar{\bar{x}}^{*i}$ and the stress variations $\delta\bar{T}^{ij}$ and $\delta\bar{\bar{T}}^{ij}$ are assumed in this boundary value problem. By (4)

$$\delta\bar{T}^{ij} = \frac{\partial^2 \Phi}{\partial(x^{*j}_{,i})\,\partial(x^{*l}_{,k})}(\delta\bar{x}^{*l})_{,k} + \frac{\partial^2 \Psi}{\partial(\dot{x}^{*j}_{,i})\,\partial(x^{*l}_{,k})}(\delta\bar{x}^{*l})_{,k} + \frac{\partial^2 \Psi}{\partial(\dot{x}^{*j}_{,i})\,\partial(\dot{x}^{*l}_{,k})}(\delta\dot{\bar{x}}^{*l})_{,k} \tag{10}$$

and the corresponding formula holds for $\delta\bar{\bar{T}}^{ij}$; in both formulae the second differentiations of the elastic potential and the dissipation function have same values, for they are composed of the given equilibrium position x^{*i}. The stress variations $\delta\bar{T}^{ij}$ and $\delta\bar{\bar{T}}^{ij}$ satisfy

$$\left.\begin{aligned} (\delta\bar{T}^{ij})_{,i} + \delta F^j &= 0 \\ \text{and} \qquad (\delta\bar{\bar{T}}^{ij})_{,i} + \delta F^j &= 0 \end{aligned}\right\} \tag{11}$$

by (7), thus

$$(\delta\bar{T}^{ij} - \delta\bar{\bar{T}}^{ij})_{,i} = 0 \tag{12}$$

holds.

On the portion S_f of S

$$\lambda_i(\delta\bar{T}^{ij} - \delta\bar{\bar{T}}^{ij}) = 0$$

and on the portion S_u of S

$$\delta\bar{x}^{*i} - \delta\bar{\bar{x}}^{*i} = 0 \tag{13}$$

hold by means of the given boundary conditions. We have

$$\lambda_i(\delta\bar{T}^{ij} - \delta\bar{\bar{T}}^{ij})(\delta\bar{x}^{*j} - \delta\bar{\bar{x}}^{*j}) = 0 \tag{14}$$

on the entire surface S at any time t' $(0\leqq t'\leqq t)$. Therefore we have

$$\int_0^t\left[\iint_S \lambda_i(\delta\bar{T}^{ij}-\delta\bar{\bar{T}}^{ij})(\delta\bar{x}^{*j}-\delta\bar{\bar{x}}^{*j})\,dS\right]_{t'}dt' = 0\,. \tag{15}$$

By means of *Gauss*' theorem and (12), we get

$$\int_0^t\left[\iiint_V (\delta\bar{T}^{ij}-\delta\bar{\bar{T}}^{ij})(\delta\bar{x}^{*j}-\delta\bar{\bar{x}}^{*j})_{,i}\,dV\right]_{t'}dt' = 0\,. \tag{16}$$

Equation (16) leads to

$$\int_0^t\left[\iiint_V\left[\frac{\partial^2\Phi}{\partial(x^{*j}_{,i})\,\partial(x^{*l}_{,k})}+\frac{\partial^2\Psi}{\partial(\dot{x}^{*j}_{,i})\,\partial(x^{*l}_{,k})}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2\Psi}{\partial(\dot{x}^{*j}_{,i})\,\partial(\dot{x}^{*l}_{,k})}\right)\right](\delta\bar{x}^{*j}-\delta\bar{\bar{x}}^{*j})_{,i}(\delta\bar{x}^{*l}-\delta\bar{\bar{x}}^{*l})_{,k}\,dV\right]_{t'}dt' = 0 \tag{17}$$

by (10) and the integration by parts with respect to time.

When the elastic potential Φ and the dissipation function Ψ have the property that the quadratic form

$$\left[\frac{\partial^2\Phi}{\partial(x^{*j}_{,i})\,\partial(x^{*l}_{,k})}+\frac{\partial^2\Psi}{\partial(\dot{x}^{*j}_{,i})\,\partial(x^{*l}_{,k})}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2\Psi}{\partial(\dot{x}^{*j}_{,i})\,\partial(\dot{x}^{*l}_{,k})}\right)\right]\eta^{ij}\eta^{kl} \tag{18}$$

is positive definite in the nine variables η^{ij} $(i, j=1, 2, 3)$ for arbitrary values of $x^{*i}_{,j}$ and $\dot{x}^{*i}_{,j}$, we can put

$$(\delta\bar{x}^{*j}-\delta\bar{\bar{x}}^{*j})_{,i} = 0 \tag{19}$$

in the region V and at any time.

Equation (19) expresses that the difference of both the displacements is a translation, and the boundary condition on the portion S_u of S guarantees the uniqueness of the infinitesimal displacement for the given variations of body force, surface force and surface displacement.

The cumulation of the unique infinitesimal displacements of the deforming process produces the unique finite displacements. Thus the criterion that the deformation of a given *Voigt* type body are unique, is that the elastic potential and the dissipation function of the body make (18) positive definite for arbitrary values of η^{ij}.

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References

1) *M. A. Biot* : J. appl. Phys. **25** 1385 (1954) ; Phys. Rev. **27** 1463 (1955).
2) *T. Tokuoka* : Ing.-Arch. **30** 385 (1961).
3) *W. Prager* : *Einführung in die Kontinuumsmechanik*, Birkhäuser Verlag, Basel/Stuttgart 1961.