Complex Representation of Basic Equations in Plasticity

By

Tatsuo TOKUOKA*

(Received July 12, 1962)

An attempt, which expresses the fundamental equations of plasticity in complex representation, is offered. When the principal axes of stress are fixed in a body element, the complex stress and the complex strain and their increments are defined in *Haigh-Westergaad*'s stress space. The yield conditions, flow rule and *Henky* and *Reuss* equations are given in complex form.

1. Introduction. In the theory of elasticity and hydrodynamics, the function theory is used as a mathematical tool. Formulae or theorems of the function theory are utilized in these regions. Basic equations, for examples stress-strain relations, complex potential etc., are expressed compactly in complex form. But such an attempt has not been presented in the theory of plasticity.

In this paper, an attempt to represent the basic equations of plasticity in complex form will be proposed.

2. Complex Representaions of States of Stress and Strain. We now restrict our discussion to the case where the principal axes of stress are fixed in a body element. In this case, the principal components of the stress-increment are equal to the increments of the principal components of stress and the principal axes of the elastic and plastic strain increments are then coincident.

If the common principal axes are chosen as the fixed axes of reference, we can describe a geometrical representation of stress and strain.

A state of stress is completely specified by the values of the three principal components of stress, so that any stress state may be represented by a bound vector in a three dimentional space (*Haigh-Westergaad*'s stress space), where the principal stresses σ_1, σ_2, σ_3 are taken as Cartesian coordinates X_1, X_2, X_3 respectively. In

* Department of Aeronautical Engineering

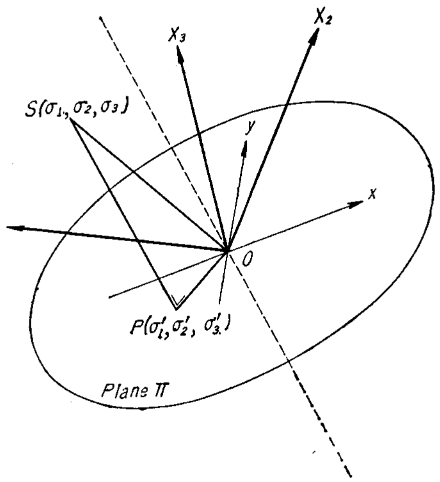

Fig. 1. Geometrical representation of a state of stress and strain.

Fig. 1, OS is the vector $(\sigma_1, \sigma_2, \sigma_3)$, while OP is the vector representing the deviatoric stress $(\sigma_1', \sigma_2', \sigma_3')$. OP always lies in the plane Π whose equation is $\sigma_1+\sigma_2+\sigma_3=0$, while PS, representing the hydrostatic component (σ, σ, σ) of the stress, is perpendicular to Π. The orthogonal projection of the X_3-axis on the plane Π is taken as the y-axis and the perpendicular to it in this plane as the x-axis. If (x, y) are the Cartesian coordinates of P with respect to these axes, then

$$\left.\begin{aligned} x &= \frac{\sigma_2-\sigma_1}{\sqrt{2}} = \frac{\sigma_2'-\sigma_1'}{\sqrt{2}}, \\ y &= \frac{2\sigma_3-\sigma_1-\sigma_2}{\sqrt{6}} = \frac{2\sigma_3'-\sigma_1'-\sigma_2'}{\sqrt{6}} = \sqrt{\frac{3}{2}}\,\sigma_3'. \end{aligned}\right\} \tag{1}$$

Now we consider the plane Π as *Gauss*' complex plane and write the state of stress as a complex quantity z:

$$z = x+iy = \frac{\sigma_2'-\sigma_1'}{\sqrt{2}}+i\sqrt{\frac{3}{2}}\,\sigma_3'. \tag{2}$$

If the state of stress at a point of the body is given, a complex quantity z is completely specified, and the latter is given, the former is specified using the relation $\sigma_1'+\sigma_2'+\sigma_3'=0$ and the hydrostatic component σ. Namely, a complex quantity z corresponds to a state of stress and vice versa. We shall call this complex quantity z a *complex stress*.

In the same way, we can obtain the *complex stress increment*

$$dz = \frac{d\sigma_2'-d\sigma_1'}{\sqrt{2}}+i\sqrt{\frac{3}{2}}\,d\sigma_3', \tag{3}$$

where $(d\sigma_1', d\sigma_2', d\sigma_3')$ is the deviatoric stress increment.

We define the *complex strain* ζ, the *complex elastic strain* ζ^e and the *complex plastic strain* ζ^p as

$$\zeta = \zeta^e+\zeta^p, \tag{4}$$

$$\zeta^e = \xi^e+i\eta^e, \quad \zeta^p = \xi^p+i\eta^p, \tag{5}$$

$$\left.\begin{aligned} \xi^e &= \frac{\varepsilon_2'^e - \varepsilon_1'^e}{\sqrt{2}}, \quad \eta^e = \frac{2\varepsilon_3'^e - \varepsilon_1'^e - \varepsilon_2'^e}{\sqrt{6}} = \sqrt{\frac{3}{2}}\,\varepsilon_3'^e, \\ \xi^p &= \frac{\varepsilon_2^p - \varepsilon_1^p}{\sqrt{2}}, \quad \eta^p = \frac{2\varepsilon_3^p - \varepsilon_1^p - \varepsilon_2^p}{\sqrt{6}} = \sqrt{\frac{3}{2}}\,\varepsilon_3^p; \end{aligned}\right\} \tag{6}$$

where $(\varepsilon_1'^e, \varepsilon_2'^e, \varepsilon_3'^e)$ is the deviatoric elastic strain and $(\varepsilon_1^p, \varepsilon_2^p, \varepsilon_3^p)$ the plastic strain, and we assume that the volume dilation does not occur in plastic deformation, namely,

$$\varepsilon_1^p + \varepsilon_2^p + \varepsilon_3^p = 0. \tag{7}$$

We can divide the strain increment $(d\varepsilon_1, d\varepsilon_2, d\varepsilon_3)$ into two parts, the elastic strain increment $(d\varepsilon_1^e, d\varepsilon_2^e, d\varepsilon_3^e)$ and the plastic strain increment $(d\varepsilon_1^p, d\varepsilon_2^p, d\varepsilon_3^p)$, and we define the *complex strain increments* $d\zeta$, $d\zeta^e$ and $d\zeta^p$ as

$$d\zeta = d\zeta^e + d\zeta^p, \tag{8}$$

$$d\zeta^e = d\xi^e + i d\eta^e, \quad d\zeta^p = d\xi^p + i d\eta^p, \tag{9}$$

$$\left.\begin{aligned} d\xi^e &= \frac{d\varepsilon_2'^e - d\varepsilon_1'^e}{\sqrt{2}}, \quad d\eta^e = \frac{2d\varepsilon_3'^e - d\varepsilon_1'^e - d\varepsilon_2'^e}{\sqrt{6}} = \sqrt{\frac{3}{2}}\,d\varepsilon_3'^e, \\ d\xi^p &= \frac{d\varepsilon_2^p - d\varepsilon_1^p}{\sqrt{2}}, \quad d\eta^p = \frac{2d\varepsilon_3^p - d\varepsilon_1^p - d\varepsilon_2^p}{\sqrt{6}} = \sqrt{\frac{3}{2}}\,d\varepsilon_3^p. \end{aligned}\right\} \tag{10}$$

The complex strain increment corresponds one-to-one with the state of strain increment for the same reason as the stress state. If we multiply the complex strain by the *Young* modulus E, the strain state can be specified by a point in *Haigh-Westergaad*'s space.

The parameters μ and ν introduced by *Lode* can be represented by the arguments of the complex stress and of the complex plastic strain increment respectively. Namely

$$\left.\begin{aligned} \mu &= \frac{2\sigma_3 - \sigma_2 - \sigma_1}{\sigma_2 - \sigma_1} = -\sqrt{3}\,\frac{y}{x} = -\sqrt{3}\,\arg z, \\ \text{and} \qquad \nu &= \frac{2d\varepsilon_3^p - d\varepsilon_1^p - d\varepsilon_2^p}{d\varepsilon_2^p - d\varepsilon_1^p} = -\sqrt{3}\,\frac{d\eta^p}{d\xi^p} = -\sqrt{3}\,\arg d\zeta^p. \end{aligned}\right\} \tag{11}$$

3. Complex Reresentations of Stress-Strain Relations.

I. ***Hencky stress-strain equations.*** The stress-strain equations, due to *Hencky*, have been frequently applied in special problems. We can easily verify that *Hencky*'s equations are equivalent to the following complex equation:

$$\zeta = \left(\phi + \frac{1}{2G}\right) z, \tag{12}$$

where ϕ is a real scalar quantity which is positive during continued loading and zero during unloading and G is the shearing modulus of a given material.

II. ***Elastic deformation.*** If a given isotropic body is subjected to *Hooke*'s law, the well-known stress-strain relations can be expressed as

$$d\zeta^e = \frac{dz}{2G}. \tag{13}$$

III. ***Yield conditions.*** The yield locus on the plane Π may be expressed as

$$f(J_2', J_3') = c^2, \tag{14}$$

where J_2' and J_3' are stress invariants and equal to

$$\left.\begin{aligned} J_2' &= \frac{1}{2}(\sigma_1'^2+\sigma_2'^2+\sigma_3'^2) = \frac{1}{2}zz^* = \frac{1}{2}|z|^2 \\ \text{and} \qquad J_3' &= \frac{1}{3}(\sigma_1'^3+\sigma_2'^3+\sigma_3'^3) = \frac{i}{6\sqrt{6}}(z^3-z^{*3}), \end{aligned}\right\} \tag{15}$$

where z^* is the complex conjugate of z and c is a real quantity specified by a given material. Thus we can write it as

$$F(z, z^*) = c^2. \tag{16}$$

According to the isotropic work-hardening rule given by *Hill*[1] and *Hodge*[2], the yield locus expands during plastic flow retaining its shape and situation with respect to the origin, and the real quantity c of (16) is a function of hydrostatic stress σ. Another rule, developed by *Prager*[3], accounts for the *Bauschinger* effect. This assumes that the yield locus is rigid but undergoes a translation in the direction of the strain increment. *Prager*'s rule was modified by *Ziegler*[4] and this modified rules can be expressed in our complex representation as

$$F[(z-A), (z^*-A^*)] = c^2, \tag{17}$$

where A is a complex quantity such as

$$A = \frac{\alpha_2-\alpha_1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}\,\alpha_3 \tag{18}$$

and $(\alpha_1, \alpha_2, \alpha_3)$ represents the total translation and satisfies

$$\alpha_1+\alpha_2+\alpha_3 = 0. \tag{19}$$

IV. ***Flow Rule.*** *Levy-Mises*' flow rule is written as

$$d\varepsilon_i^p = \frac{\partial f}{\partial \sigma_i'}d\lambda, \quad (i = 1, 2, 3), \quad \begin{cases} d\lambda > 0 & \text{for loading,} \\ d\lambda = 0 & \text{for unloading.} \end{cases} \tag{20}$$

We shall express the above flow rule in complex form. Equation (2) may be rewritten such as

$$\left.\begin{aligned} \sigma_1' &= -\frac{z+z^*}{2\sqrt{2}}+i\frac{z-z^*}{2\sqrt{6}}, \\ \sigma_2' &= \frac{z+z^*}{2\sqrt{2}}+i\frac{z-z^*}{2\sqrt{6}}, \\ \sigma_3' &= -i\frac{z-z^*}{\sqrt{6}}. \end{aligned}\right\} \tag{21}$$

Therefore we have

$$\begin{aligned} \frac{\partial F}{\partial z^*} &= \frac{\partial f}{\partial \sigma_1'}\frac{\partial \sigma_1'}{\partial z^*}+\frac{\partial f}{\partial \sigma_2'}\frac{\partial \sigma_2'}{\partial z^*}+\frac{\partial f}{\partial \sigma_3'}\frac{\partial \sigma_3'}{\partial z^*} \\ &= \frac{d\varepsilon_2^p-d\varepsilon_1^p}{2\sqrt{2}\,d\lambda}+i\frac{2d\varepsilon_3^p-d\varepsilon_1^p-d\varepsilon_2^p}{2\sqrt{6}\,d\lambda} \\ &= \frac{1}{2}\frac{d\zeta^p}{d\lambda}, \end{aligned}$$

namely

$$d\zeta^p = 2\frac{\partial F}{\partial z^*}d\lambda, \qquad \begin{cases} d\lambda > 0 & \text{for loading,} \\ d\lambda = 0 & \text{for unloading.} \end{cases} \tag{22}$$

This is a required flow rule represented in complex form.

V. ***Von-Mises' Body.*** *Von-Mises'* yield condition is specified by

$$F \equiv J_2' = c^2,$$

namely

$$\frac{1}{2}zz^* = c^2. \tag{23}$$

The flow rule is, therefore, in this case

$$d\zeta^p = zd\lambda, \qquad \begin{cases} d\lambda > 0 & \text{for loading,} \\ d\lambda = 0 & \text{for unloading.} \end{cases} \tag{24}$$

According to (13) and (24), we obtain the stress-strain relation as

$$d\zeta = zd\lambda+\frac{dz}{2G}, \qquad \begin{cases} d\lambda > 0 & \text{for loading,} \\ d\lambda = 0 & \text{for unloading.} \end{cases} \tag{25}$$

which denotes the *Reuss* equations.

If we adopt (17) as the yield condition, (23) is reduced to

$$\frac{1}{2}(z-A)(z^*-A^*) = c^2, \tag{26}$$

and the flow rule is, in this case, expressed as

$$d\zeta^p = (z-A)d\lambda, \qquad \begin{cases} d\lambda > 0 & \text{for loading,} \\ d\lambda = 0 & \text{for unloading.} \end{cases} \tag{27}$$

Thus the stress-strain relation can be written as

$$d\zeta = (z-A)d\lambda+\frac{dz}{2G}, \qquad \begin{cases} d\lambda > 0 & \text{for loading,} \\ d\lambda = 0 & \text{for unloading.} \end{cases} \tag{28}$$

4. Conclusion. When the principal axes of stress are fixed in a body element, the complex representations of the state of stress and strain were defined in *Haigh-Westergaad*'s stress space. It was proved that such defined complex stress and complex strain and their increments correspond to the state of stress and strain and vice versa.

By means of these quantities we obtained the complex forms of fundamental relations in plasticity, namely *Hencky* equations, elastic deformation formula, yield conditions, flow rule and *Reuss* equations.

References

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