Statistical Studies on the Response of Non-Linear Time-Variant Control Systems Subjected to a Suddenly Applied Stationary Gaussian Random Input

By

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In the previous paper, the authors have treated the response of non-linear control systems with time-invariant or -variant characteristics subjected to a suddenly applied stationary gaussian random input. However, recent trends in automatic control systems have required much attention to random changes of circuit parameters depending on the environment.

The description is divided into two parts. In Part I, an analytical approach on the statistical evaluation of the response of non-linear control systems with randomly time-variant characteristics is described. Part II is concerned with the stability of the response of non-linear time-variant control systems under the excitation of a random input signal. First, the stability conditions of systems with or without a random excitation are established from general points of view. Second, the stability of the system containing an on-off or a saturated characteristic is explored respectively. Finally, the influence of the random excitations on the stability of non-linear time-variant control systems is considered in detail.

List of Principal Symbols

t : time variable

t_0 : any initial time

$v(t)$: desired signal to the system

$z(t)$ and $y(t)$: input and output of a non-linear element of the zero-memory type respectively

$u(t)$: disturbance to the system

$y_0(t)$ and $x(t)$: input and output of the controlled system respectively

$A_0(t)$: gaussian random coefficient

m and $c(t)$: mean value and randomly fluctuating portion of $A_0(t)$ respectively

$R_c(t_1, t_2)$, $R_u(t_1, t_2)$ and $R_z(t_1, t_2)$: auto-correlation functions of $c(t)$, $u(t)$ and $z(t)$ respectively

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κ : equivalent gain of a non-linear element of the zero-memory type
$f_p(z, t)$: time dependent gaussian probability density function of $z(t)$
$W_1(t)$: weighting function
$\psi_z(t)$: variance of $z(t)$ at the time t
ψ_z : variance of $z(t)$ in the steady state
k and T: system parameters
E: symbol representing the ensemble average

Introduction

Recent trends in automatic control systems have required much attention to random changes of circuit parameters depending on the environment. A central problem in control theory is the evaluation of the non-stationary response of non-linear time-variant control systems with stationary or non-stationary random inputs.

The authors have treated the evaluation of the non-stationary response of non-linear control systems subjected to a suddenly applied random input in our earlier paper. The present paper develops the two aspects of the study of non-stationary response. The first is the evaluation of the response of non-linear control systems with randomly time-variant characteristics. An extensive approach is established. The second is the mean square stability of the response of non-linear time-variant control systems.

Such systems arise, for example, in the study of servomechanism and process control systems in which several parameters are undergoing various changes of environments.

Part I: The Response of Non-Linear Control Systems with Randomly Time-Variant Characteristics

1. Evaluation of the Mean Squared Value of the Response

We consider, as shown in Fig. 1, a typical automatic control system containing a non-linear element with zero-memory characteristic of which the relation between the input $z(t)$ and the output $y(t)$ is given by $y=f(z)$. We assume that the dynamical characteristic of the controlled element is expressed as

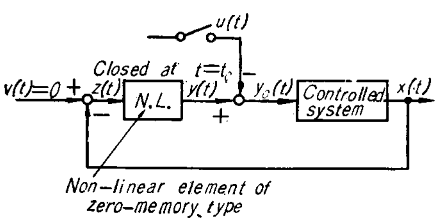


Fig. 1. Block diagram of a non-linear control system with randomly time-variant characteristics.

a randomly time-variant system of the following form:

$$\sum_{i=1}^{N} A_i \frac{d^i x(t)}{dt^i} + A_0(t)\, x(t) = k y_0(t)\,, \tag{1.1}$$

where A_i's are constant coefficients and $A_0(t)$ is a purely random coefficient with the form of

$$R_{A_0}(t_1, t_2) = D^2 \delta(t_1 - t_2)\,, \qquad (D\text{: constant}) \tag{1.2}$$

as its auto-correlation function. The equation of this non-linear control system becomes

$$\sum_{i=1}^{N} A_i \frac{d^i z(t)}{dt^i} + A_0(t)\, z(t) + kf[z(t)] = ku(t)\,. \tag{1.3}$$

In the control system under consideration, by applying the statistical linearization technique,[1] we may replace the non-linear element by a linear one with equivalent gain, κ, as shown in Fig. 2. The relation between the input signal $u(t)$ and the response $z(t)$ of the equivalent linearized system can therefore be expressed as

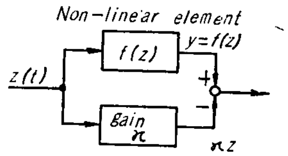


Fig. 2. Illustration of the analytical characterization of a non-linear element.

$$\sum_{i=1}^{N} A_i \frac{d^i z(t)}{dt^i} + A_0(t)\, z(t) + k\kappa z(t) = ku(t)\,. \tag{1.4}$$

In Eq. (1.4), the equivalent gain, κ, is defined by

$$\kappa[\psi_z(t)] = \int_{-\infty}^{\infty} z f(z) f_p(z, t)\, dz \Big/ \int_{-\infty}^{\infty} z^2 f_p(z, t)\, dz\,, \tag{1.5}$$

where $f_p(z, t)$ is the gaussian probability density function of $z(t)$ at the time t. Since Eq. (1.4) can, therefore, be written as

$$\sum_{i=1}^{N} A_i \frac{d^i z(t)}{dt^i} + k\kappa z(t) = V(t)\,, \tag*{(1.6)$_1$}$$

where

$$V(t) = ku(t) - A_0(t)\, z(t)\,, \tag*{(1.6)$_2$}$$

if we formally express the unit-impulse-response for the equivalent linearized control system governed by Eq. $(1.6)_1$ as $W_1(t, \xi)$, then the response $z(t)$ at the time t may be expressed as

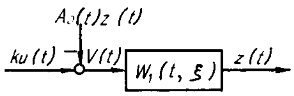


Fig. 3. Block diagram of a linearized control system shown in Eq. $(1.6)_1$.

$$z(t) = \int_{t_0}^{t} W_1(t, \xi)\, V(\xi)\, d\xi\,, \tag{1.7}$$

where t_0 is the time which the input $u(t)$ is applied. Therefore, we have

$$z(t) = F(t) - \int_{t_0}^{t} K(t, \xi)\, z(\xi)\, d\xi , \tag{1.8}$$

where

$$F(t) = k\int_{t_0}^{t_1} W_1(t, \xi)\, u(\xi)\, d\xi \tag{1.9}$$

and $K(t, \xi)$ is a kernel expressed by

$$K(t, \xi) = W_1(t, \xi)\, A_0(\xi) . \tag{1.10}$$

In order to determine the second product moment of $z(t)$, we multiply Eq. (1.8) at two time instants t_1, $t_1' = t_1 + \Delta t_1$ and average to give

$$\begin{aligned} \langle z(t_1)\, z(t_1')\rangle_{\text{av.}} &= \langle F(t_1)\, F(t_1')\rangle_{\text{av.}} \\ &- \int_{t_0}^{t_1} \langle K(t_1, \xi)\, F(t_1')\, z(\xi)\rangle_{\text{av.}}\, d\xi - \int_{t_0}^{t_1'} \langle K(t_1', \xi)\, F(t_1)\, z(\xi)\rangle_{\text{av.}}\, d\xi \\ &+ \int_{t_0}^{t_1}\int_{t_0}^{t_1'} \langle K(t_1, \xi_1)\, K(t_1', \xi_2)\, z(\xi_1)\, z(\xi_2)\rangle_{\text{av.}}\, d\xi_1 d\xi_2 . \end{aligned} \tag{1.11}$$

We assume[2)]

$$\left.\begin{aligned} \langle K(t_1, \xi)\, F(t_1')\, z(\xi)\rangle_{\text{av.}} &= \langle K(t_1, \xi)\rangle_{\text{av.}} \langle F(t_1')\, z(\xi)\rangle_{\text{av.}} \\ \langle K(t_1', \xi)\, F(t_1)\, z(\xi)\rangle_{\text{av.}} &= \langle K(t_1', \xi)\rangle_{\text{av.}} \langle F(t_1)\, z(\xi)\rangle_{\text{av.}} \end{aligned}\right\} \tag{1.12}$$

and

$$\langle K(t_1, \xi_1)\, K(t_1', \xi_2)\, z(\xi_1)\, z(\xi_2)\rangle_{\text{av.}} = \langle K(t_1, \xi_1)\, K(t_1', \xi_2)\rangle_{\text{av.}} \langle z(\xi_1)\, z(\xi_2)\rangle_{\text{av.}} \tag{1.13}$$

Since it is easily seen that $\langle A_0(t)\rangle_{\text{av.}} = 0$ from Eq. (1.2) by using Eqs. (1.12), (1.13) and $\langle K(t, \xi)\rangle_{\text{av.}} = 0$, we have

$$\begin{aligned} R_z(t_1, t_1') &= R_F(t_1, t_1') \\ &+ \int_{t_0}^{t_1}\int_{t_0}^{t_1'} \langle K(t_1, \xi_1)\, K(t_1', \xi_2)\rangle_{\text{av.}}\, R_z(\xi_1, \xi_2)\, d\xi_1 d\xi_2 , \end{aligned} \tag{1.14}$$

where $R_z(t_1, t_1')$ and $R_F(t_1, t_1')$ are auto-correlation functions of $z(t)$ and $F(t)$ respectively. Since $A_0(t)$ is a purely random signal, we have

$$\langle K(t_1, \xi_1)\, K(t_1', \xi_2)\rangle_{\text{av.}} = D^2 W_1(t_1, \xi_1)\, W_1(t_1', \xi_2)\, \delta(\xi_1 - \xi_2) . \tag{1.15}$$

Therefore, setting $t_1 = t_1' = t$ and carrying out the integration with respect to ξ_2, Eq. (1.14) becomes

$$\psi_z(t) = \psi_F(t) + D^2 \int_{t_0}^{t} W_1(t, \xi)^2 \psi_z(\xi)\, d\xi , \tag{1.16}$$

where

$$\psi_F(t) = \langle F(t)^2\rangle_{\text{av.}} = k^2 \int_{t_0}^{t}\int_{t_0}^{t} W_1(t, \xi_1)\, W_1(t, \xi_2)\, R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2 . \tag{1.17}$$

Since the integrand, $W_1(t, \xi)$, in Eq. (1.17) involves an unknown function, $\kappa[\psi_z(t)]$,[1)] it is impossible to evaluate directly $\psi_z(t)$ by using Eq. (1.17). Therefore, as we have already shown,[1] let $t_{j+1}-t_j=\Delta_j$ $(j=0, 1, 2,\cdots)$. It can thus be considered that the equivalent gain, κ, is kept a constant independent of time in these infinitesimally small time intervals, Δ_j's. We denote this value by κ_j at the time $t=t_j$.

If we introduce the equivalent gain, κ_j, into Eq. (1.17), then all parameters involved in $W_1(t, \xi)$ become constant in the time interval, Δ_j. Therefore, $W_1(t, \xi)$ depends on the time interval, $(t-\xi)$, between the application of an impulsive signal as the input and the observation of the output in this infinitesimally small time interval Δ_j. From this point of view, if we use the conventional expression,

$$W_1(t, \xi)\Big|_{\text{at } t=t_1} = W_1(t_1-\xi)\Big|_{\kappa=\kappa_0}, \tag{1.18}$$

then, from Eqs. (1.16) and (1.17), the integral equation determining the mean squared value, $\psi_z(t)$, at the time $t=t_1$ becomes as follows;

$$\psi_z(t_1) = \psi_F(t_1) + D^2\int_{t_0}^{t_1} W_1(t_1-\xi)^2\Big|_{\kappa=\kappa_0} \psi_z(\xi)\, d\xi \tag{1.19}$$

and

$$\psi_F(t_1) = k^2\int_{t_0}^{t_1}\int_{t_0}^{t_1} W_1(t_1-\xi_1)\, W_1(t_1-\xi_2)\Big|_{\kappa=\kappa_0} R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2. \tag{1.20}$$

In Eqs. (1.19) and (1.20), we must consider that

$$\kappa_0 = \kappa[\psi_z(t_0)]. \tag{1.21}$$

By substituting Eq. (1.19) into Eq. (1.5), the equivalent gain, κ_1, corresponding to the second time interval, Δ_1, becomes

$$\kappa_1 = \kappa[\psi_z(t_1)]. \tag{1.22}$$

Therefore, by using the values of κ_0 and κ_1, we have

$$\psi_z(t_2) = \psi_F(t_2) + D^2\left\{\int_{t_0}^{t_1} W_1(t_2-\xi)^2\Big|_{\kappa=\kappa_0} \psi_z(\xi)\, d\xi + \int_{t_1}^{t_2} W_1(t_2-\xi)^2\Big|_{\kappa=\kappa_1} \psi_z(\xi)\, d\xi\right\}, \tag{1.23}$$

where

$$\begin{aligned}\psi_F(t_2) = &\int_{t_0}^{t_1}\int_{t_0}^{t_1} W_1(t_2-\xi_1)\, W_1(t_2-\xi_2)\Big|_{\kappa=\kappa_0} R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \\ &+2\int_{t_0}^{t_1} W_1(t_2-\xi_1)\Big|_{\kappa=\kappa_0} d\xi_1 \int_{t_1}^{t_2} W_1(t_2-\xi_2)\Big|_{\kappa=\kappa_1} R_u(\xi_1, \xi_2)\, d\xi_2 \\ &+\int_{t_1}^{t_2}\int_{t_1}^{t_2} W_1(t_2-\xi_1)\, W_1(t_2-\xi_2)\Big|_{\kappa=\kappa_1} R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2.\end{aligned} \tag{1.24}$$

By applying the above procedure, the integral equation determining the mean squared value, $\psi_z(t_n)$, at the time $t=t_n$ is given as follows;

$$\psi_z(t_n) = \psi_F(t_n) + D^2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} W_1(t_n-\xi)^2 \Big|_{\kappa=\kappa_j} \psi_z(\xi)\, d\xi\,, \tag{1.25}$$

where

$$\psi_F(t_n) = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} W_1(t_n-\xi_1)\, W_1(t_n-\xi_2) \Big|_{\kappa=\kappa_i} R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2$$
$$+ \sum_{\substack{i,j=0 \\ (j\neq i)}}^{n-1} \int_{t_i}^{t_{i+1}} W_1(t_n-\xi_1) \Big|_{\kappa=\kappa_i} \int_{t_j}^{t_{j+1}} W_1(t_n-\xi_2) \Big|_{\kappa=\kappa_j} R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \tag{1.26}$$

and

$$\kappa_j = \kappa[\psi_z(t_j)]\,, \qquad (j = 0, 1, 2, \cdots, n-1)\,. \tag{1.27}$$

2. The Method of an Approximate Calculation

Since it is complex and tedious to calculate the value of the response, $\psi_z(t)$, by using Eqs. (1.25) and (1.27), the method of an approximate calculation is described in this section. When the difference between the values of equivalent gains κ_{j-1} and κ_j very slowly change with respect to the variation of time, Eqs. (1.25) and (1.26) may be approximately written as[1]

$$\psi_z(t_n) = \psi_F(t_n) + D^2 \int_{t_0}^{t_n} W_1(t_n-\xi)^2 \Big|_{\kappa=\kappa_{n-1}} \psi_z(\xi)\, d\xi \tag{2.1}$$

and

$$\psi_F(t_n) = k^2 \int_{t_0}^{t_n} \int_{t_0}^{t_n} W_1(t_n-\xi_1)\, W_1(t_n-\xi_2) \Big|_{\kappa=\kappa_{n-1}} R_u(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \tag{2.2}$$

where

$$\kappa_{n-1} = \kappa[\psi_z(t_{n-1})]\,. \tag{2.3}$$

When the response of the system becomes the stationary state as the time further increases, we use another method of calculating the response. The mean squared value in the steady state is calculated by using $\psi_z \equiv \psi_z(t)|_{t=\infty}$, i.e.,

$$\psi_z = \psi_F + \int_{t_0}^{\infty} \int_{t_0}^{\infty} \Big\{ W_1(t, \xi_1)\, W_1(t, \xi_2)\, \langle A_0(\xi_1) A_0(\xi_2)\rangle_{\text{av.}} \langle z(\xi_1)\, z(\xi_2)\rangle_{\text{av.}} \Big\} d\xi_1 d\xi_2\,, \tag{2.4}$$

where

$$\psi_F = k^2 \int_{t_0}^{\infty} \int_{t_0}^{\infty} W_1(t, \xi_1)\, W(t, \xi_2)\, \langle u(\xi_1)\, u(\xi_2)\rangle_{\text{av.}}\, d\xi_1 d\xi_2\,. \tag{2.5}$$

By noting $\langle z(\xi_1) z(\xi_2)\rangle_{\text{av.}}|_{\xi_1=\xi_2\to\infty} = \psi_z$ and considering Eq. (1.2), Eq. (2.4) becomes

$$\psi_z = \psi_F + D^2 \psi_z \int_{t_0}^{\infty} W_1(t, \xi)^2 d\xi\,. \tag{2.6}$$

If we use the following expression;[1]

$$W_1(t, \xi) \equiv W_1(t-\xi) \Big|_{\kappa=\kappa_\infty}\,, \tag{2.7}$$

where

$$\kappa_\infty \equiv \kappa[\psi_z(t) \Big|_{t=\infty}] = \kappa[\psi_z]\,, \tag{2.8}$$

then the mean squared value ψ_z is given by

$$\psi_z = \psi_F \Big/ \left[1 - D^2 \int_{t_0}^{\infty} W_1(t-\xi)^2 \Big|_{\kappa=\kappa_\infty} d\xi\right], \tag{2.9}$$

where

$$1 - D^2 \int_{t_0}^{\infty} W_1(t-\xi)^2 \Big|_{\kappa=\kappa_\infty} d\xi \neq 0. \tag{2.10}$$

The values of ψ_z and κ_∞ can, thus, be determined by solving Eqs. (2.8) and (2.9) simultaneously. The required instant of time t_s for the response of the system to reach the steady state is determined as already shown in the previous paper.[1] We, therefore, divide the time interval (t_0, t_s) into contiguous disjointed intervals of Δ_{s-j} $(t_{s-j} - t_{s-j-1} = \Delta_{s-j}, j=0, 1, 2, \cdots)$, and by a similar method described in section 1, we successively evaluate the values $\psi_z(t_s)$, κ_s, $\psi_z(t_{s-1})$, $\kappa_{s-1} \cdots$ as shown in Fig. 4. Thus, by noting that the respective conditions, $\kappa_0 = \kappa_0$ at the time $t = t_0$ and $\kappa = \kappa_s$ at the time $t = t_s$, we can calculate the value of $\psi_z(t)$.

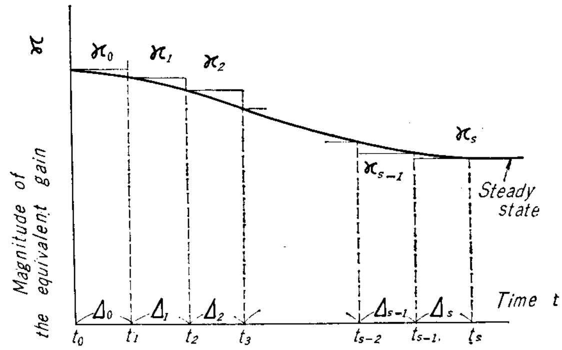


Fig. 4. Equivalent gain when the time t is divided into contiguous disjointed intervals of $\Delta_j = t_{j+1} - t_j$.

3. Examples

Example 1

As the first example, we assume that $N=1$ and $A_1 = T$ in Eq. (1.1) for the convenience of mathematical calculations. The controlled system is, in this case, expressed as

$$T\frac{dx(t)}{dt} + A_0(t)\, x(t) = ky_0(t). \tag{3.1}$$

We assume that the stationary gaussian random disturbance which is given by

$$R_u(t_1, t_2) = \psi_u \delta(t_1 - t_2) \tag{3.2}$$

as its auto-correlation function is applied to the system at the time $t=0$. The equation of the control system corresponding to Eqs. (1.6) becomes

$$T\frac{dz(t)}{dt} + k\kappa z(t) = ku(t) - A_0(t)\, z(t). \tag{3.3}$$

Since, from Eq. (1.18), the one-sided Green function associated with the left hand side of Eq. (3.3) is given by

$$W_1(t_1-\xi) = \frac{1}{T}\exp\left\{-\frac{k\kappa_0}{T}(t_1-\xi)\right\} \qquad (t_1>\xi)\,, \tag{3.4}$$

then the mean squared value, $\psi_z(t)$ at the time $t=t_1$ becomes

$$\psi_z(t_1) = \psi_F(t_1) + \frac{D^2}{T^2}\int_0^{t_1}\exp\left\{-\frac{2k\kappa_0}{T}(t_1-\xi)\right\}\psi_z(\xi)\,d\xi\,, \tag{3.5}$$

where

$$\psi_F(t_1) = \frac{k\psi_u}{2\kappa_0 T}\left\{1-\exp\left(-\frac{2k\kappa_0}{T}t_1\right)\right\}. \tag{3.6}$$

Therefore, we have

$$T^2\dot{\psi}_z(t_1) + (2k\kappa_0 T - D^2)\,\psi_z(t_1) = k^2\psi_u\,, \tag{3.7}$$

where

$$\dot{\psi}_z(t_1) = \frac{d\psi_z(t_1)}{dt_1}.$$

Considering the initial condition, $\psi_z(t_1)=0$ at the time $t_1=0$, we have the following solution,

$$\psi_z(t_1) = \frac{k^2\psi_u}{2k\kappa_0 T - D^2}\left[1-\exp\left\{-\left(\frac{2k\kappa_0}{T}-\frac{D^2}{T^2}\right)t_1\right\}\right]. \tag{3.8}$$

By substituting Eq. (3.8) into Eq. (1.5), the equivalent gain, κ_1, corresponding to Δ_1 is calculated as shown in Eq. (1.22). Since Eq. (1.23) is expressed as

$$\begin{aligned}\psi_z(t_2) \cong \psi_F(t_2) + \frac{D^2}{T^2}\Big[&\exp\left\{-2\frac{k\kappa_0}{T}\left(t_2-\frac{t_1}{2}\right)\right\}\psi_z\left(\frac{t_1}{2}\right)\cdot t_1 \\ &+\exp\left\{-2\frac{k\kappa_1}{T}\left(\frac{t_1+t_2}{2}\right)\right\}\psi_z\left(\frac{t_1+t_2}{2}\right)\cdot(t_2-t_1)\Big],\end{aligned} \tag{3.9}$$

then, by applying the following relation,

$$\psi_z\left(\frac{t_1+t_2}{2}\right) \cong \left\{\psi_z(t_1)+\psi_z(t_2)\right\}\Big/2 \tag{3.10}$$

into Eq. (3.9), $\psi_z(t_2)$ is easily calculated. By a similar method, the mean squared value, $\psi_z(t_n)$, can be obtained by

$$\begin{aligned}\psi_z(t_n) \cong{}& \psi_F(t_n) \\ &+\frac{D^2}{T^2}\sum_{j=0}^{n-1}\left\{\exp\left\{-2\frac{k\kappa_j}{T}\left(t_n-\frac{t_j+t_{j+1}}{2}\right)\right\}\psi_z\left(\frac{t_j+t_{j+1}}{2}\right)(t_{j+1}-t_j)\right\},\end{aligned} \tag{3.11}$$

where

$$\psi_z\left(\frac{t_j+t_{j+1}}{2}\right) \cong \frac{1}{2}\left\{\psi_z(t_{j+1})+\psi_z(t_j)\right\} \tag{3.12}$$

and

$$\kappa_j = \kappa[\psi_z(t_j)]\,, \qquad (j=0,1,2,\cdots,n-1)\,. \tag{3.13}$$

If we use the method of the approximate calculation described in section 2, the mean squared value, $\psi_z(t_n)$, from Eq. (2.1), becomes

$$\psi_z(t_n) = \frac{k^2\psi_u}{2k\kappa_{n-1}T-D^2}\Big[1-\exp\Big\{-\Big(\frac{2k\kappa_{n-1}}{T}-\frac{D^2}{T^2}\Big)t_n\Big\}\Big]. \qquad (3.14)$$

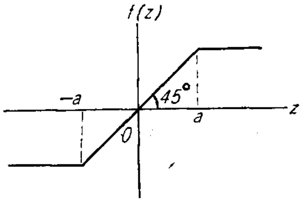


Fig. 5. The saturated non-linear characteristic.

As shown in Fig. 5, if the non-linear characteristic is given by

$$f(z) = \begin{cases} a & (z>a) \\ z & (|z|<a) \\ -a & (z<-a), \end{cases} \qquad (3.15)$$

then, it is easily shown that

$$\kappa[\psi_z(t)] = 2\phi\{a/\sqrt{\psi_z(t)}\}, \qquad (3.16)$$

where

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}}\int_0^\xi \exp\Big(-\frac{\eta^2}{2}\Big)d\eta. \qquad (3.17)$$

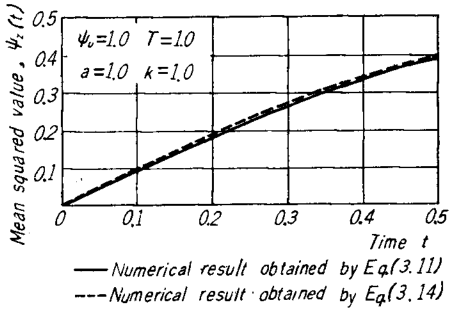


Fig. 6. The mean squared value of the response of the system shown in example 1.

The mean squared value, $\psi_z(t)$, can be computed by Eq. (3.11) or Eqs. (3.14) and (3.12). Fig. 6 shows numerical results of the response of the control system containing a non-linear element given by Eq. (3.15).

Example 2

Let the statistical characteristic of a stationary gaussian random input be

(i) mean value, $m_u=0$

(ii) auto-correlation function,

$$R_u(t_1, t_2) = \psi_u \exp\Big\{-\beta(t_1-t_2)\Big\}, \qquad (\beta>0) \qquad (3.19)$$

where ψ_u is the mean squared value of random disturbance. The control equation is the same as Example 1.

By substituting Eq. (3.19) into Eq. (2.2), with the help of Appendix-B, we have

$$\psi_F(t_n) = \frac{k\psi_u}{\kappa_{n-1}}\Big[\frac{1}{\beta T+k\kappa_{n-1}}+\frac{2k\kappa_{n-1}}{(\beta T+k\kappa_{n-1})(\beta T-k\kappa_{n-1})}$$
$$\times\exp\Big\{-\Big(\beta+\frac{k\kappa_{n-1}}{T}\Big)t_n\Big\}-\frac{1}{\beta T-k\kappa_{n-1}}\exp\Big(-2\frac{k\kappa_{n-1}}{T}t_n\Big)\Big]. \qquad (3.20)$$

Therefore, from Eqs. (3.5) and (3.19), it follows that

$$T^2\dot{\psi}_z(t_n)+(2k\kappa_{n-1}T-D^2)\psi_z(t_n) = U(t_n)\,, \tag{3.21}$$

where

$$U(t_n) = \frac{2k^2\psi_u T}{(\beta T+k\kappa_{n-1})}\left[1-\exp\left\{-\left(\beta+\frac{k\kappa_{n-1}}{T}\right)t_n\right\}\right]. \tag{3.22}$$

Therefore, the solution of Eq. (3.21) under the initial condition, $\psi_z(t_n)\big|_{t_n=0}=0$, is given by

$$\psi_z(t_n) = \frac{2k^2\psi_u}{\beta T+k\kappa_{n-1}}\left[\frac{1}{2k\kappa_{n-1}T-D^2}-\frac{1}{k\kappa_{n-1}T-\beta T-D^2}\exp\left\{-\left(\beta+\frac{k\kappa_{n-1}}{T}\right)t_n\right\}\right]. \tag{3.23}$$

From Eq. (3.23), it is easily shown that the steady solution becomes

$$\psi_z = \frac{2k^2\psi_u}{(\beta T+k\kappa_{n-1})(2kT\kappa_{n-1}-D^2)}\,, \tag{3.24}$$

where

$$\kappa_{n-1} = \kappa[\psi_z(t_{n-1})]. \tag{3.25}$$

By using the similar method as mentioned above, we can thus calculate the value of $\psi_z(t)$ successively. Fig. 7 shows the result of the numerical calculation of the response of the system containing the non-linear element given by Eq. (3.15).

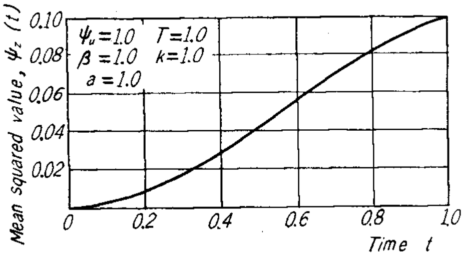


Fig. 7. The mean squared value of the response of the system shown in example 2.

Part II: On the Stability of Non-linear Randomly Time-Variant Control Systems

4. The Basic Concept of Stability

Further developments of studies in this field are requiring the exploration of the stability of randomly time-variant non-linear control systems with or without a random excitation. Let the equation of the control system, in general, be given by the following differential equation of the first order;

$$\dot{z}(t) = F[A_0(t), z(t)\; u(t)]\,, \quad \left[\dot{z}(t) = \frac{dz(t)}{dt}\right]. \tag{4.1}$$

When no excitation to the system exists, i. e., $u(t)=0$ in Eq. (4.1), the system is stated to be stable, if the solution $z(t)$ satisfies the following relation for any initial condition, i. e., $t=t_0$;

$$\lim_{t\to\infty} E[z(t)^2] = 0 . \tag{4.2}$$

When an excitation to the system exists, i. e., $u(t) \neq 0$ in Eq. (4.1), the system is stated to be stable, if the solution $z(t)$ satisfies the following relation;

$$\lim_{t\to\infty} E[z(t)^2] < \beta^2 , \tag{4.3}$$

where β is an arbitrary constant.

5. The Stability Condition of the Control System without a Random Excitation

We consider a typical control system as shown in Fig. 1. We assume that the dynamical characteristics of the controlled element is expressed as Eq. (3.1) and $A_0(t)$ is a stationary gaussian random coefficient expressed by

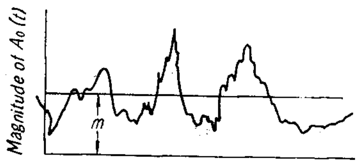


Fig. 8. Gaussian random coefficient $A_0(t)$ with the mean valve m.

$$A_0(t) = m + c(t) , \tag{5.1}$$

where m is the mean value of $A_0(t)$ as shown in Fig. 8. Let, moreover, the auto-correlation function $R_c(t_1, t_2)$ of $c(t)$ be

$$R_c(t_1, t_2) = D^2\delta(t_1 - t_2) . \tag{5.2}$$

The equation of this non-linear control system becomes

$$T\frac{dz(t)}{dt} + \left\{m + c(t)\right\} z(t) + kf[z(t)] = 0 , \tag{5.3}$$

For the convenience of the analysis, if the non-linear element is replaced by a linear one with the equivalent gain, κ, as shown in Fig. 2, we have

$$T\frac{dz(t)}{dt} + (m + k\kappa) z(t) = V(t) , \tag{5.4$_1$}$$

where

$$V(t) = -c(t) z(t) . \tag{5.4$_2$}$$

Under the assumption that κ is a constant, since the one-sided Green function associated with the left hand of Eq. $(5.4)_1$ becomes

$$W_1(t-\xi) = \frac{1}{T} \exp\left\{-\frac{m + k\kappa}{T}(t-\xi)\right\}, \qquad (t-\xi > 0) \tag{5.5}$$

and the fundamental solution of Eq. $(5.4)_1$ is given by $-\exp\{-(m+k\kappa)t/T\}$, then we have

$$z(t) = z_0 \exp\left\{-\frac{m+k\kappa}{T}(t-t_0)\right\} + \frac{1}{T}\int_{t_0}^{t} \exp\left\{-\frac{(m+k\kappa)}{T}(t-\xi)\right\} V(\xi)\, d\xi , \tag{5.6}$$

where $z=z_0$ is the initial value at the time $t=t_0$. Therefore, the mean squared value of $z(t)$ is given by the following equation;

$$\psi_z(t) = z_0^2 \exp\left\{-2\frac{m+k\kappa}{T}(t-t_0)\right\} + \frac{D^2}{T^2}\int_{t_0}^{t} \exp\left\{-2\frac{m+k\kappa}{T}(t-\xi)\right\}\psi_z(\xi)\,d\xi. \tag{5.7}$$

Differentiating Eq. (5.7) with respect to the time t, we obtain

$$T^2\dot{\psi}_z(t)+\left\{2(m+k\kappa)\,T-D^2\right\}\psi_z(t) = 0, \tag{5.8}$$

where $\dot{\psi}_z(t)\equiv d\psi_z(t)/dt$. By using the definition given by Eq. (4.2), we have the following result as the stability condition,

$$2(m+k\kappa)\,T-D^2>0. \tag{5.9}$$

where κ is the value of a stationary equivalent gain of non-linear characteristics.

6. The Stability Condition of the Control System with a Random Excitation

When the control system is excited by the stationary gaussian random input, the mean squared value $\psi_z(t)$ is given by

$$\psi_z(t) = \psi_F(t)+\frac{D^2}{T^2}\int_{t_0}^{t} \exp\left\{-\frac{2}{T}(m+k\kappa)(t-\xi)\right\}\psi_z(\xi)\,d\xi, \tag{6.1}$$

where, under the assumption that the initial condition $z(t_0)=0$ and $\tau=\xi_1-\xi_2$ $(\tau>0)$,

$$\psi_F(t) = \frac{\kappa^2}{T^2}\int_{t_0}^{t} R_u(\xi_1)\,d\xi_1\int_{t_0}^{t-\tau} \exp\left\{-\frac{m+k\kappa}{T}(t-\xi_2)\right\} \times\exp\left\{-\frac{m+k\kappa}{T}(t-\tau-\xi_2)\right\}d\xi_2. \tag{6.2}$$

From Eq. (6.1), we have

$$\dot{\psi}_z(t)+\left\{\frac{2(m+k\kappa)}{T}-\frac{D^2}{T^2}\right\}\psi_z(t) = U(t), \tag{6.3}$$

where

$$U(t) = \dot{\psi}_F(t)+\frac{2(m+k\kappa)}{T}\psi_F(t), \tag{6.4}$$

and $\dot{\psi}_F(t)=d\psi_F(t)/dt$. By using the definition established in Eq. (4.3), the following inequality must be satisfied

$$2k\kappa T+2mT>D^2. \tag{6.5}$$

Therefore, if Eq. (6.5) holds, it can be concluded that the control system given by Eqs. (5.4) is stable.

7. Examples

As the first example, the on-off relay characteristic given by

$$f(z)=\begin{cases} a & (z>0) \\ -a & (z<0) \end{cases} \tag{7.1}$$

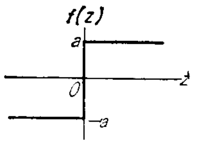


Fig. 9. The on-off relay characteristic.

as shown in Fig. 9 is considered. From Eq. (4.5), the equivalent gain in the steady state becomes

$$\kappa=\sqrt{\frac{2}{\pi}}\frac{a}{\sqrt{\psi_z}}, \tag{7.2}$$

where

$$\psi_z=\lim_{t\to\infty}\psi_z(t). \tag{7.3}$$

If we assume that auto-correlation function of the random disturbance to the system is

$$R_u(t_1, t_2)=\psi_u\delta(t_1-t_2), \tag{7.4}$$

then, under the initial condition $\psi_z(t)\Big|_{t=t_0}=0$, the solution $\psi_z(t)$ of Eq. (6.3) becomes

$$\psi_z(t)=\frac{k^2\psi_u}{2T(m+k\kappa)-D^2}\left[1-\exp\left\{-\left(2\frac{m+k\kappa}{T}-\frac{D^2}{T^2}\right)t\right\}\right]. \tag{7.5}$$

Therefore, we have

$$\psi_z=k^2\psi_u\Big/\left\{2T(m+k\kappa)-D^2\right\}. \tag{7.6}$$

It is easily seen that Eq. (7.6) satisfies the condition Eq. (6.5). In order to obtain the region of stability and instability, from Eqs. (7.2) and (7.6), we have

$$\pi(D^2-2mT)^2\psi_z^2+2k^2\left\{\pi\psi_u(D^2-2mT)-4a^2T^2\right\}\psi_z+\pi k^4\psi_u^2=0. \tag{7.7}$$

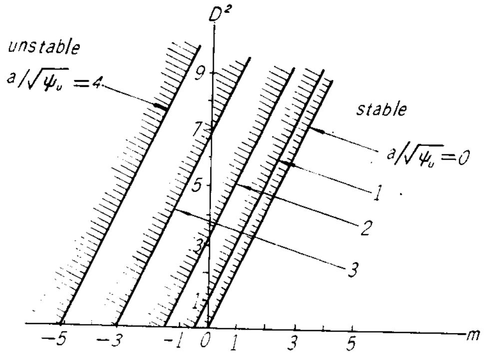
Since the above equation is of the second order with respect to variable, ψ_z, the condition that these roots have the positive real value is

$$2a^2T^2-\pi\psi_u(D^2-2mT)>0. \tag{7.8}$$

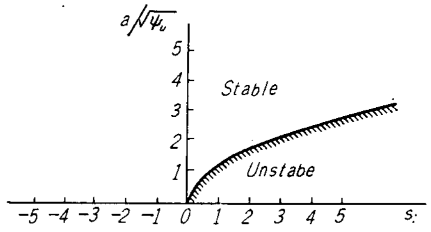
If the system parameters satisfy Eq. (7.8), the control system expressed by Eq. (5.5) is stable. When $T=1$, Eq. (7.8) becomes

$$\frac{2}{\pi}\left(\frac{a}{\sqrt{\psi_u}}\right)^2>D^2-2m. \tag{7.9}$$

The relation given by Eq. (7.9) is shown in Fig. 10. We treat the second case of the system containing a non-linear characteristic given by Eq. (3.16). By using Eqs. (7.6) and (3.16), with the help of the graphical method, we can determine the region of the stability. This result is shown in Fig. 11.

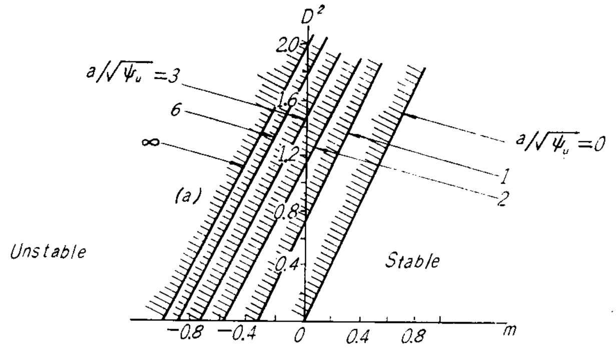


(a) The relation between m and D^2

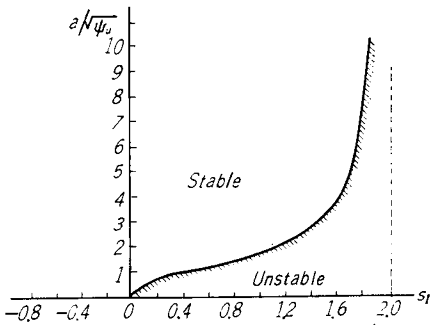


(b) The relation between $a/\sqrt{\psi_u}$ and s_1, where $s_1=D^2-2m$

Fig. 10. Stability plot for a non-linear control system with randomly time-variant characteristic containing an on-off relay element.



(a) The relation between m and D^2



(b) The relation $a/\sqrt{\psi_u}$ and s_1, where $s_1=D^2-2m$

Fig. 11. Stability plot for a non-linear control system with randomly time-variant characteristic containing a saturated element.

Conclusions

The statistical evaluation of the response of non-linear control systems with randomly time-variant characteristic subjected to a suddenly applied stationary gaussian random input is described. Although it is complex and tedious to calculate the value of the response, $\psi_x(t)$, by using Eqs. (1.25) and (1.26), the procedure described in section 3 is a remarkably effective tool for calculating the response and for decreasing the numerical error. It is, in general, difficult to find out the conditions defining the stability of non-linear time-variant control systems. In the particular case considered here, it is, however, possible to obtain the stability condition.

In the case of linear control systems, the value of ψ_u does not depend on the stability of the system.[3] On the contrary, for non-linear control systems, the influence of ψ_u on the stability of the system is shown in detail with various kinds of interesting results.

The procedure described here can be extended to non-linear control systems governed by the higher order differential equation with randomly time-variant coefficients.

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Appendix A: Numerical Calculation of the Integral Equation given by Eq. (1.25)

As the method of calculating numerically the integral equation as shown in Eq. (1.25), the method of MacLaurin, Newton-Cotes, Tchebyscheff or Gauss will be considered. Since the MacLaurin's formula is very simple and has good accuracy when time t is not so large, we apply this procedure in this paper.

$$\int_0^h F_2(\xi)\,d\xi = F_2\left(\frac{1}{2}h\right)\cdot h \tag{A-1}$$

$$\int_0^{2h} F_2(\xi)\,d\xi = \left\{\frac{1}{2}F_2\left(\frac{1}{2}h\right)+\frac{1}{2}F_2\left(h+\frac{1}{2}h\right)\right\}\cdot 2h \qquad \text{(A-2)}$$

$$\int_0^{3h} F_2(\xi)\,d\xi = \left\{\frac{3}{8}F_2\left(\frac{1}{2}h\right)+\frac{2}{8}F_2\left(h+\frac{1}{2}h\right)+\frac{3}{8}F_2\left(2h+\frac{1}{2}h\right)\right\}3h \qquad \text{(A-3)}$$

$$\int_0^{4h} F_2(\xi)\,d\xi = \left\{\frac{13}{48}F_2\left(\frac{1}{2}h\right)+\frac{11}{48}F_2\left(h+\frac{1}{2}h\right)+\frac{11}{48}F_2\left(2h+\frac{1}{2}h\right)+\frac{13}{48}F_2\left\{3h+\frac{1}{2}h\right)\right\}4h \qquad \text{(A-4)}$$

and

$$\int_0^{5h} F_2(\xi)\,d\xi = \left\{\frac{275}{1152}F_2\left(\frac{1}{2}h\right)+\frac{100}{1152}F_2\left(h+\frac{1}{2}\right)+\frac{402}{1152}F_2\left(2h+\frac{1}{2}h\right)+\frac{100}{1152}F_2\left(3h+\frac{1}{2}h\right)+\frac{272}{1152}F_2\left(4h+\frac{1}{2}h\right)\right\}5h. \qquad \text{(A-5)}$$

By applying the above equations, we can successively evaluate the response $z(t)$ of the non-linear control system. For example, applying Eq. (A-1) to Eq. (1.25), we have

$$\psi_z(t_n) = \psi_F(t_n)+D^2\sum_{j=0}^{n-1} W_1\left(t_n-\frac{t_{j+1}+t_j}{2}\right)^2\Big|_{\kappa=\kappa_j}\cdot\psi_z\left(\frac{t_{j+1}+t_j}{2}\right)\Delta_j, \qquad \text{(A-6)}$$

where $\Delta_j=t_{j+1}-t_j$. By noting Eq. (3.11), (A-6) becomes

$$\psi_z(t_n) = \psi_F(t_n) + D^2\sum_{j=0}^{n-1} W_1\left\{t_n-\frac{(t_j+t_{j+1})}{2}\right\}^2\Big|_{\kappa=\kappa_j}\cdot\left\{\psi_z(t_j)+\psi_z(t_{j+1})\right\}\frac{\Delta_j}{2}. \qquad \text{(A-7)}$$

Therefore, we have

$$\psi_z(t_n) = \frac{\psi_F(t_n)+K_1}{1-D^2W_1\{t_n-t_{n-1})/2\}^2\big|_{\kappa=\kappa_{n-1}}\cdot\Delta_{n-1}/2}, \qquad \text{(A-8)}$$

where

$$K_1 = D^2\sum_{j=0}^{n-1} W_1\left\{t_n-\frac{(t_j+t_{j+1})}{2}\right\}^2\Big|_{\kappa=\kappa_j}\cdot\left\{\psi_z(t_j)+\psi_z(t_{j+1})\right\}\frac{\Delta_j}{2} \qquad \text{(A-9)}$$

and

$$1-D^2W_1\left\{\frac{(t_n-t_{n-1})}{2}\right\}^2\Big|_{\kappa=\kappa_{n-1}}\cdot\frac{\Delta_{n+1}}{2}\neq 0. \qquad \text{(A-10)}$$

Appendix B: Evaluation of $\psi_F(t)$ given by Eq. (3.20)

Since

$$\int_0^t\int_0^t d\tau_1 d\tau_2 = \int_0^t\int_0^{t-\tau} d\tau\, d\tau_2+\int_{-\tau}^0\int_0^{t+\tau} d\tau\, d\tau_1 \qquad \text{(B-1)}$$

it follows that

$$\psi_F(t) = k^2 \int_0^t \int_0^{t-\tau} W_1(t, \tau_2+\tau) W_1(t, \tau_2) R_u(\tau) d\tau d\tau_2 + k^2 \int_{-t}^0 \int_0^{t+\tau} W_1(t, \tau_1) W_1(t, \tau_1-\tau) R_u(\tau) d\tau d\tau_1 . \quad \text{(B-2)}$$

Changing the sign of the second term, Eq. (B-2) becomes

$$\psi_F(t) = k^2 \int_0^t \int_0^{t-\tau} W_1(t, \tau_2+\tau) W_1(t, \tau_2) R_u(\tau) d\tau d\tau_2 + k^2 \int_0^t \int_0^{t-\tau} W_1(t, \tau_1) W_1(t, \tau_1+\tau) R_u(-\tau) d\tau d\tau_1 . \quad \text{(B-3)}$$

By noting the relation, $R_u(\tau)=R_u(-\tau)$, the above equation becomes

$$\psi_F(t) = 2k^2 \int_0^t R_u(\tau) d\tau \int_0^{t-\tau} W_1(t, \tau_2+\tau) W_1(t, \tau_2) d\tau_2 . \quad \text{(B-4)}$$

In Eq. (B-4), since the following function,

$$\Psi(t, \tau) \equiv \int_0^{t-\tau} W_1(t, \tau_2+\tau) W_1(t, \tau_2) d\tau_2 , \quad \text{(B-5)}$$

does not depend on the statistical properties of random input, we can easily evaluate this value from the control equation. The mean squared value, $\psi_F(t)$, can be evaluated as follows;

$$\psi_F(t) = 2k^2 \int_0^t R_u(\tau) \Psi(t, \tau) d\tau . \quad \text{(B-6)}$$