Self-excited Oscillation of Relay Servomechanism with Coulomb Friction and its Graphic Solution

By

Michiyoshi KUWAHARA* and Kazumasa HIRAI**

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In this paper, the authors treat the relay servomechanism containing Coulomb friction as one of the control systems with multiple nonlinearities, and are particularly concerned with the self-excited oscillation of this relay servomechanism, discussing the effect of the Coulomb friction on its performance.

In the case where the relay has hysteresis but no dead zone in its characteristics, the self-excited oscillation occurs in the servomechanism and the waveform and the frequency of the oscillation are greatly affected by the Coulomb friction. The output waveform and the freqency in this state are analyzed exactly, using the fact that the Coulomb friction characteristic in a periodic state can be treated as the ideal relay characteristic, and the effects of the Coulomb friction on them are considered.

Moreover, the authors develop the graphic method of solving the self-excited oscillation of the relay servomechanism. Using this approximation method, the frequency of the self-excited oscillation can be obtained graphically through the vector locus of the transfer function of the linear part of this servomechanism plotted on a complex plane.

The result obtained by this approximation method shows a good agreement with the theoretical one.

1. Introduction

The performance of relay control systems whose elements are linear except for relays has already been investigated in detail and reported in many papers. Actual control systems, however, usually include nonlinear elements besides relays, and these nonlinear elements have a great influence on the behavior of the control systems. In an electromechanical servomechanism with a DC servomotor operated by relay amplifier, for example, the behavior of the servomechanism is greatly affected by the relay characteristic and the Coulomb friction existing between the commutator and the brushes of the servomotor.

In this paper, the authors are particularly concerned with the self-excited oscillation of the relay servomechanism containing the Coulomb friction whose

* Automation Research Laboratory
** Department of Electrical Engineering

relay has hysteresis but no dead zone. First the angular frequency and waveforms of the relay servomechanism will be accurately analyzed, and the effects of the Coulomb friction on its performance will be discussed. Next, the graphic method of solving the self-excited oscillation of the relay servomechanism by means of the characteristic quantity of the relay control system will be developed and the result obtained by this approximation method is compared with the theoretical one.

2. Analysis of Self-excited Oscillation

Self-excited oscillation may occur in a simple relay servomechanism with Coulomb friction, if the relay has hysteresis characteristics. Consider the self-excited oscillation in the relay servomechanism in Fig. 1, in which R represents the relay characteristic shown in Fig. 2 and N represents the characteristic of the Coulomb friction torque shown in Fig. 3. Let the output torque of the relay

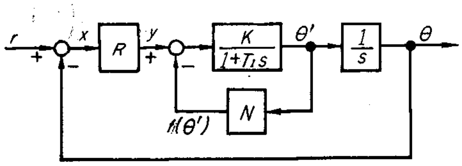

Fig. 1. Block diagram of relay servomechanism with Coulomb friction.

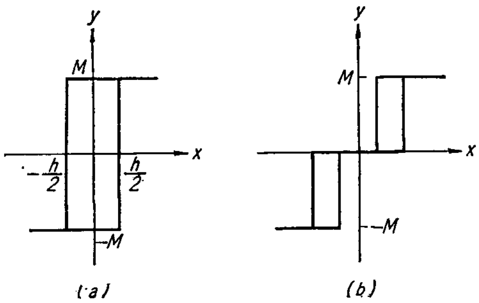

Fig. 2. Relay characteristics.

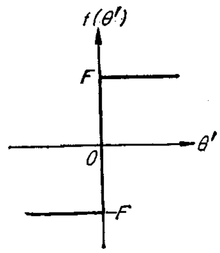

Fig. 3. Coulomb friction characteristic.

element be $\pm M$ and the width of the hysteresis h as shown in Fig. 2a. In the following discussion, it is assumed that the Coulomb friction torque, F, is not larger than the output of the relay element, M, i.e., $F \leqq M$, for the servomechanism does not work if $F > M$. If $F < M$, the torque input to the controlled member never becomes zero, provided the relay has no dead zone, and the output of the servomechanism does not become sluggish. On the other hand, when the relay has a more general characteristic like the one shown in Fig. 2b, a self-excited oscillation may take place in the system, but it is very difficult to find the output waveforms accurately in this case because of the sluggishness of the output caused by the Coulomb friction torque and the dead zone in the relay characteristic. Therefore only a system with a relay of the characteristic shown in Fig. 2a is discussed here.

Suppose that a self-excited oscillation of angular frequency ω_0 may occur in the relay servomechanism shown in Fig. 1. If the instant at which the relay output changes from negative to positive in periodic state is taken as the time origin, various quantities of the system vary as shown in Fig. 4.

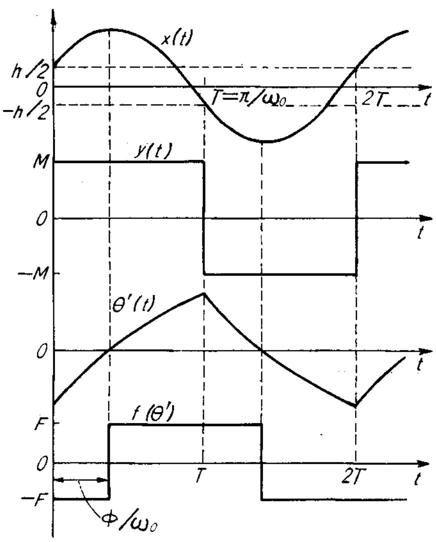

Fig. 4. Variation of quantities of the relay servomechanism in periodic state.

The switching of the relay occurs once in a half period, $T=\pi/\omega_0$, so that

$$\left.\begin{aligned} x(0) &= \frac{h}{2} \\ x\left(\frac{\pi}{\omega_0}\right) &= -\frac{h}{2} \end{aligned}\right\} \tag{1}$$

where $x(t)$ is the input to the relay.

At the instant of switching,

$$\left.\begin{aligned} x'(0) &> 0 \\ x'\left(\frac{\pi}{\omega_0}\right) &< 0 \end{aligned}\right\} \tag{2}$$

where $x'(t)=dx(t)/dt$.

On the other hand

$$x(t) > -\frac{h}{2}, \quad 0 < t < \frac{\pi}{\omega_0} \tag{3}$$

for the switching of the relay does not occur in one period $2T$ except at $t=0$, π/ω_0, $2\pi/\omega_0$.

From these conditions we obtain the angular frequency ω_0 and the waveforms of output $\theta(t)$ and output speed $\theta'(t)$ of the self-excited oscillation of the relay servomechanism. The output of the relay, $y(t)$, is a periodic rectangular wave of half period $T=\pi/\omega_0$ and Laplace transformation of $y(t)$ yields

$$Y(s) = \frac{M}{s}\frac{1-\exp(-sT)}{1+\exp(-sT)} \tag{4}$$

in a periodic state. When the output does not become sluggish, the waveform of $f(\theta')$ becomes periodic rectangular wave of half period T, too. So that if ϕ is the phase difference between the two waves, Laplace transformation of $f(\theta')$ gives

$$\mathcal{L}\{f(\theta')\} = \frac{F}{s}\frac{1-\exp(-sT)}{1+\exp(-sT)}\exp\left(-\frac{s\phi}{\omega_0}\right) \tag{5}$$

As is evident from Fig. 1

$$\left.\begin{aligned}\mathcal{L}\{\theta(t)\} &= KG(s)[Y(s)-\mathcal{L}\{f(\theta')\}]\\ G(s) &= \frac{1}{s(1+T_1 s)}\end{aligned}\right\} \tag{6}$$

Substitution of Eqs. (4) and (5) into Eq. (6), and subsequent inverse Laplace transformation of the result give the output, $\theta(t)$, of the relay servomechanism in the steady state as follows:

(i) In the interval $0\leqq t\leqq \phi/\omega_0$

$$\frac{\theta(t)}{KM} = \left(t-\frac{T}{2}\right)-T_1\left\{1-\left(1+\tanh\frac{T}{2T_1}\right)\exp\left(-\frac{t}{T_1}\right)\right\}$$
$$+\frac{F}{M}\left[\left(t+\frac{T}{2}-\frac{\phi}{\omega_0}\right)-T_1\left\{1-\left(1+\tanh\frac{T}{2T_1}\right)\exp\left(-\frac{t+T-\phi/\omega_0}{T_1}\right)\right\}\right] \tag{7}$$

(ii) In the interval $\phi/\omega_0\leqq t\leqq T$

$$\frac{\theta(t)}{KM} = \left(t-\frac{T}{2}\right)-T_1\left\{1-\left(1+\tanh\frac{T}{2T_1}\right)\exp\left(-\frac{t}{T_1}\right)\right\}$$
$$-\frac{F}{M}\left[\left(t-\frac{T}{2}-\frac{\phi}{\omega_0}\right)-T_1\left\{1-\left(1+\tanh\frac{T}{2T_1}\right)\exp\left(-\frac{t-\phi/\omega_0}{T_1}\right)\right\}\right] \tag{8}$$

And the output speed, $\theta'(t)$, is given by differentiating Eqs. (7) and (8) with respect to time t as follows:

(i) In the interval $0\leqq t\leqq \phi/\omega_0$

$$\frac{\theta'(t)}{KM} = \left(1+\frac{F}{M}\right)-\left(1+\tanh\frac{T}{2T_1}\right)\left\{1+\frac{F}{M}\exp\left(-\frac{T-\phi/\omega_0}{T_1}\right)\right\}\exp\left(-\frac{t}{T_1}\right) \tag{9}$$

(ii) In the interval $\phi/\omega_0\leqq t\leqq T$

$$\frac{\theta'(t)}{KM} = \left(1-\frac{F}{M}\right)-\left(1+\tanh\frac{T}{2T_1}\right)\left\{1-\frac{F}{M}\exp\left(\frac{\phi}{\omega_0 T_1}\right)\right\}\exp\left(-\frac{t}{T_1}\right) \tag{10}$$

On the other hand, the polarity of the wave of $f(\theta')$ is reversed at the instant $\theta'(t)=0$.

Hence

$$\theta'\left(\frac{\phi}{\omega_0}\right) = 0 \tag{11}$$

From Eqs. (9) to (11), we obtain

$$\frac{\phi}{\omega_0} = T_1\log\frac{1+\tanh\dfrac{T}{2T_1}}{1+\dfrac{F}{M}\tanh\dfrac{T}{2T_1}} \tag{12}$$

From Eq. (1),

$$x\left(\frac{\pi}{\omega_0}\right) = -\theta\left(\frac{\pi}{\omega_0}\right) = -\frac{h}{2} \quad (13)$$

The angular frequency, ω_0, of the self-excited oscillation and the phase difference, ϕ, between $y(t)$ and $f(\theta')$ are obtained from Eqs. (8) and (12) by using Eq. (13) and the relationship $T=\pi/\omega_0$.

By way of example, ω_0 and ϕ are plotted for various values of F/M in the case where $KM=50$, $h=10$, $T_1=0.1$ as shown in Fig. 5. Figure 5 shows that the angular frequency, ω_0, of the self-excited oscillation and the phase difference, ϕ, decreases as the ratio of the Coulomb friction torque to the relay output, F/M, increases from zero. The waveforms of output and output speed in the interval $0 \leqq t \leqq T$ is to be found by means of Eqs. (7) to (10). The waveforms in $T \leqq t \leqq 2T$ are the same as in $0 \leqq t \leqq T$, but the polarity is reversed. Figure 6 shows an example of the waveform of output speed which is plotted for $KM=50$, $T_1=0.1$, $h=10$ and $F/M=0.4$.

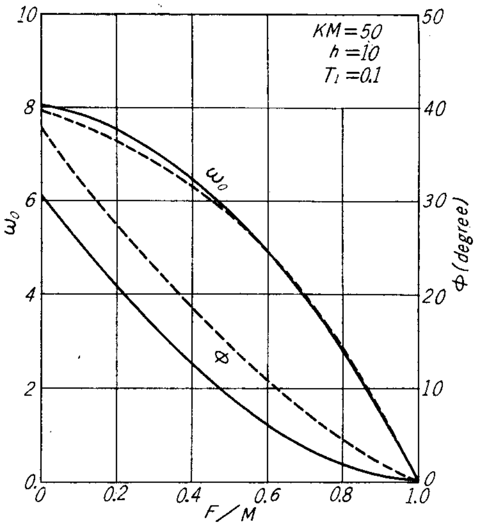

Fig. 5. Angular frequency of self-excited oscillation and phase difference.

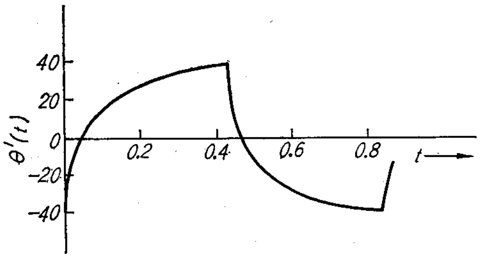

Fig. 6. Waveform of output speed ($KM=50$, $T_1=0.1$, $h=10$, $F/M=0.4$).

3. Graphic Analysis of Self-excited Oscillation

3.1 Principle of graphic analysis

In the case where a symmetric periodic oscillation occurs at an angular frequency ω in a relay servomechanism with Coulomb friction shown in Fig. 1, the relay output, $y(t)$, and the Coulomb friction torque, $f(\theta')$, of the servomechanism are periodic rectangular waves of angular frequency ω. If we take the time origin at the instant at which the relay output, $y(t)$, changes from nagative to positive in periodic state, the Coulomb friction torque, $f(\theta')$, reverses its polarity from negative to positive at the instant corresponding to the phase difference ϕ. Then these periodic rectangular waves are generally expressed as follows:

$$y(t) = \frac{4M}{\pi}\sum_{n}\frac{\sin n\omega t}{n} \tag{14}$$

$$f(\theta') = f(t) = \frac{4F}{\pi}\sum_{n}\frac{\sin n\omega(t-\phi/\omega)}{n} \tag{15}$$

where $\sum_{n}$ indicates the summation for all positive odd values of n.

The output, $\theta(t)$, in the case where $y(t)$ and $f(t)$ are impressed as the input to the linear part, $KG(s)$, of the servomechanism can be expressed as

$$\mathcal{L}\{\theta(t)\} = KG(s)\mathcal{L}\{y(t)-f(t)\} \tag{16}$$

For a sinusoidal input of angular frequency ω

$$G(j\omega) = A(\omega)\exp[j\Phi(\omega)] \tag{17}$$

so that from Eqs. (14) and (15) we obtain

$$\begin{aligned}\theta(t) &= \frac{4KM}{\pi}\sum_{n}\frac{1}{n}A(n\omega)\sin[n\omega t+\Phi(n\omega)]\\ &\quad -\frac{4KF}{\pi}\sum_{n}A(n\omega)\sin\left[n\omega\left(t-\frac{\phi}{\omega}\right)+\Phi(n\omega)\right]\end{aligned} \tag{18}$$

Now define complex quantities $J_1(\omega)$ and $J_2(\omega)$ as follows:

$$J_1(\omega) = -\frac{1}{\omega}\theta'\left(\frac{\pi}{\omega}\right)-j\theta\left(\frac{\pi}{\omega}\right) \tag{19}$$

$$J_2(\omega) = -\frac{1}{\omega^2}\theta''\left(\frac{\pi}{\omega}\right)-j\frac{1}{\omega}\theta'\left(\frac{\pi}{\omega}\right) \tag{20}$$

where $\theta'(t)$ is the output speed and $\theta''(t)$ in the acceleration of the output.

On the other hand, $G(j\omega)$ can be rewritten from Eq. (17) in the form:

$$G(jn\omega) = U(n\omega)+jV(n\omega) \tag{21}$$

where

$$\left.\begin{aligned}U(n\omega) &= A(n\omega)\cos\Phi(n\omega)\\ V(n\omega) &= A(n\omega)\sin\Phi(n\omega)\end{aligned}\right\} \tag{22}$$

Substitution of Eq. (21) into Eqs. (18), (19) and (20) and subsequent rearrangements give

$$\left.\begin{aligned}\operatorname{Im}J_1(\omega) &= \frac{4KM}{\pi}\left\{\sum_{n}\frac{1}{n}V(n\omega)+\frac{F}{M}\sum_{n}\frac{1}{n}|G(jn\omega)|\sin(n\phi-\alpha_n)\right\}\\ \operatorname{Re}J_1(\omega) &= \frac{4KM}{\pi}\left\{\sum_{n}U(n\omega)-\frac{F}{M}\sum_{n}|G(jn\omega)|\cos(n\phi-\alpha_n)\right\}\\ \operatorname{Im}J_2(\omega) &= \frac{4KM}{\pi}\left\{-\sum_{n}|G(jn\omega)|\cos(n\phi+\alpha_n)+\frac{F}{M}\sum_{n}U(n\omega)\right\}\\ \operatorname{Re}J_2(\omega) &= \frac{4KM}{\pi}\left\{\sum_{n}n|G(jn\omega)|\sin(n\phi+\alpha_n)-\frac{F}{M}\sum_{n}nV(n\omega)\right\}\end{aligned}\right\} \tag{23}$$

where

$$\alpha_n = \tan^{-1}\frac{V(n\omega)}{U(n\omega)} \tag{24}$$

From Eqs. (1), (2) and the requirements at the instant at which the polarity of the Coulomb friction torque reverses, the conditions for the existence of a symmetric self-excited oscillation with angular frequency ω are

$$\theta\left(\frac{\pi}{\omega}\right) = -\frac{h}{2}, \quad \theta'\left(\frac{\pi}{\omega}\right) > 0 \tag{25}$$

$$\theta'\left(\frac{\phi}{\omega}\right) = 0, \quad \theta''\left(\frac{\phi}{\omega}\right) > 0 \tag{26}$$

Rewriting Eqs. (25) and (26) by means of Eqs. (19) and (20),

$$\mathrm{Im}\,J_1(\omega) = -\frac{h}{2}, \quad \mathrm{Re}\,J_1(\omega) < 0 \tag{27}$$

$$\mathrm{Im}\,J_2(\omega) = 0, \quad \mathrm{Re}\,J_2(\omega) < 0 \tag{28}$$

The angular frequency, ω, and the phase difference, ϕ, that satisfy these conditions at the same time are the required ω_0 and ϕ of the self-excited oscillation. Then the angular frequencies at the intersections of $J_1(\omega)$ locus with the $-h/2$ horizontal line on the negative half plane and of $J_2(\omega)$ locus with the negative real axis are to be found for various values of ϕ and F/M as parameters. If the angular frequencies obtained from $J_1(\omega)$ and $J_2(\varphi)$ loci are identical, it satisfies both Eqs. (27) and (28), and hence gives the angular frequency ω_0 of the self-excited oscillation and its phase difference ϕ: that is, the angular frequency and phase difference of the self-excited oscillation for a value of F/M are given by ω and ϕ corresponding to the intersecting point of curves on the ω, ϕ plane for that value of F/M. Each group of curves on ω, ϕ plane corresponds to $J_1(\omega)$ and $J_2(\omega)$ loci with F/M as a parameter. In practice we cannot plot the vectors $J_1(\omega)$ and $J_2(\omega)$ given by Eq. (23) for infinite n, and a definite number of terms are taken. As the first approximation, let $n=1$, then

$$J_1(\omega) = \frac{4KM}{\pi}\left\{G(j\omega) - \frac{F}{M}|G(j\omega)|\exp[j(\alpha_1-\phi)]\right\} \tag{29}$$

$$jJ_2(\omega) = \frac{4KM}{\pi}\left\{-\frac{F}{M}G(j\omega) + |G(j\omega)|\exp[j(\alpha_1+\phi)]\right\} \tag{30}$$

The vector loci of $J_1(\omega)$ and $J_2(\omega)$ of the above expression are easily drawn by plotting the vector locus, $G(j\omega)$, of the linear part of the servomechanism, as is shown concretely by the following example.

3.2 An example of numerical calculations

Let $KM=50$, $T_1=0.1$, $h=10$ in the servomechanism of Fig. 1. The angular frequency ω_0 and phase difference ϕ of self-excited oscillation of the system are to be found by the use of Eqs. (29) and (30).

First the vector locus of the frequency transfer function of the linear part of the system,

$$G(j\omega)=\frac{1}{j\omega(1+j\omega T_1)}$$

is plotted on a complex plane in order to draw vector loci of $J_1(\omega)$ and $J_2(\omega)$.

In Fig. 7, take an arbitrary point A on the vector locus of $G(j\omega)$ and let the angular frequency at the point be ω_1. Equation (29) shows that the vector $J_1(\omega_1)$ is given by the difference of vector $\frac{F}{M}|G(j\omega_1)|\exp[j(\alpha_1-\phi)]$ from vector $G(j\omega_1)$. $\overline{OA}=|G(j\omega_1)|$, and α_1 is the phase angle of vector $G(j\omega_1)$. Take $\overline{OB}=\frac{F}{M}|G(j\omega_1)|$ on $\overline{OA}$ and rotate $\overline{OB}$ clockwise by ϕ, and we obtain $\overline{OB'}=\frac{F}{M}|G(j\omega_1)|\exp[j(\alpha_1-\phi)]$. Hence the vector difference between $\overline{OA}$ and $\overline{OB}$ is $\overline{B'A}$. As $\overline{B'A}=\overline{OC}$, $4KM/\pi$ times $\overline{OC}$ is $\overline{OD}$, i.e. $J_1(\omega_1)$, and point D is one point on the $J_1(\omega)$ locus. Repeating similar processes the $J_1(\omega)$ locus is easily drawn.

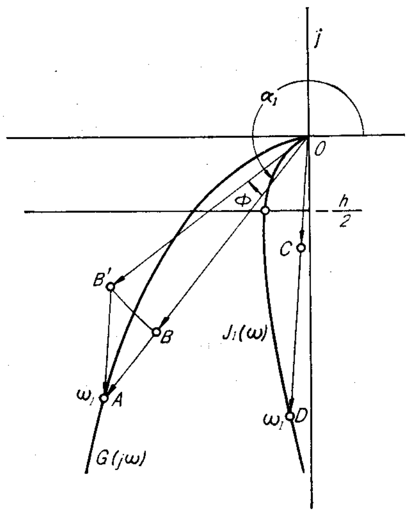

Fig. 7. Construction of $J_1(\omega)$ locus.

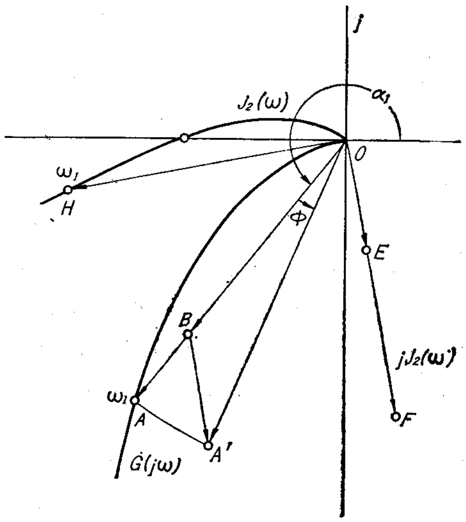

Fig. 8. Construction of $J_2(\omega)$ locus.

Secondly the $J_2(\omega)$ locus is to be drawn by Eq. (30). In Fig. 8, take an arbitrary point A on the vector locus of $G(j\omega)$ and let $\overline{OB}=\frac{F}{M}|G(j\omega_1)|$ on $\overline{OA}$ as in the case of the construction of the $J_1(\omega)$ locus. Then rotating $\overline{OA}$ counterclockwise by ϕ we obtain vector $\overline{OA'}$ and the difference $\overline{BA'}$ between $\overline{OA'}$ and $\overline{OB}$. Taking $\overline{OE}$ so that $\overline{BA'}=\overline{OE}$, and multiplying this by $4KM/\pi$, we obtain $\overline{OF}$, i.e. $jJ_2(\omega_1)$ as is evident from Eq. (30). Hence the required vector $J_2(\omega_1)$ is equal to $\overline{OH}$, obtained by rotating $\overline{OF}$ clockwise by $\pi/2$. Point H is one point on the $J_2(\omega)$ locus. The $J_2(\omega)$ locus is easily drawn by repeating similar processes.

Thus plotting many loci of $J_1(\omega)$ and $J_2(\omega)$ with F/M as parameter for a value of ϕ, the intersecting points of $J_1(\omega)$ loci with the $-h/2$ horizontal line on the negative half plane and the points where the loci of $J_2(\omega)$ intersect the negative real axis determine the angular frequency ω. Then similar processes are repeated by varying ϕ, and plotting ω as a function of ϕ with F/M as parameter, two groups of curves corresponding to $J_1(\omega)$ and $J_2(\omega)$ are drawn as shown in Fig. 9. The values of ω and ϕ given by the intersecting points of these two curves for the same value of F/M satisfy Eqs. (27) and (28) at the same time and hence are the required angular frequency ω_0 and phase difference ϕ of the self-excited oscillation. ω_0 and ϕ thus obtained are plotted as functions of F/M and are shown by the dotted line in Fig. 5. Comparison of this result with the theoretical one shows that there is some error in phase difference ϕ, but the angular frequencies of self-excited oscillation are in very good agreement.

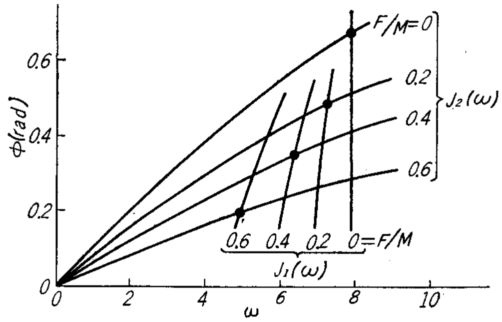

Fig. 9. Determination of the frequency of self-excited oscillation and phase difference.

4. Conclusion

The authors have analyzed the self-excited oscillation taking place in a simple relay servomechanism with Coulomb friction and found analytically the angular frequency of the self-excited oscillation and its precise output speed waveform. Furthermore the approximation method of obtaining the angular frequency and the phase difference between the output speed and the relay output by plotting the vector locus of the linear part of the servomechanism on the complex plane is explained. Also by a numerical example the authors have shown that the results obtained by the approximation method and the precise analysis are in good agreement.