Uniqueness Criterion of Stress State of Maxwell Type Viscoelastic Body

By

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The paper deals with a criterion of uniqueness between stress state and deformation of a *Maxwell* type viscoelastic body, of which stress-strain relations are of the *Lagrang*ian form. If the stress state is given, the body force and the surface displacement are uniquely specified by means of generalized *Hamilton*'s principle. Conversely, the uniqueness criterion for the infinitesimal increment of stress of the given infinitesimal variations of body force, surface force and surface displacement, is that a function of the elastic potential and the dissipation function is positive definite.

1. Iotroduction. The *Lagrang*ian form of the stress-strain relations of the *Maxwell* type viscoelastic body was formulated by the author[1]. In that paper, generalized *Hamilton*'s principle was applied to the equation of energy conservation, which denotes that the sum of the elastic and the dissipative energy of the body is equal to the work done by the body and the surface force on it. On the condition that the elastic potential is a function only of stress and the dissipation function is a function of rate of stress and stress, it was reduced necessarily that such a body is of the *Maxwell* type.

The uniqueness between the stress state and the deformation of hyper-elastic media has been discussed by *Prager*[2], and that of plastic media by *Hill* and *Drucker*[3]. We investigated the uniqueness criterion of deformation of the *Voigt* type body[4]. In that paper we proved that, when the deformed state is given, then the forces acting on the surface and in the body are uniquely determined; conversely, for a given infinitesimal variation of the specified body force and for those of the specified surface force and surface displacement, the uniqueness of the infinitesimal displacement is that the expression of the elastic potential Φ and the dissipation function Ψ:

$$\left[\frac{\partial^2\Phi}{\partial(x^{*j}{}_{,i})\partial(x^{*l}{}_{,k})}+\frac{\partial^2\Psi}{\partial(\dot{x}^{*j}{}_{,i})\partial(x^{*l}{}_{,k})}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2\Psi}{\partial(\dot{x}^{*j}{}_{,i})\partial(\dot{x}^{*l}{}_{,k})}\right)\right]\eta^{ij}\eta^{kl} \qquad (1)$$

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is positive difinite in the nine variables η^{ij} $(i, j=1, 2, 3)$.

In this paper we will discuss the uniqueness of the stress-strain correspondence of a *Maxwell* type viscoelastic body having the *Lagrang*ian form proposed by us.

2. General Stress-Strain Relations. A *Maxwell* type viscoelastic body is in homogeneous undeformed initial state at time $t=0$, which is bounded by a simply connected finite region V with a sufficiently regular surface S. The body is subjected to the body force $\boldsymbol{F}$ in V and its boundary S is subjected to the surface force $\boldsymbol{f}$ on the portion S_f of S and to the displacement $\boldsymbol{u}$ on the portion S_u of S; then it deformes viscoelastically to time t.

The body in the deformed state is in a state of equilibrium. When the elastic potential Φ as a function of stress σ_{ij} is given, and when the dissipation function Ψ as a function of rate of stress $\dot{\sigma}_{ij}$ and stress σ_{ij}, having the viscoelastic conditions

$$\frac{d}{dt}\left(\frac{\partial \Psi}{\partial \dot{\sigma}_{ij}}\right)-\frac{\partial \Psi}{\partial \sigma_{ij}}=0 \qquad (i, j = 1, 2, 3)\,, \tag{2}$$

is given; then the strain e_{ij} is expressed by generalized *Hamilton*'s principle as[1)]

$$e_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}}+\frac{\partial \Psi}{\partial \dot{\sigma}_{ij}} \qquad (i, j = 1, 2, 3) \tag{3}$$

or

$$\dot{e}_{ij} = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \sigma_{ij}}\right)+\frac{\partial \Psi}{\partial \sigma_{ij}} \qquad (i, j = 1, 2, 3)\,, \tag{4}$$

where the strain component e_{ij} is denoted by the displacement vector u_i as

$$e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}) \qquad (i, j = 1, 2, 3)\,. \tag{5}$$

3. Uniqueness Criterion. In the first place, we will investigate the uniqueness of the body force and the surface displacement; when the stress satisfying the equilibrium equation

$$\sigma_{ij,j}+F_i = 0 \qquad (i = 1, 2, 3) \quad \text{in } V \tag{6}$$

and the boundary condition

$$\lambda_i\sigma_{ij} = f_j \qquad (j = 1, 2, 3) \quad \text{on } S_f \tag{7}$$

is given, where F_i and f_i are the body force per unit volume and the surface force per unit area and λ_i is the exterior normal on the boundary.

A volume element, which has the volume dV, is subjected to the force

$$F_i\,dV = -\sigma_{ij,j}\,. \tag{8}$$

Now we apply generalized *Hamilton*'s principle to the *Maxwell* type viscoelastic deformation. Then the tensor quantity $\bar{e}_{ij}=\frac{\partial\Phi}{\partial\sigma_{ij}}+\frac{\partial\Psi}{\partial\dot{\sigma}_{ij}}$ must be subjected to

$$\delta^{ikm}_{jln}\bar{e}_{kl,mn}=0 \qquad \text{in } V \tag{9}$$

under the conditions (2) and (6), where δ^{ikm}_{jln} is a generalized *Kronecker* delta. Equation (9) is the compatibility equations, when $\bar{e}_{ij}$ is considered as a component of strain. Therefore (9) shows the existence of three single valued quantities $\bar{u}_i$ having the relations

$$\bar{e}_{ij}=\frac{1}{2}(\bar{u}_{i,j}+\bar{u}_{j,i}), \tag{10}$$

if V is a simply connected region[5]. Generalized *Hamilton*'s principle requires also the condition under the relations (10) and (7),

$$\bar{u}_i=u_i \qquad \text{on } S_u, \tag{11}$$

where u_i is a given displacement vector on the portion S_u of S.

In the second place, we will find a condition for the uniqueness of the equilibrium stress state σ_{ij} for given body force and given boundary conditions.

When the equilibrium stress state is given with specified body force $\boldsymbol{F}$, surface force $\boldsymbol{f}$ and surface displacement $\boldsymbol{u}$, the uniqueness of the infinitesimal increment of stress $d\sigma_{ij}$ will be discussed for a given infinitesimal variation δF_i of the specified body force and for given infinitesimal variations δf_i and δu_i of the specified surface force and of the surface displacement.

The displacement variations $\delta\bar{u}_i$ and $\delta\bar{\bar{u}}_i$ and the stress variations $\delta\bar{\sigma}_{ij}$ and $\delta\bar{\bar{\sigma}}_{ij}$ are assumed in this boundary value problem. By (3)

$$\delta\bar{e}_{ij}=\frac{\partial^2\Phi}{\partial\sigma_{ij}\partial\sigma_{kl}}\delta\bar{\sigma}_{kl}+\frac{\partial^2\Psi}{\partial\dot{\sigma}_{ij}\partial\sigma_{kl}}\delta\bar{\sigma}_{kl}+\frac{\partial^2\Psi}{\partial\dot{\sigma}_{ij}\partial\dot{\sigma}_{kl}}\delta\dot{\bar{\sigma}}_{kl} \tag{12}$$

and the corresponding formula holds for $\delta\bar{\bar{e}}_{ij}$; in both formulae the second differentiations of the elastic potential and the dissipation function have the same values, for they are composed of the given equilibrium stress state. The stress variations $\delta\bar{\sigma}_{ij}$ and $\delta\bar{\bar{\sigma}}_{ij}$ satisfy

$$\left.\begin{aligned}(\delta\bar{\sigma}_{ij})_{,j}+\delta F_i&=0\\(\delta\bar{\bar{\sigma}}_{ij})_{,j}+\delta F_i&=0\end{aligned}\right\} \tag{13}$$

by (6), thus

$$(\delta\bar{\sigma}_{ij}-\delta\bar{\bar{\sigma}}_{ij})_{,j}=0 \tag{14}$$

holds.

By means of given boundary conditions, we have and

$$\left.\begin{aligned}\lambda_i(\delta\bar{\sigma}_{ij}-\delta\bar{\bar{\sigma}}_{ij}) &= 0 \qquad \text{on } S_f\\ \delta\bar{u}_i-\delta\bar{\bar{u}}_i &= 0 \qquad \text{on } S_u\,.\end{aligned}\right\} \tag{15}$$

Then

$$\lambda_i(\delta\bar{\sigma}_{ij}-\delta\bar{\bar{\sigma}}_{ij})(\delta\bar{u}_j-\delta\bar{\bar{u}}_j) = 0 \tag{16}$$

holds on the entire surface S at any time t' $(0 \leqq t' \leqq t)$. Therefore we have

$$\int_0^t \left[\iint_S \lambda_i(\delta\bar{\sigma}_{ij}-\delta\bar{\bar{\sigma}}_{ij})(\delta\bar{u}_j-\delta\bar{\bar{u}}_j)dS\right]_{t'} dt' = 0\,. \tag{27}$$

By means of *Gauss*' theorem and (14), we get

$$\int_0^t \left[\iiint_V (\delta\bar{\sigma}_{ij}-\delta\bar{\bar{\sigma}}_{ij})(\delta\bar{e}_{ij}-\delta\bar{\bar{e}}_{ij})dV\right]_{t'} dt' = 0\,, \tag{18}$$

where

and

$$\left.\begin{aligned}\delta\bar{e}_{ij} &= \frac{1}{2}\{(\delta\bar{u}_i)_{,j}+(\delta\bar{u}_j)_{,i}\}\\ \delta\bar{\bar{e}}_{ij} &= \frac{1}{2}\{(\delta\bar{\bar{u}}_i)_{,j}+(\delta\bar{\bar{u}}_j)_{,i}\}\end{aligned}\right\} \tag{19}$$

and we use the symmetry property of the stress tensor.
Equation (18) leads to

$$\int_0^t \left[\iiint_V \left[\frac{\partial^2\Phi}{\partial\sigma_{ij}\partial\sigma_{kl}}+\frac{\partial^2\Psi}{\partial\dot{\sigma}_{ij}\partial\sigma_{kl}}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2\Psi}{\partial\dot{\sigma}_{ij}\partial\dot{\sigma}_{kl}}\right)\right]\times \right.$$
$$\left. \times(\delta\bar{\sigma}_{ij}-\delta\bar{\bar{\sigma}}_{ij})(\delta\bar{\sigma}_{kl}-\delta\bar{\bar{\sigma}}_{kl})dV\right]_{t'} dt' = 0 \tag{20}$$

by (12) and the integration by parts with respect to time.

When the elastic potential Φ and the dissipation function Ψ have the property that the quadratic form

$$\left[\frac{\partial^2\Phi}{\partial\sigma_{ij}\partial\sigma_{kl}}+\frac{\partial^2\Psi}{\partial\dot{\sigma}_{ij}\partial\sigma_{kl}}-\frac{1}{2}\frac{d}{dt}\left(\frac{\partial^2\Psi}{\partial\dot{\sigma}_{ij}\partial\dot{\sigma}_{kl}}\right)\right]\eta_{ij}\eta_{kl} \tag{21}$$

is positive definite in the nine variables η_{ij} $(i, j=1, 2, 3)$ for arbitrary values of σ_{ij} and $\dot{\sigma}_{ij}$, we can put

$$\delta\bar{\sigma}_{ij} = \delta\bar{\bar{\sigma}}_{ij} \qquad (i, j = 1, 2, 3) \tag{22}$$

in the region V and at any time.
Equation (22) experesses the fact that the assumed two variations of stress $\delta\bar{\sigma}_{ij}$ and $\delta\bar{\bar{\sigma}}_{ij}$ for δF_i, δf_i and δu_i are identical.

The cumulation of the unique infinitesimal increments of stress produces the unique stress. Thus the criterion that the stress state of a given *Maxwell* type viscoelastic body are unique, is that the elastic potential and the dissipation function of the body make (21) positive definite for arbitrary values of η_{ij}.

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