Improvement of the Non-Stationary Response by the Introduction of a Non-Linear Element of Zero-Memory Type

By

Yoshikazu SAWARAGI*, Yoshifumi SUNAHARA* and Takayoshi NAKAMIZO*

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A method of improving the response of control systems subjected to suddenly applied random inputs by introducing a non-linear element is presented. The technique described here is based on the concept of non-stationary equivalent linearization.

The description begins with the calculation of optimum equivalent gain to minimize the mean squared error. From this result, the principal line of attack is to determine the introduced non-linear characteristic and to evaluate how much better it can be expected to perform than its linear counterpart. It is also shown that an intentional non-linear control system gives a considerable reduction of mean squared error. Detailed aspects of the numerical procedure are illustrated by a typical example.

List of Symbols

$v(t)$ and $u(t)$: desired signal and random disturbance respectively

$z(t)$ and $y(t)$: input and output of a non-linear element respectively

$e(t)$: error signal in the system

$f(z)$: non-linear transfer characteristic

κ_t : non-stationary equivalent gain

κ_∞ : stationary equivalent gain

$G(s)$: transfer function of the controlled system

$W_v(t;\kappa_t)$ and $W_u(t;\kappa_t)$: impulse responses of equivalent linearized control system subjected to the signals $v(t)$ and $u(t)$ respectively

$\psi_e(t)$: mean squared value of the error signal $e(t)$

$\phi_v(\tau)$ and $\phi_u(\tau)$: auto-correlation functions of the desired signal $v(t)$ and the disturbance $u(t)$ respectively

$p(z;t)$: time-dependent probability density function of the signal $z(t)$

* Department of Applied Mathematics and Physics.

1. Introduction

Various works for improving the performance of control systems subjected to stationary random inputs by introducing a non-linear element of zero-memory type have been presented by J. L. Douce, R. E. King[1] and the authors.[2] In practice, there are many cases where the characteristics of the random environment suddenly change and as a result the system is excited by a suddenly applied random signal. As we have already pointed out in our earlier paper,[3] the transient effect of statistical behavior of the response must be taken into account as the non-stationary time series. For the purpose of evaluating the non-stationary response of control systems containing a non-linear element of zero-memory type, the non-stationary linearization technique was established by the author.[4] Results of the analytical studies suggest the possibility of remarkable improvement of the control performance by utilizing the non-linear characteristic in both stationary and non-stationary aspects.

Before proceeding with the present discussion, we consider briefly the statistical behavior of a linear control system as shown in Fig. 1 which is excited by a suddenly applied random signal contaminated by a random disturbance. Fig. 2 shows the mean squared value of the error response with the value of

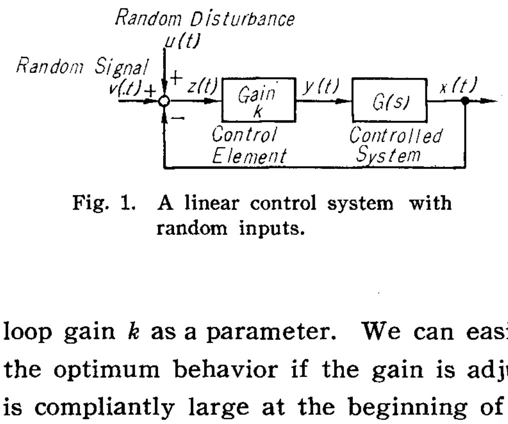

Fig. 1. A linear control system with random inputs.

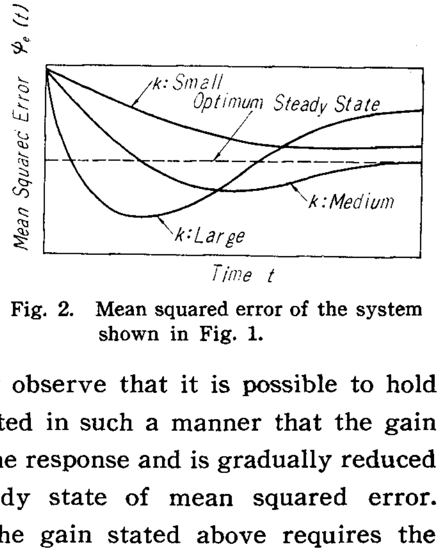

Fig. 2. Mean squared error of the system shown in Fig. 1.

loop gain k as a parameter. We can easily observe that it is possible to hold the optimum behavior if the gain is adjusted in such a manner that the gain is compliantly large at the beginning of the response and is gradually reduced to a certain value to minimize the steady state of mean squared error. Although the manipulation of changing the gain stated above requires the design of a time-variant control system, it is considerably difficult to solve the integral equation which appears in the theoretical procedure. Remarkable characteristics of non-linearities explored by the authors give the same contribution as stated above on the improvement of control performance. Central problems are, therefore, to determine what types of non-linear characteristics should be introduced into control systems with known random environments and to show how much better they can be expected to perform than their linear counterparts.

2. Minimization of the Mean Squared Error

The basic non-linear control system to be considered here is illustrated in Fig. 3 in which $f(z)$ is an intentionally introduced non-linear characteristic and $G(s)$ is a transfer function of the controlled system. Since both the random signal $v(t)$ and the disturbance $u(t)$ are suddenly applied at time $t=0$, then the intentionally introduced non-linear element, $f(z)$, can be analytically replaced by the non-stationary equivalent gain κ_t defined by[5)]

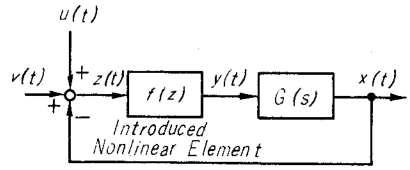

Fig. 3. A non-linear control system with random inputs.

$$\kappa_t = \frac{\int_0^t dt \int_{-\infty}^{\infty} zf(z)p(z\,;t)dz}{\int_0^t dt \int_{-\infty}^{\infty} z^2 p(z\,;t)dz}\,. \tag{2.1}$$

Now the error signal $e(t)$ of the equivalent linearized control system can be expressed as

$$\begin{aligned} e(t) &= v(t)-x(t) \\ &= \int_0^t W_v(t-\tau\,;\kappa_t)v(\tau)d\tau - \int_0^t W_u(t-\tau\,;\kappa_t)u(\tau)d\tau \end{aligned} \tag{2.2}$$

where

$$\left.\begin{aligned} W_u(\tau\,;\kappa_t) &= \mathcal{L}^{-1}\left[\frac{\kappa_t G(s)}{1+\kappa_t G(s)}\right] \\ W_v(\tau\,;\kappa_t) &= \mathcal{L}^{-1}\left[\frac{1}{1+\kappa_t G(s)}\right] \end{aligned}\right\}. \tag{2.3}$$

If the desired signal $v(t)$ and the random disturbance $u(t)$ are statistically independent, then the mean squared error $\psi_e(t)$ becomes

$$\psi_e(t) = \psi_{ev}(t)+\psi_{eu}(t)\,, \tag{2.4}$$

in which the respective portions can be represented as

$$\psi_{ev}(t) = \int_0^t W_v(t-\tau_1\,;\kappa_t)d\tau_1 \int_0^t W_v(t-\tau_2\,;\kappa_t)\phi_v(\tau_1-\tau_2)d\tau_2 \tag{2.5}$$

and

$$\psi_{eu}(t) = \int_0^t W_u(t-\tau_1\,;\kappa_t)d\tau_1 \int_0^t W_u(t-\tau_2 : \kappa_t)\phi_u(\tau_1-\tau_2)d\tau_2\,. \tag{2.6}$$

The result of Eqs. (2.4), (2.5) and (2.6) may be generally evaluated as a function of κ_t as well as of time t, i.e.,

$$\psi_e(t) = g_e(\kappa_t\,;\,t)\,. \tag{2.7}$$

Plotting Eq. (2.7) with values of κ_t as a parameter, we can obtain the optimum response represented by its envelope shown in Fig. 4. From Fig. 4, the optimum equivalent gain to minimize the mean squared error for all time instants can, therefore, be determined as a function of time t.

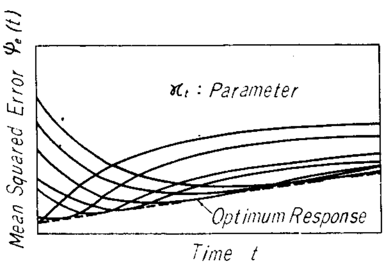

Fig. 4. Evaluation of the mean squared error.

3. Determination of the Intentional Non-Linearity

It is already shown in Fig. 4 that the control system will have the minimum mean squared error for all time instants, if the intentionally introduced non-linear element with the desired equivalent gain is determined. The equivalent gain, κ_t, defined by Eq. (2.1) depends on the average power of the signal $z(t)$ and then on time instant t.

Since the signal $z(t)$ can be represented by

$$z(t) = \int_0^t W_v(t-\tau\,;\,\kappa_t)\{v(\tau)+u(\tau)\}d\tau\,, \tag{3.1}$$

the mean squared value $\psi_z(t)$ is expressed as

$$\psi_z(t) = \int_0^t W_v(t-\tau_1\,;\,\kappa_t)d\tau_1 \int_0^t W_v(t-\tau_2\,;\,\kappa_t)\{\phi_v(\tau_1-\tau_2)+\phi_u(\tau_1-\tau_2)\}d\tau_2 \tag{3.2}$$

and more compactly

$$\psi_z(t) = g_z(\kappa_t\,;\,t)\,. \tag{3.3}$$

Let the intentional non-linear characteristic be of the form

$$f(z) = \sum_n c_n z^{2n-1}\,, \tag{3.4}$$

then the equivalent gain κ_t yields

$$\kappa_t = \sum_n c_n \{\prod_{k=1}^{n}(2k-1)\} \frac{\overline{\psi_z^n(t)}}{\psi_z(t)}\,, \tag{3.5}$$

where

$$\overline{\psi_z^m(t)} = \int_0^t \psi_z^m(\tau)d\tau\,. \tag{3.6}$$

Substituting Eq. (3.3) for $\psi_z(t)$ in Eq. (3.5), the equivalent gain κ_t can be calculated as a function of time t by using a simple graphical procedure, i.e.,

$$\kappa_t = g_\kappa(c_n\,;t)\,. \tag{3.7}$$

The coefficients c_n's can be determined by comparing Eq. (3.7) with the opitimum value obtained in the previous section. Detailed procedure will be shown in the following section.

4. An Illustrative Example

A block diagram of the control system is shown in Fig. 5. The system is at rest prior to time $t=0$ and after that time is excited by two gaussian inputs of which one is a desired signal $v(t)$ and the other is a disturbance $u(t)$, the spectral densities being respectively as follows;

$$S_v(\omega) = \frac{2}{1+\omega^2} \tag{4.1}$$

and

$$S_u(\omega) = \frac{16\omega^2}{(1+\omega^2)(9+\omega^2)}\,. \tag{4.2}$$

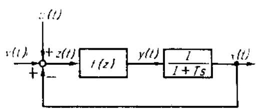

Fig. 5. Block diagram of the illustrative example.

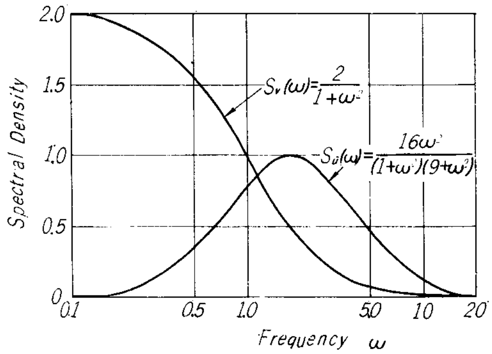

Fig. 6. Spectral densities of the system inputs.

The spectral densities are shown in Fig. 6.

4.1. Calculation of the Optimum Equivalent Gain

Letting $T=1$ for the convenience of calculation, we then have

$$W_u(\tau\,;\kappa_t) = \kappa_t \exp\{-(1+\kappa_t)\tau\} \tag{4.3}$$

$$W_v(\tau\,;\kappa_t) = \delta(\tau)-\kappa_t \exp\{-(1+\kappa_t)\tau\}\,. \tag{4.4}$$

With the help of Appendix, the respective portions of the mean squared error given by Eqs. (2.5) and (2.6) become

$$\psi_{ev}(t) = \frac{1}{1+\kappa_t}[1+\kappa_t \exp\{-2(1+\kappa_t)t\}] \tag{4.5}$$

and

$$\psi_{eu}(t) = \frac{2\kappa_t^2}{(2+\kappa_t)(4+\kappa_t)}+\frac{2\kappa_t}{\kappa_t-2}\exp\{-2(1+\kappa_t)t\}+\frac{2\kappa_t}{\kappa_t+2}\exp\{-(\kappa_t+2)t\}$$
$$-\frac{6\kappa_t^2}{(\kappa_t-2)(\kappa_t+4)}\exp\{-(4+\kappa_t)t\}\,. \tag{4.6}$$

The mean squared error yields

$$\begin{aligned}\psi_e(t) &= \psi_{ev}(t)+\psi_{eu}(t)\\ &= \frac{2\kappa_t^3+3\kappa_t^2+6\kappa_t+8}{(1+\kappa_t)(2+\kappa_t)(4+\kappa_t)}+\frac{3\kappa_t^2}{(1+\kappa_t)(\kappa_t-2)}\exp\{-2(1+\kappa_t)t\}\\ &+\frac{2\kappa_t}{2+\kappa_t}\exp\{-2(1+\kappa_t)t\}-\frac{6\kappa_t^2}{(4+\kappa_t)(\kappa_t-2)}\exp\{-(4+\kappa_t)t\}\,. \end{aligned} \tag{4.7}$$

Numerical results of Eq. (4.7) with the value of κ_t as a parameter are shown in Fig. 7 by solid curves. The dotted curve can be determined as the required

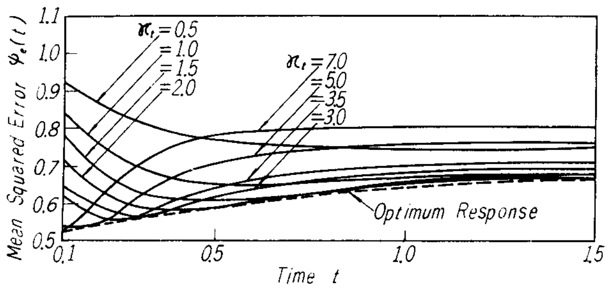

Fig. 7. Mean squared error of the system shown in Fig. 5.

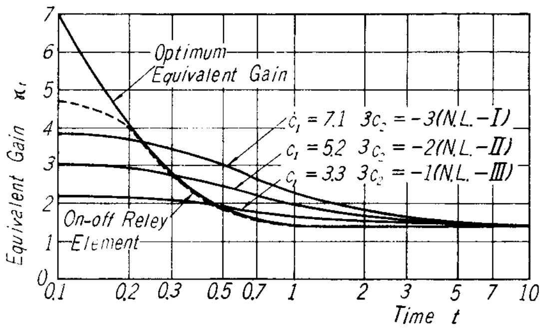

Fig. 8. Graphical procedure for determining non-linearities to be introduced.

optimum response to the random system inputs whose spectral densities are given by Eqs. (4.1) and (4.2). The optimum equivalent gain κ_t can, therefore, be calculated as shown in Fig. 8.

4.2. Determination of the Non-linearity

The remainder of the present problem is to determine a non-linear transfer characteristic of which the equivalent gain agrees with the result obtained above. Let the intentional non-linear characteristic be of the form

$$f(z) = c_1 z + c_2 z^3\,, \tag{4.8}$$

which is considered as the truncated series of Eq. (3.4). From Eq. (3.5), the non-stationary equivalent gain of such a non-linear element is given by

$$\kappa_t = c_1 + 3c_2\, h(t)\,, \tag{4.9}$$

where

$$h(t) = \overline{\psi_z^2(t)}/\overline{\psi_z(t)}\,. \tag{4.10}$$

On the other hand, from Eq. (3.2), the mean squared value of the signal $z(t)$ becomes

$$\frac{1}{3}\psi_z(t) = \frac{3\kappa_t+4}{(1+\kappa_t)(4+\kappa_t)} - \frac{4\kappa_t}{(4+\kappa_t)(\kappa_t-2)} \exp\{-(4+\kappa_t)t\} + \frac{\kappa_t^2}{(1+\kappa_t)(\kappa_t-2)} \exp\{-2(1+\kappa_t)t\}\,. \tag{4.11}$$

Eq. (4.10) can, therefore, be evaluated as shown in Fig. 9, by substituting the value of optimum equivalent gain for κ_t in Eq. (4.11).

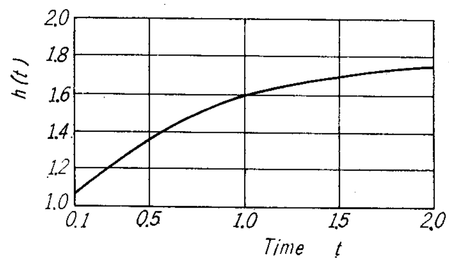

Fig. 9. Numerical plot of Eq. (4.10).

In order to determine the coefficients, c_1 and c_2, we must take the following two conditions into account:

i) $c_2<0$; This is readily obtained from the comparison of Fig. 8 and Fig. 9.

ii) Since the optimum behavior is also required in the steady state, we have

$$\kappa_\infty = c_1 + 3c_2\, h(t)\,|_{t=\infty}\,, \tag{4.12}$$

where κ_∞ is the stationary equivalent gain.[6] From the condition (ii), since $\kappa_\infty=1.4$ and $h(t)\,|_{t=\infty}=1.898$, we obtain

$$\kappa_t = c_1 + \frac{1.4-c_1}{1.898}h(t) \tag{4.13}$$

and from the condition (i), we have

$$c_1 > 1.4\,.$$

The result of Eq. (4.13) is shown in Fig. 8 with the value of c_1 (or c_2) as a parameter. Three kinds of non-linear characteristics corresponding to the above are shown in Fig. 10.

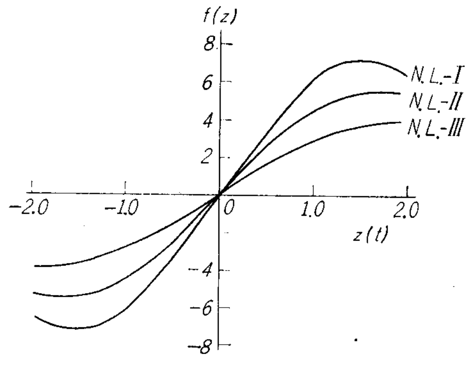

Fig. 10. Non-linear characteristics to be introduced.

4.3. An Alternative Method of Determination of Non-Linearity

The optimum equivalent gain can be shown on $\kappa_t \sim h(t)$ plane by a solid curve in Fig. 11. We can approximate it by the straight line of the form given by Eq. (4.9) taking the previous two conditions into account. Three kinds of non-linearities shown in Fig. 10 can be plotted by chain lines in Fig. 11.

The feature of Fig. 8 or Fig. 11 reveals that a considerably large value of optimum equivalent gain is required at the beginning of the response. On the basis of this fact, the form of the non-linear characteristic is adequately assumed to be an on-off relay i.e.,

$$f(z) = a\,\mathrm{sgn}\,(z)\,, \qquad (4.14)$$

where a is an adjustable value. The non-stationary equivalent gain of the on-off relay element is given by

$$\kappa_t = \sqrt{\frac{2}{\pi}}\,a\,\overline{\psi_z^{1/2}(t)}/\overline{\psi_z(t)} \qquad (4.15)$$

which expresses a linear relation with gradient $\sqrt{\frac{2}{\pi}}a$ on $\kappa_t \sim \overline{\psi_z^{1/2}(t)}/\overline{\psi_z(t)}$ plane. By a graphical procedure as shown in Fig. 12, we can readily determine the clipping level a of the on-off relay as

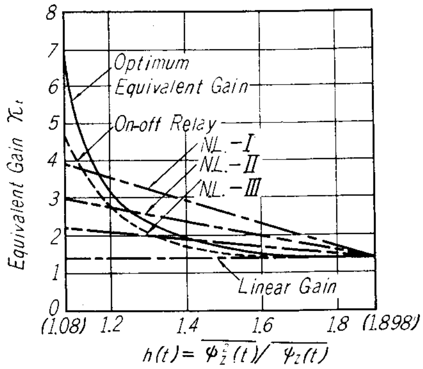

Fig. 11. Equivalent gain vs. $h(t)$.

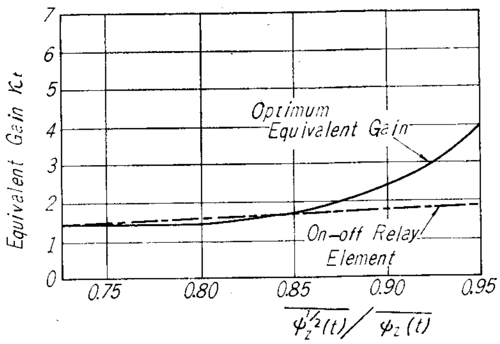

Fig. 12. Graphical procedure for determining the on-off clipping level.

$$a = 2.42\,.$$

The equivalent gain of such an on-off relay element is plotted in Figs. 8 and 11, from which the on-off relay element seems to be a good approximation.

4.4. Discussion

We can thus determine the transfer characteristic of the intentionally introduced non-linearity, and the non-linear control system can be constructed in such a way that the mean squared error is minimized. Here we will direct our attention to the important matter of how the actual control performance is improved by introducing such a non-linear element of zero-memory type. Replacing the non-linear element by the non-stationary equivalent gain as defined by Eq. (2.1), the mean squared error of the intentional non-linear control system can be expressed in the form of Eq. (4.7). On the other hand,

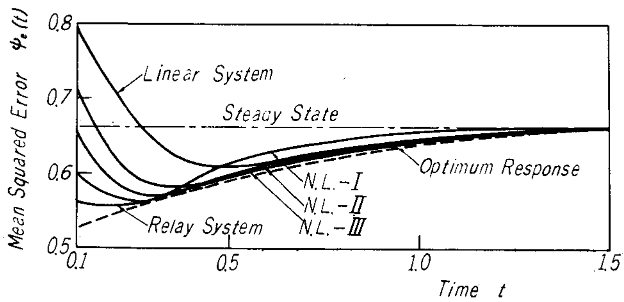

Fig. 13. Comparison of the control performances with the desired response.

by using Eq. (4.11), the mean squared error can be evaluated, as already presented by the authors.[6] Fig. 13 shows the comparison of the time dependent features of the mean squared error corresponding to the respective non-linear characteristics.

5. Conclusions

This paper shows the synthetical procedure of improving the control performance of the system excited by suddenly applied random inputs utilizing a non-linear element of zero-memory type. The procedure is briefly summarized in the following two steps;

(1) graphical determination of optimum non-stationary equivalent gain

(2) derivation of the non-linear characteristic from the result of step (1).

Although the fundamental concept is of the equivalent linearization technique, and the effect of distortion due to an introduced non-linear element is not taken into account, it has been shown that an intentional introduced non-linear element can effect considerable improvement to the transient behavior of the control system subjected to a random signal contaminated by a random disturbance. In our example considered in this paper, it is very interesting that the on-off relay element produces a good effect to the improvement of control performance. There is a possibility that the method may be extended to a case where system input consists of a deterministic function contaminated by a random noise.

Appendix

Let us consider the case where the system with its impulse response $W(\tau;\kappa_t)$, is at rest prior to time $t=0$ and after that time receives a stationary random input with its spectral density:

$$S(\omega) = \frac{2\beta\phi_i}{\beta^2+\omega^2}. \tag{A.1}$$

The mean squared value of the output at time t is given by[7]

$$\frac{\psi_o}{\psi_i} = F^2(t) + 2\beta\int_0^t F^2(\tau)d\tau \tag{A.2}$$

where

$$F(t) = \int_0^t W(t-\tau\,;\kappa_t)\exp(-\beta\tau)d\tau\,. \tag{A.3}$$

Using this relation, Eq. (4.7) can be obtained by applying Eqs. (4.3) and (4.4) to Eq. (A.2). Substituting Eq. (4.3) for $W(\tau\,;\kappa_t)$ in Eq. (A.3), we have

$$F(t) = K_\beta\,[\exp(-\beta t)-\exp\{-(1+\kappa_t)t\}], \tag{A.4}$$

where

$$K_\beta = \frac{1}{1+\kappa_t-\beta}\,. \tag{A.5}$$

Then we have also

$$\begin{aligned}\int_0^t F^2(\tau)d\tau = {} & K_\beta^2\left[\frac{1}{2\beta}-\frac{2}{1+\kappa_t+\beta}+\frac{1}{2(1+\kappa_t)}\right]\\ & -K_\beta^2\left[\frac{1}{2\beta}\exp(-\beta t)-\frac{2}{1+\kappa_t+\beta}\exp\{-(1+\kappa_t+\beta)t\}\right.\\ & \left.+\frac{1}{2(1+\kappa_t)}\exp\{-2(1+\kappa_t)t\}\right].\end{aligned} \tag{A.6}$$

Therefore, from Eq. (A.2), we obtain the following result:

$$\begin{aligned}\frac{\psi_o}{K_\beta^2\psi_i} = {} & \frac{1+\kappa_t+\beta}{1+\kappa_t}-\frac{4\beta}{1+\kappa_t+\beta}-\frac{2(1+\kappa_t-\beta)}{1+\kappa_t+\beta}\exp\{-(1+\kappa_t+\beta)t\}\\ & +\frac{1+\kappa_t-\beta}{1+\kappa_t}\exp\{-2(1+\kappa_t)t\}\,.\end{aligned} \tag{A.7}$$

Now, Eq. (4.2) can be rewritten as

$$S_u(\omega) = \frac{18}{9+\omega^2}-\frac{2}{1+\omega^2}\,. \tag{A.8}$$

From Eq. (A.7), we have respectively

$$\left.\begin{aligned}\psi_{o1} = {} & \frac{\kappa_t^2}{(1+\kappa_t)(2+\kappa_t)}-\frac{2\kappa_t}{\kappa_t+2}\exp\{-(\kappa_t+2)t\}+\frac{\kappa_t}{1+\kappa_t}\exp\{-2(1+\kappa_t)t\}\\ \frac{\psi_{o3}}{3} = {} & \frac{\kappa_t^2}{(1+\kappa_t)(4+\kappa_t)}-\frac{2\kappa_t^2}{(\kappa_t-2)(\kappa_t+4)}\exp\{-(\kappa_t+4)t\}\\ & +\frac{\kappa_t^2}{(1+\kappa_t)(\kappa_t-2)}\exp\{-2(1+\kappa_t)t\}\end{aligned}\right\}. \tag{A.9}$$

Therefore we get finally Eq. (4.6) by computing

$$\psi_{eu}(t) = \psi_{o3}-\psi_{o1}\,. \tag{A.10}$$

Eq. (4.11) can, of course, be accomplished in a similar way without any difficulties.

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