Existing Conditions for Self-Oscillation in an Improved On-Off Control System

By

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This paper deals with the self-oscillation occurring in an improved on-off control system, in which the on-off controller is compensated by two first-order linear elements and the controlled system is a first-order linear element with a dead time. The existing conditions for self-oscillation in this system are exactly derived by the periodicity of the time responses of various variables under some assumptions.

The present analysis has revealed the followings :

a) in respect to the oscillated amplitude of a controlled variable, the performance of a usual on-off control system is improved by introducing compensating elements

b) as the value of K_p/K_f is changed monotonously, one mode of self-oscillation changes into another one, that is, the amplitude and period of self-oscillation change discontinuously

c) for some values of K_p/K_f, two or more solutions satisfy the existing conditions for self-oscillation

d) the self-oscillation in this improved on-off control system never vanishes with an extremely small value of K_p/K_f, but the amplitude and period of the oscillated variables become very small values.

1. Introduction

It is well known that on-off or relay control methods are very useful for process controls and servomechanisms. There are several reasons for this. First, the relay represents the simplest possible amplifier. Second, on-off operation of the power element or actuator represents the most economical use of the installed capacity. Third, with on-off control the design and construction, indeed the entire principle of operation of the output element, may be simplified with consequent improvement in reliability and economy and saving in weight. Since, however, the output of on-off control elements are switched abruptly in accordance with an input, this operation results in an unsatisfactory oscillation of controlled variables.

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There are some possible methods of compensating for the disadvantage just mentioned:

a) introduction of proportional and derivative characteristics into the measuring element

b) application of a sufficient high frequency signal from a source external to the system

c) compensation for the on-off control element by some linear elements

d) endowment of negative hysteresis characteristics with the on-off control element

Many workers[1] have studied the transient or frequency response of improved on-off controllers or control systems by method (c), however their papers did not mention the existing conditions and their stability in regard to steady self-oscillations occurring in the improved on-off control systems.

This paper deals with existing conditions for self-oscillations occurring in the improved on-off control system shown in Fig. 1, where a symmetrical on-off relay element compensated by two first-order linear elements constructs an on-off controller. The controlled system is a first-order linear element with a dead time, and the final values of controlled variables corresponding to the only two outputs from the on-off controller, $\pm A$, are assumed to be symmetrical by the set point.

In order to analyze self-oscillations in such a nonlinear control system, the phase plane method, the describing function method and the method[2] by the characteristic property of relay control system are useful. If, however, the characteristics of linear part of nonlinear control system contains a dead time, it is troublesome to use these method for analysis, furthermore the results of the analysis are not very accurate in spite of the laborious work.

In this paper the existing conditions for steady self-oscilations in the given

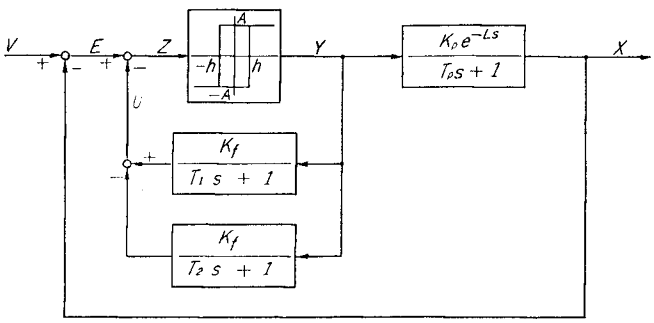

Fig. 1 Block diagram of the improved on-off control system

on-off control system are described by use of the periodicity and symmetricity of steady self-oscillations in the time domain. The next paper will deal with the stability of the self-oscillations in the on-off control system shown in Fig. 1.

2. Analysis

To determine the wave shapes of self-oscillations in the on-off control system shown in Fig. 1, it is sufficient to study the responses of oscillated variables in either the on- or off-period, because the on- and off-periods of the symmetrical relay element are equal to each other and therefore the responses of the controlled variable in the on- and off-periods are symmetrical by the set point.

If the time origin is chosen at the instant when the controlled variable is switched from the increasing to the decreasing process, the response of the controlled variable in the decreasing process is given by

$$x(\tau) = x(0)\, e^{-\tau/T_p} - AK_p(1-e^{-\tau/T_p}) \tag{1}$$

where T_p and K_p are the time constant and the proportional gain of the controlled system, respectively, and $x(0)$ is the initial value and equal to the amplitude of self-oscillation of x_m, namely,

$$x(0) = x_m \tag{2}$$

Now, letting P be the period of self-oscillation in the system, the next equation is derived by Eq. (1) and the symmetricity of $x(\tau)$

$$x\left(\frac{P}{2}\right) = x_m e^{-P/2T_p} - AK_p(1-e^{-P/2T_p}) = -x_m$$

Therefore, the relation between the period of self-oscillation and the amplitude of oscillation of the controlled variable is rewritten by the above equation as

$$\frac{x_m}{AK_p} = \tanh \frac{P}{4T_p} \tag{3}$$

and is illustrated in Fig. 2[3].

2.1 Case without compensating elements[3]

Let us consider the behaviour of a usual on-off control system without the compensating elements shown in Fig. 3.

Because of the dead time, L, of the controlled element, the changing time of direction of the controlled variable is delayed by L from the switching time of the relay element, namely, the switching time of the relay element from

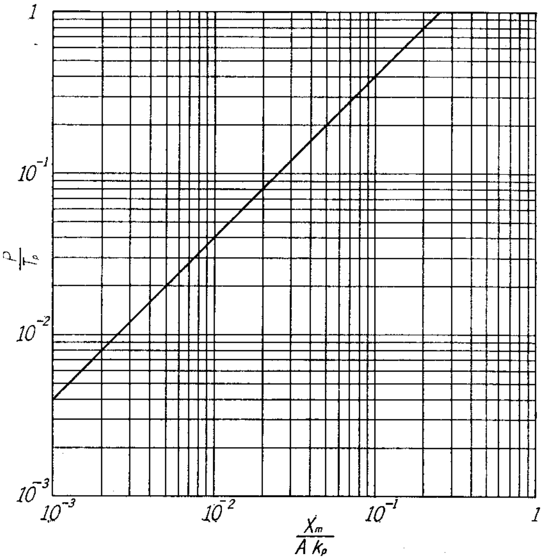

Fig. 2 Relation between the period of self-oscillation and the oscillated amplitude of the controlled variable

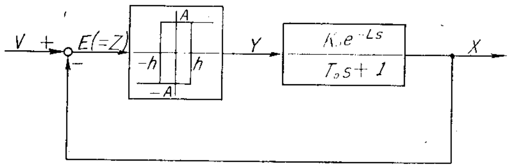

Fig. 3 Block diagram of a usual on-off control system

"off" to "on" is expressed as

$$\tau = P/2-L$$

at which the negative value of the controlled variable is equal to the value of the hysteresis of the positive side in the relay element. This physical expression is given in a mathematical form

$$h = -x_m e^{-P/2T_p+L/T_p} + AK_p(1-e^{-P/2T_p+L/T_p})$$

from Eqs. (1) and (2), or in nondimensional form

$$\frac{h}{AK_p} = 1-\left(1+\frac{x_m}{AK_p}\right)e^{-P/T_p+L/T_p}$$

This formula may be rewritten by Eq. (3) as

$$\frac{x_m}{AK_p} = 1-\left(1-\frac{h}{AK_p}\right)e^{-L/T_p} \tag{4}$$

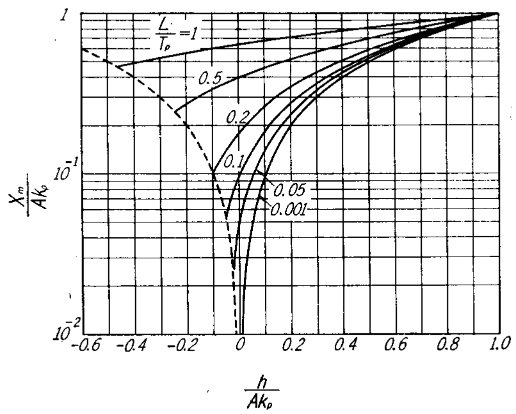

Fig. 4 Relation between the oscillated amplitude of the controlled variable and the width of hysteresis of the relay element

This relation is plotted for various values of L/T_p in Fig. 4, where the dotted line expresses that the amplitude of the controlled variable is never smaller than $|h|$.

By Eqs. (3) and (4), in the self-oscillation of the system shown in Fig. 2 the correlations between the period, the amplitude of the controlled variable, the width of the hysteresis in the relay element and the dead time of the controlled element are determined.

In this case, the condition of the switching time of the on-off control element is given by Eq. (4), and the condition of the switching direction in the process above-mentioned is

$$\left[\frac{dx}{d\tau}\right]_{\tau=P/2-L} < 0 \qquad (5)$$

Since, however, the controlled element in the given system is a first-order linear element with a dead time, Eq. (6) is sure to be satisfied.

2.2 Case with compensating elements (c.f. Fig. 1)

If the origin of time t is chosen at the instant when the relay element switches from "off" to "on" the output from compensating elements, $u(t)$, in an increasing process is given by

$$u(t) = u_1(0)\, e^{-t/T_1} + u_2(0)\, e^{-t/T_2} - AK_f(e^{-t/T_1} - e^{-t/T_2}) \qquad (6)$$

where T's and K_f are time constants and the proportional gain of the compensating elements, and $u(0)$'s are the initial values of $u(t)$.

By the reason described in 2.1, the wave shapes of $u(t)$ are symmetrical by the set point. Therefore, the value of $u(t)$ at the discontinuous points given by

$$u(0) = u_1(0)+u_2(0)$$

$$u\left(\frac{P}{2}\right) = u_1(0)e^{-P/2T_1}+u_2(0)e^{-P/2T_2}-AK_f(e^{-P/2T_1}-e^{-P/2T_2})$$

$$= -u_1(0)-u_2(0)$$

namely,

$$u_1(0)(1+e^{-P/2T_1})+u_2(0)(1+e^{-P/2T_2})-AK_f(e^{-P/2T_1}-e^{-P/2T_2}) = 0 \qquad (7)$$

And if the differentiation in respect to time is taken to Eq. (6), the values of incline of $u(t)$ at the discontinuous points are given as

$$\dot{u}(0) = -\frac{u_1(0)}{T_1}-\frac{u_2(0)}{T_2}+AK_f\left(\frac{1}{T_1}-\frac{1}{T_2}\right)$$

$$\dot{u}\left(\frac{P}{2}\right) = -\frac{u_1(0)}{T_1}e^{-P/2T_1}-\frac{u_2(0)}{T_2}e^{-P/2T_2}+AK_f\left(\frac{1}{T_1}e^{-P/2T_1}-\frac{1}{T_2}e^{-P/2T_2}\right)$$

The jump value of incline of $u(t)$ at discontinuous points corresponding to the switching of relay elemnt is expressed

$$\dot{u}(0)+\dot{u}\left(\frac{P}{2}\right) = 2AK_f\left(\frac{1}{T_1}-\frac{1}{T_2}\right)$$

(see Appendix).

The above 3 equations can be simplified as

$$\frac{u_1(0)}{T_1}(1+e^{-P/2T_1})+\frac{u_2(0)}{T_2}(1+e^{-P/2T_2})-AK_f\left(\frac{1}{T_1}e^{-P/2T_1}-\frac{1}{T_2}e^{-P/2T_2}\right) = 0 \qquad (8)$$

From Eqs. (7) and (8), the initial values of $u(t)$ yield

$$\left.\begin{aligned} u_1(0) &= 2AK_f\frac{e^{-P/2T_1}}{1+e^{-P/2T_1}} \\ &\text{and} \\ u_2(0) &= 2AK_f\frac{e^{-P/2T_2}}{1+e^{-P/2T_2}} \end{aligned}\right\} \qquad (9)$$

respectively.

By the correlation between the value of dead time and the one for the period of oscillation, it is considered that the controlled variable is on whether during the increasing or decreasing process.

First, let us consider the case that the dead time of the controlled element is

$$0 \leqslant L < \frac{P}{2}$$

In this case, at the switching instant of relay element from the on to the off, the controlled variable is on a decreasing process

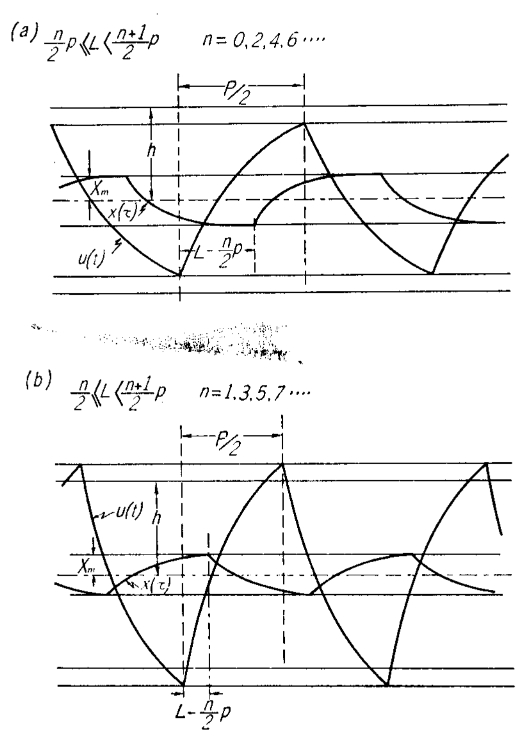

Fig. 5 Periodical responses of linear elements of the improved on-off control system

$$x(\tau) = x_m e^{-\tau/T_p} - AK_p(1-e^{-\tau/T_p})$$

where the origin of τ is chosen when the controlled variable switches from an increasing to a decreasing process (see Fig. 5a). Therefore the time when $u(t)$ turns from a decreasing to an increasing process is expressed as

$$\tau = \frac{P}{2} - L$$

at which the controlled variable is given by

$$x(P/2-L) = x_m e^{-P/2T_p+L/T_p} - AK_p(1-e^{-P/2T_p+L/T_p}) \tag{10}$$

so that the condition of the switching time of the on-off control element can be written as follows:

$$x(\tau = P/2-L) + u(t=0) = -h$$

Applying Eqs. (3), (7), (9) and (10), the above equation may be rewritten as

$$\frac{K_p}{K_f}\left\{\frac{h}{AK_p} - 1 + \left(1-\tanh\frac{P}{4T_p}\right)e^{-L/4T}\right\} - \tanh\frac{P}{4T_1} + \tanh\frac{P}{4T_2} = 0 \tag{11}$$

And since the condition of switching direction of on-off control element is

$$\dot{x}(\tau = P/2-L)+\dot{u}(t = P/2) < 0$$

this condition yields as

$$\frac{1}{2}\frac{K_p}{K_f}\frac{P}{T_p}\left(1-\tanh\frac{P}{4T_p}\right)e^{L/T_p}+\frac{P}{T_1}\left(1-\tanh\frac{P}{4T_1}\right)-\frac{P}{T_2}\left(1-\tanh\frac{P}{4T_2}\right)>0 \tag{12}$$

In the case where the dead time is

$$P/2 \leqslant L < P$$

the controlled variable is on an increasing process when the on-off control element is switched from "on" to "off" that is,

$$x(\tau) = -x_m e^{-\tau/T_p}+AK_p(1-e^{-\tau/T_p})$$

where the origin of τ is chosen at the instant when the controlled variable is turned from a decreasing to an increasing process (see Fig. 5b). Hence the time when $u(t)$ is changed from a decreasing to an increasing process is

$$\tau = P-L$$

at which the controlled variable is

$$x(P-L) = -x_m e^{-P/T_p+L/T_p}+AK_p(1-e^{-P/T_p+L/T_p})$$

In this case the condition of the switching time of the on-off control element is expressed

$$x(\tau = P-L)+u(t = 0) = -h$$

namely

$$\frac{K_p}{K_f}\left\{\frac{h}{AK_p}+1-\left(1-\tanh\frac{P}{4T_p}\right)e^{-P/T_p+L/T_p}\right\}-\tanh\frac{P}{4T_1}+\tanh\frac{P}{4T_2}=0 \tag{13}$$

And the condition of switching direction is as follows:

$$\dot{x}(\tau = P-L)+\dot{u}(t = 0) < 0$$

that is

$$\frac{1}{2}\frac{K_p}{K_f}\frac{P}{T_p}\left(1-\tanh\frac{P}{4T_p}\right)e^{-P/2T_p+L/T_p}$$
$$-\frac{P}{T_1}\left(1-\tanh\frac{P}{4T_1}\right)+\frac{P}{T_2}\left(1-\tanh\frac{P}{4T_2}\right)<0 \tag{14}$$

In the general case,

$$nP/2 \leqslant L < (n+1)P/2 \qquad (n = 0, 1, 2, 3, \cdots)$$

by the method used in the previous cases, the conditions of switching time and direction of the on-off control element are derived as

$$\frac{K_p}{K_f}\left\{\frac{h}{AK_p}+(-1)^{n+1}+(-1)^n\left(1-\tanh\frac{P}{4T_p}\right)e^{-nP/2T_p+L/T_p}\right\}$$
$$-\tanh\frac{P}{4T_1}+\tanh\frac{P}{4T_2}=0 \tag{15}$$

and

$$\frac{(-1)^n}{2}\frac{K_p}{K_f}\frac{P}{T_p}\left(1-\tanh\frac{P}{4T_p}\right)e^{-nP/2T_p+LT_p}$$
$$+\frac{P}{T_1}\left(1-\tanh\frac{P}{4T_1}\right)-\frac{P}{T_2}\left(1-\tanh\frac{P}{4T_2}\right)>0 \tag{16}$$

respectively.

The general existing conditions for self-oscillation in the on-off control system shown in Fig. 1 are given by Eqs. (15) and (16).

3. Numerical Examples

The examples of numerical solutions of self-oscillation occurring in the on-off control system shown in Fig. 1 are plotted for various values of n in Fig. 6, where the values of constants in the system are as follows:

$$(\mathrm{a})\left\{\begin{aligned} h/AK_p &= 0.05 \\ L/T_p &= 0.3 \\ T_p/T_1 &= 10 \\ T_p/T_2 &= 1 \end{aligned}\right. \qquad (\mathrm{b})\left\{\begin{aligned} h/AK_p &= 0.05 \\ L/T_p &= 0.3 \\ T_p/T_1 &= 5 \\ T_p/T_2 &= 2 \end{aligned}\right.$$

and the small circles with an arrow indicate the solutions for an infinitive value of K_p/K_f, that is, the solutions are ones of self-oscillation in the usual on-off control system shown in Fig. 3.

4. Conclusion

The existing conditions for self-oscillation occurring in the on-off control system compensated by linear elements are derived precisely, furthermore by numerical examples the solutions of self-oscillation of the given system are compared with one of the usual on-off control systems.

As a result of the analysis mentioned above, the following are revealed:

1. in respect to the oscillated amplitude of the controlled variable, the perormance of a usual on-off control system is improved by introducing compensating elements

2. as the value of K_p/K_f is changed monotonously, one mode of self-oscillation changes into another one, that is, the amplitude and period of self-

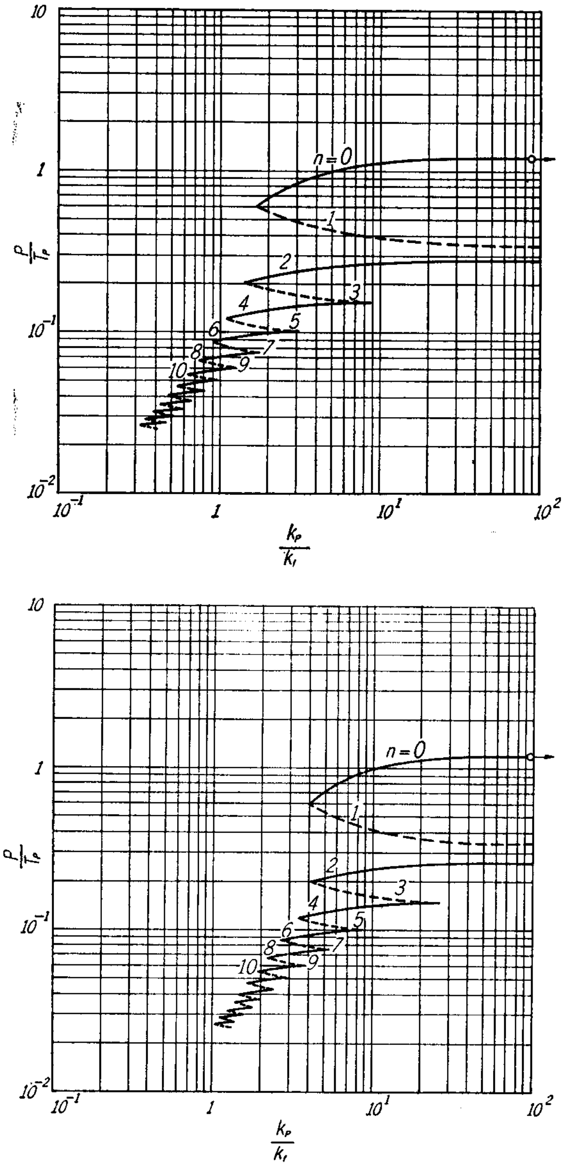

Fig. 6 Self-oscillations in the improved on-off control system

oscillation change discontinuously

3. for some values of K_p/K_f, two or more solutions satisfy the existing conditions for self-oscillation in the given on-off control system

4. self-oscillation in the improved on-off control system never vanishes even with an extremely small value of K_p/K_f, but the amplitude and period of the oscillated variables become very small values.

Therefore, in this improved on-off control system, by a large value of K_f the amplitude and period of self-oscillation can be reduced. Since, however, it is supposed that the undesirable transient response of the controlled variable for the very large value of K_f is caused by the changing of set point or the introducing of disturbance, an adequate value of K_f in the compensating element should be chosen corresponding to the characteristics of the controlled element in order that the responses in the transient as well as steady states are desirable.

The stability of self-oscillations in the given system should be determined, but it will be dealt with in the next paper.

References

1) For instance
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C. Kessler : Regelungstech., **4**, 339 (1955)
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2) Ja.S. Zypkin : "Theorie der Relaissysteme der Automatischen Regelung", R. Oldenburg Verlag, Muenchen, p. 170 (1958)
3) R.C. Oldenbourg und H. Sartorius (Japanese Translation by Y. Takahashi and K. Izawa). "Dynamik Selbsttaetiger Regelungen, (Zidoseigyo no Rikigaku)", Seibundo Shinkosha, Tokyo, p. 178 (1953)

Appendix

From the transfer function of compensating elements

$$\mathcal{L}\left\{\frac{u(t)}{y(t)}\right\} = \frac{K_f}{T_1 s+1} - \frac{K_f}{T_2 s+1} = \frac{K_f(T_2-T_1)s}{T_1T_2s^2+(T_1+T_2)s+1}$$

the relation between the input and output in the form of a differential equation is given by

$$T_1T_2\ddot{u}(t)+(T_1+T_2)\dot{u}(t)+u(t) = K_f(T_2-T_1)\dot{y}(t) \qquad \text{(A-1)}$$

where $\cdot$ denotes d/dt.

Now, when the input to this system is a step function, namely,

$$\begin{aligned} y(t) &= 0 & t &\leqslant 0 \\ &= 2A\mathbf{1}(t) & t &> 0 \end{aligned}$$

let us assume the time response of the output near to $t=0$ is as follows:

$$u(t) = u_0(t)+\alpha\mathbf{1}(t)+\beta t\mathbf{1}(t)$$

Hence the derivative of $y(t)$ and $u(t)$ in respect to time are derived as

$$\dot{y}(t) = 2A\delta(t)$$
$$\dot{u}(t) = \dot{u}_0(t) + \alpha\delta(t) + \beta\mathbf{1}(t)$$
$$\ddot{u}(t) = \ddot{u}_0(t) + \alpha\dot{\delta}(t) + \beta\delta(t)$$

where $\delta(t)$ is a delta-function.
Substituting these relations into Eq. (A-1), the resultant is

$$T_1T_2\{\ddot{u}_0(t) + \alpha\dot{\delta}(t) + \beta\delta(t)\} + (T_1 + T_2)\{\dot{u}_0(t) + \alpha\delta(t) + \beta\mathbf{1}(t)\}$$
$$+ u_0(t) + \alpha\mathbf{1}(t) + \beta t\mathbf{1}(t) = 2AK_f(T_2 - T_1)\delta(t)$$

Since from the above identity α and β are given by

$$\alpha = 0$$

and

$$\beta = 2AK_f\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

respectively, the response of the output near to $t=0$ for the step input is expressed as

$$u(t) = u_0(t) + 2AK_f\left(\frac{1}{T_1} - \frac{1}{T_2}\right)t\mathbf{1}(t) \tag{A-2}$$

Therefore the jump value of the incline of $u(t)$ is

$$2AK_f\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \tag{A-3}$$