Heat Transfer in a Turbulent Boundary Layer with Pressure Gradient

By

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The relation between the volocity and temperature distributions in a turbulent boundary layer with pressure gradient and the effect of the pressure gradient on Stanton or Nusselt number are investigated theoretically. The result is summarized as follows. The velocity distribution and, hence, the skin friction coefficient are affected by the pressure gradient while the temperature distribution and Stanton number are almost independent of the pressure gradient.

1. Introduction

The relation between the velocity and temperature distributions in a turbulent boundary layer accompanied with pressure gradient and also the effect of the pressure gradient on Stanton or Nusselt number are not well established as in the case of flow without pressure gradient or pipe flow. In this paper, therefore, this problem is treated theoretically and the results of numerical calculation are shown.

2. Notations

x, y: orthogonal co-ordinates;
u, v: velocity components in direction of x, y;
ρ: density of fluid;
ν: kinematic viscosity of fluid;
c_p: specific heat of fluid;
g: acceleration of gravity;
δ: boundary layer thickness;
θ: momentum thickness;
l: mixing length of turbulent flow;
τ: shear stress in fluid;
c_f: skin friction coefficient i.e. $c_f = 2\tau_w/\rho u_0^2$;
u^*: friction velocity i.e. $\sqrt{\tau_w/\rho}$;
T: temperature of fluid;
q: heat flux per unit area;
T^*: friction temperature i.e. $q_w/\rho g c_p u^*$;
Re_θ: Reynolds number, built with θ;
Nu_θ: Nusselt number, built with θ;
Pr: Prandtl number;
St_θ: Stanton number i.e. $Nu_\theta/Pr \cdot Re_\theta$.

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Subscripts w: values on the wall surface;
o: values outside the boundary layer.

3. Theoretical Relations

In the region of fully turbulent flow, it is assumed after Prandtl and Reynolds,

$$\tau = \rho l^2 \left(\frac{du}{dy}\right)^2, \tag{1}$$

$$q = -\rho g c_p l^2 \frac{du}{dy} \cdot \frac{dT}{dy}. \tag{2}$$

If the physical properties of the fluid are constant, then the equations of momentum and energy in the case of the steady state become as follows,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_0 \frac{du_0}{dx} + \frac{\partial}{\partial y}\left(l^2 \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y}\right), \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(l^2 \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y}\right). \tag{4}$$

In the last equation, the term of energy dissipation is neglected assuming low speed flow. When there is pressure gradient along the flow, u_0 is not constant, hence it can be easily seen that the velocity and temperature distributions are not similar by comparing eqs. (3) and (4).

Let $\eta = y/\delta$, $l(\eta) = l/\delta$, $\tau(\eta) = \tau/\tau_w$ and $q(\eta) = q/q_w$, then from eqs. (1) and (2),

$$f_1(\eta) = \frac{u_0 - u}{u^*} = \int_\eta^1 \frac{\sqrt{\tau(\eta)}}{l(\eta)} d\eta, \tag{5}$$

$$f_2(\eta) = \frac{T - T_0}{T^*} = \int_\eta^1 \frac{q(\eta)}{l(\eta)\sqrt{\tau(\eta)}} d\eta, \tag{6}$$

and $f_1(\eta) = f_2(\eta)$ only when $q(\eta) = \tau(\eta)$. But from eqs. (3) and (4)

$$\frac{\partial \tau}{\partial y} = -\rho u_0 \frac{du_0}{dx} \quad \text{and} \quad \frac{\partial q}{\partial y} = 0,$$

on the wall surface where $u = v = 0$, so $q(\eta)$ is not equal to $\tau(\eta)$, hence $f_2(\eta)$ can not be equal to $f_1(\eta)$ unless $u_0 = \text{const.}$

Now, if the mixing length is assumed $l(\eta) = 0.4\eta$, then, in the neighbourhood of $\eta = 0$, $f_1(\eta)$ and $f_2(\eta)$ can be expressed as follows,

$$f_1(\eta) = -5.75 \log_{10} \eta + c_0 + c_1 \eta + \cdots\cdots, \tag{7}$$

$$f_2(\eta) = -5.75 \log_{10} \eta + d_0 + d_1 \eta + \cdots\cdots, \tag{8}$$

where c_0, c_1, d_0 and d_1 are constants and they depend on

$$a_1 = -\frac{\rho u_0 \delta}{\tau_w} \cdot \frac{du_0}{dx}. \tag{9}$$

But in the case of smooth wall, it is known that

$$\frac{u}{u^*} = A + 5.75 \log_{10} \eta + 5.75 \log_{10} \frac{u^* \delta}{\nu}, \quad A = 5.5, \tag{10}$$

hence comparing eqs. (7) and (10)

$$\sqrt{\frac{c_f}{2}} \cdot \frac{u_0 \delta}{\nu} = \exp\left[0.4\left(\sqrt{\frac{2}{c_f}} - A - c_0\right)\right].$$

So let $F_1 = \int_0^1 f_1(\eta) d\eta$ and $F_2 = \int_0^1 \{f_1(\eta)\}^2 d\eta$, then

$$Re_\theta = \left(F_1 - \sqrt{\frac{c_f}{2}} F_2\right) \exp\left[0.4\left(\sqrt{\frac{2}{c_f}} - A - c_0\right)\right]. \tag{11}$$

Also, in the neighbourhood of the wall surface

$$\frac{T_w - T}{T^*} = A + 5.75 \log_{10} \eta + 5.75 \log_{10} \frac{u^* \delta}{\nu} + K, \tag{12}$$

where K is the function given by v. Kármán i.e. $K = 5[Pr - 1 + 2.3 \log_{10}\{(5Pr + 1)/6\}]$.
Comparing eqs. (8) and (12)

$$\frac{T_w - T_0}{T^*} = \sqrt{\frac{2}{c_f}} - (c_0 - d_0) + K,$$

hence

$$St_\theta = \frac{c_f/2}{1 - \sqrt{c_f/2}\,(c_0 - d_0 - K)}. \tag{13}$$

4. Results of Numerical Calculation

In this calculation, the shear stress and the heat flux or the functions of $\tau(\eta)$ and $q(\eta)$ are approximated by two power series of η. Since $\partial \tau / \partial y = -\rho u_0 du_0/dx$ and $\partial q / \partial y = 0$ on the wall surface, these power series are determined by the following conditions,

$$\tau(0) = 1, \quad \tau'(0) = a_1, \quad q(0) = 1, \quad q'(0) = 0,$$
$$\tau(1) = 0, \quad \tau'(1) = 0, \quad q(1) = 0 \quad \text{and} \quad q'(1) = 0,$$

where $\tau'(\eta) = \partial \tau(\eta) / \partial \eta$ and $q'(\eta) = \partial q(\eta) / \partial \eta$. Hence

$$\tau(\eta) = 1 - 3\eta^2 + 2\eta^3 + a_1 \eta (1 - \eta)^n, \quad n > 1, \tag{14}$$
$$q(\eta) = 1 - 3\eta^2 + 2\eta^3, \tag{15}$$

in which n is assumed to be 2 in this numerical calculation.

The mixing length distribution $l(\eta)$ is assumed according to the experimental results obtained in the case of flow along a flat plate and is shown in fig. 1.

Integration of eqs. (5) and (6) was carried out by the numerical method and thence c_0 and d_0 were determined and the result is as follows,

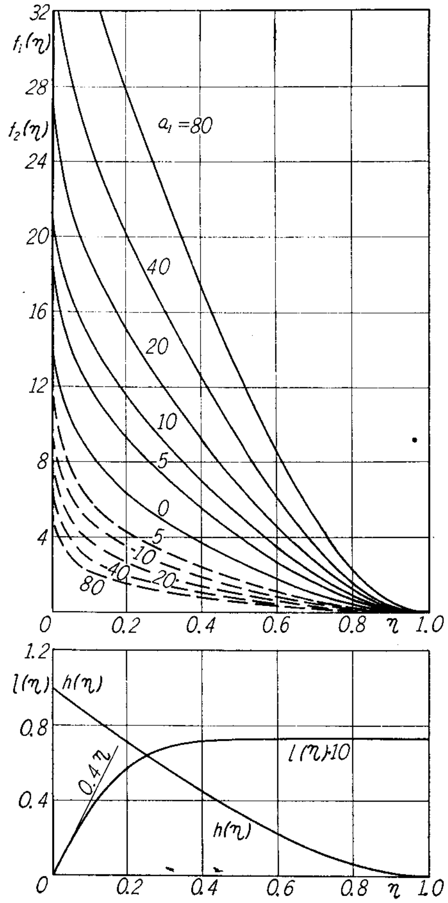

Fig. 1.

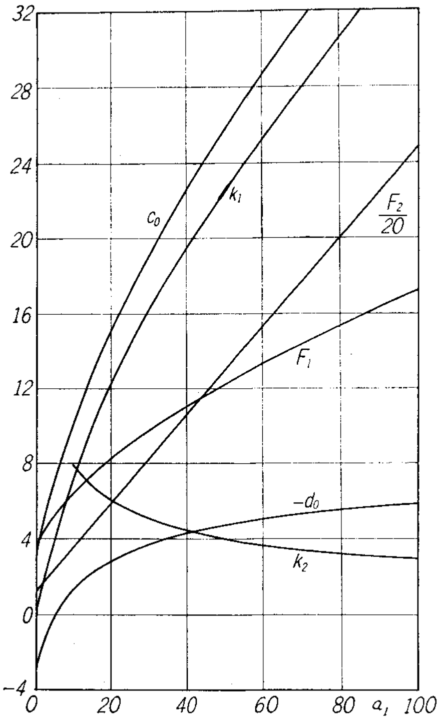

Fig. 2.

In fig. 1, the lines show $f_1(\eta)$ and the broken lines show $f_2(\eta)$ for several values of a_1. Let $f_0(\eta)$ denotes $f_1(\eta)$ when $a_1=0$, then $f_1(\eta)$ can be expressed approximately by

$$f_1(\eta)=f_0(\eta)+k_1h(\eta)\,. \tag{16}$$

The function $h(\eta)$ is shown in fig. 1 and k_1 in fig. 2. Also $f_2(\eta)$ can be expressed approximately by

$$f_2(\eta)=-k_2\log_{10}\eta\,,\qquad 0.1<\eta<1.0\,,$$

where k_2 is a function of a_1 as shown in fig. 2. In fig. 2, values of c_0, d_0, F_1 and F_2 are shown.

The broken lines in fig. 3 show the relation between the skin friction coefficient and Reynolds number for several values of a_1 calculated by eq. (11)

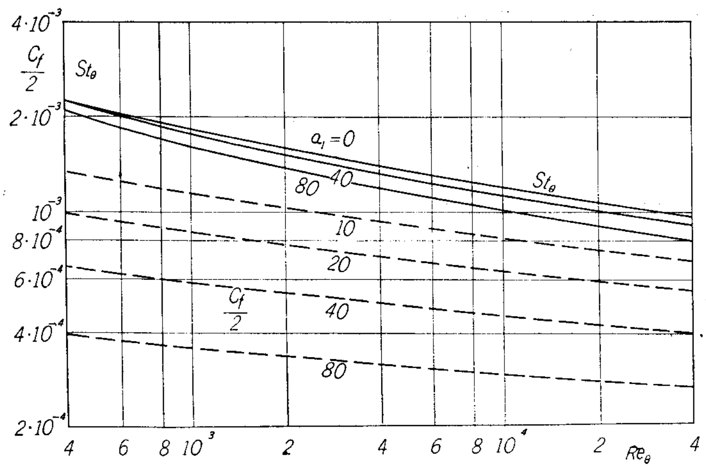

Fig. 3.

and the lines show the Stanton number calculated by eq. (13) when $Pr=1$. Unless a_1 is large, the lines indicating Stanton number coincide with each other, hence it can be said that Stanton and Nusselt numbers are almost independent of the pressure gradient. This result coincides with existing experimental results of Hool[1] and Romanenko and others[2] at least qualitatively.

An example of comparison between the velocity and temperature distributions of an air flow is shown in fig. 4, in which the thick lines show the results of the present calculation and the fine lines are the experimental results. Such a relation between the two distributions is also given by the above-mentioned authors. Hence it can be said that the velocity distribution varies appreciably

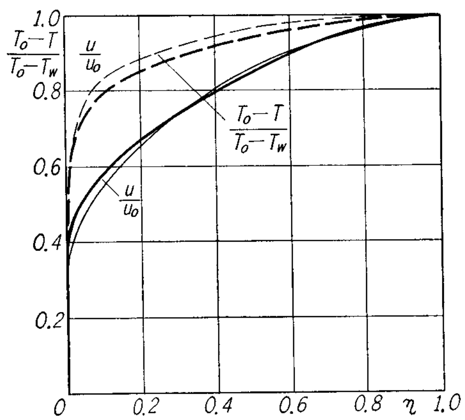

Fig. 4.

by the pressure gradient, while the temperature distribution is not affected much and it becomes flatter as the adverse pressure gradient increases.

Acknowledgment

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2) P.N. Romanenko, A.I. Leont'ev and A.N. Oblivin : Inter. J. Heat and Mass Transfer, **5**, p. 541-557, (1962)