The Meson-Nucleon Phenomena and the Many Body Problem

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The dynamical theory of the many body problem is used to account for the meson-meson resonance and to investigate the behavior of the nucleon pair from the standpoint of a composite model of pion.

1. Introduction and Summary

Recently the dynamical theory of the many body problem has been developed by many authors and applied with great success to nuclear and solid state physics. We should like to use this theory to explain the collective behavior of pions and nucleons. Such an application is made possible because of the existence of short range correlation between meson and meson or nucleon and antinucleon. We discuss the meson-meson resonance in §2 following the Fukuda-Wada report about the treatment method of the many body problem and the behavior of nucleon-antinucleon pairs in §3 along with the treatment of plasma oscillation formally.

It is interesting whether the meson-meson resonance state obtained in §2 corresponds to the ρ-meson in the case of pion-pion scattering and the result in §3 seems to be useful for our purpose, that is to make the difference between the customary meson theory and the composite model of the pion clear.

2. The Meson-Meson Resonance

The recently-observed multi-pion resonances have been investigated by many authors and in particular various methods have been used to study the pion-pion resonance (the so-called ρ-meson) from the conventional field theory. The importance of pion-pion interaction in the pion-pion scattering problem is emphasized by them and therefore we also account for the pion-pion resonance in terms of a self-coupled pion field, without explicit introduction of a pion-nucleon

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interaction. But one great difference is that we explain this problem from the standpoint of the many particle problem. Thus far, the work is confined to the neutral scalar field, although it is expected that the same methods will be applied to the pion field.

The total Hamiltonian of our system is given by

$$H = H_0 + V, \tag{1}$$

$$H_0 = \frac{1}{2}\int (\pi(\boldsymbol{x})^2 + (\nabla\varphi(\boldsymbol{x}))^2 + \mu^2\varphi(\boldsymbol{x})^2)d^3x, \tag{2}$$

$$V = \frac{1}{4}\lambda\int (\varphi(\boldsymbol{x})^2 d^3x)^2, \tag{3}$$

where $\varphi(\boldsymbol{x})$ is the meson field, $\pi(\boldsymbol{x})$ its canonical conjugate, μ the meson mass, λ the coupling constant. Units are chosen such that $\hbar = c = 1$.

We start with the following considerations. In the four meson interaction it is considered that the four mesons are all free mesons, but in meson-meson scattering it is advantageous to consider that two of them are free and the other two are the meson source (i. e. c-number). Then the meson-meson interaction (3) is rewrited as follows,

$$V = \frac{1}{2}\lambda\int (\rho(\boldsymbol{x})\varphi(\boldsymbol{x})d^3x)^2, \tag{4}$$

where $\rho(x)$ is the source function of the meson which depends only on $r = |\boldsymbol{x}|$. If the interaction Hamiltonian is linearized in this way, the total Hamiltonian of our system becomes the same form as one in the neutral scalar meson pair theory first discussed by Wentzel.[1] As is well known, this is an example of a many body system that can be exactly solved. Fukuda and Wada[2] studied this problem in detail by the new Tamm-Dancoff method developed by Dyson.[3] Then we make use of this calculation method by Fukuda and Wada but we must notice that the ground state is the physical nucleon state in the case of meson pair theory and our ground state is the one physical meson state. When Ψ_0 and E_0 represent the ground state (one meson state) and its energy and Ψ and E the excited state (two meson state) and its energy respectively, after an ingenious calculation the energy shift of the ground state is obtained as follows,

$$\Delta E_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty} du \log\left(1 + \frac{\lambda}{2\pi^2}\int_0^{\infty} dp\,\frac{p^2 v(p)^2}{\omega_p^2 + u^2}\right) \tag{5}$$

where

$$v(k) = \int \rho(r) e^{i\boldsymbol{k}\boldsymbol{x}} d^3x, \qquad \omega_k = \sqrt{k^2 + \mu^2}. \tag{6}$$

Furthermore, from the equation

$$(E - H)\Psi = 0, \tag{7}$$

we can derive

$$1=\frac{\lambda}{\Omega}\sum_k\frac{v(k)^2}{(E-E_0)^2-\omega_k^2}, \tag{8}$$

where Ω is the normalization volume. When we solve equation (8), we have the eigenvalue $E-E_0$. If we choose $v(k)=0$ for $k>k_{\max}$, the so-called plasma solution Ω_{pl} appears in addition to the ordinary scattering solution Ω_k.

First of all, we put the scattering solution $\Omega_k=\omega_k+\varDelta\omega_k$ and expand φ and π in terms of spherical waves instead of plane waves, containing the system a large sphere with radius R. Then we see that if $v(k)$ is the spherical symmetric function only the s-wave meson can interact with the source and the energy shift $\varDelta\omega_k$ (i. e. the shift of the excitation energy $\varDelta E_{\mathrm{ex}}$) is given by

$$\varDelta\omega_k=\frac{k}{R\omega_k}\tan^{-1}\frac{\lambda\beta_k}{1+\lambda\alpha_k}, \tag{9}$$

where

$$\alpha_k=\frac{1}{2\pi^2}P\int dk'\frac{k'^2v(k')^2}{k'^2-k^2},$$

$$\beta_k=\frac{1}{4\pi}kv(k)^2, \tag{10}$$

and P denotes the principal value of the integral.

Next the plasma solution Ω_{pl} is derived by the application of the Chew-Low-Wick method

$$\Omega_{pl}^2=\omega_{\max}^2+\frac{1}{2\pi^3}\int_0^\lambda d\lambda\int_0^\infty dk\frac{k^2v(k)^2}{|1+f(\omega_k+i\varepsilon)|^2}\times$$

$$\times\left\{2Ref(\omega_k+i\varepsilon)+|f(\omega_k+i\varepsilon)|^2\right\}, \tag{11}$$

where

$$f(z)=\frac{\lambda}{\Omega}\sum_k\frac{v(k)^2}{2\omega_k}\left\{\frac{1}{\omega_k+z}+\frac{1}{\omega_k-z}\right\}. \tag{12}$$

Finally we will obtain the phase shift of the meson scattering by the meson source. According to Lippmann and Schwinger[4]

$$\tan\delta^s(p)=\frac{-\lambda\beta_p}{1+\lambda\alpha_p}, \tag{13}$$

which shows, as already noticed, that the s-wave can only be scattered. When we examine the resonance scattering which occurs around at $\omega_p\sim\Omega_{pl}$, the cross section near the resonance is given by

$$\sigma(p)=\frac{4\pi}{C^2k_{pl}^2}\cdot\frac{\lambda^2\beta_{pl}^2}{(\omega_p-\Omega_{pl})^2+\lambda^2\beta_{pl}^2/C^2}, \tag{14}$$

where

$$C = -\frac{1}{k_{pl}} + \frac{\lambda k^3_{\max}}{2\pi^2 k_{pl}(k^2_{pl} - k^2_{\max})}. \tag{15}$$

The resonance half-width is obtained by

$$\Gamma = \frac{2\lambda \beta_{pl}}{C}. \tag{16}$$

The plasmon energy is not related with the normalization volume. The plasmon mode behaves like the bound state localized around the source.

3. The Behavior of the Nucleon-Antinucleon Pair

When we investigated the pion-nucleon interaction in the composite model of the pion, we had an interest in finding out the ratio of the pion state (bound nucleon pair state) to the unbound pair state in the all nucleon pair state.

We first write the total Hamiltonian of our system by means of second quantized operators as follows,

$$\begin{aligned} H &= H_0 + H_{\text{int}}, \\ H_0 &= \int E_p(a_p^* a_p + b_p^* b_p)\, dp, \\ H_{\text{int}} &= \frac{1}{(2\pi)^3} \sum_i \frac{g}{M^2} \int (-a^*_{p+q} B^i b^*_p + b_{p+q} C^i a_p) \times \\ &\quad \times (-a^*_{p'-q} B^i b^*_{p'} + b_{p'-q} C^i a_{p'})\, dp\, dp'\, dq, \end{aligned} \tag{17}$$

where

$$B^i = (\tau_i \tau_2)\sigma_2, \qquad C^i = (\tau_2 \tau_i)\sigma_2, \tag{18}$$

and a_p^* and b_p^* are creation operators and a_p and b_p annihilation operators for the nucleon and antinucleon respectively. σ_i's are the spin operators, τ_i's the isotopic spin operators and g is the coupling constant, M the nucleon mass. To construct the eigenvalue equation for one pair state, we take for Ψ some eigenstate of the total Hamiltonian with eigenvalue E and for Ψ_0 the exact ground state (no nucleon pairs when the interaction vanishes) with energy E_0 and so we start with

$$(H - E)\Psi = 0. \tag{19}$$

Then we can derive

$$\begin{aligned} &(\Psi, (H-E)(-a^*_{p+q} B^i b^*_p)\Psi_0) \\ &= (E_{p+q} + E_p - E + E_0)(\Psi, (-a^*_{p+q} B^i b^*_p)\Psi_0) \\ &\quad - \frac{8}{(2\pi)^3} \frac{g}{M^2} \int dp' (\Psi, (-a^*_{p'+q} B^i b^*_{p'} + b_{p'+q} C^i a_{p'})\Psi_0) = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &(\Psi, (H-E)(b_{p+q} C^i a_p)\Psi_0) \\ &= (-E_{p+q} - E_p - E + E_0)(\Psi, (b_{p+q} C^i a_p)\Psi_0) \\ &\quad + \frac{8}{(2\pi)^3} \frac{g}{M^2} \int dp' (\Psi, (-a^*_{p'+q} B^i b^*_{p'} + b_{p'+q} C^i a_{p'})\Psi_0) = 0. \end{aligned} \tag{21}$$

From (20) and (21) we get the eigenvalue equation

$$1=\frac{8}{(2\pi)^3}\frac{g}{M^2}\int d\boldsymbol{p}\left(\frac{1}{E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}-\omega_{\boldsymbol{q}}}+\frac{1}{E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{q}}}\right), \tag{22}$$

where $E-E_0=\omega_{\boldsymbol{q}}$.

This equation has two kinds of solution. One is

$$\omega_{\boldsymbol{q}}=E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}, \tag{23}$$

which corresponds to a free nucleon-antinucleon pair and the other is

$$\omega_{\boldsymbol{q}}=\sqrt{\boldsymbol{q}^2+\mu^2}, \tag{24}$$

which corresponds to a bound state.

Thus, the following expansion is possible for one pair state:

$$\int d\boldsymbol{p}(-a^*_{\boldsymbol{p}+\boldsymbol{q}}B^ib^*_{\boldsymbol{p}}+b_{\boldsymbol{p}+\boldsymbol{q}}C^ia_{\boldsymbol{p}})\Psi_0$$
$$=\int d\boldsymbol{p}C_s(\boldsymbol{p})\Psi^{i(+)}_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}}+C_\pi(\boldsymbol{q})\Psi^i_\pi(\boldsymbol{q}). \tag{25}$$

$\Psi^i_\pi(\boldsymbol{q})$ is the normalized wave function of the pion[5] given by

$$\Psi^i_\pi(\boldsymbol{q})=\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\int d\boldsymbol{p}\left(\frac{a^*_{\boldsymbol{p}+\boldsymbol{q}}B^ib^*_{\boldsymbol{p}}}{E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}-\omega_{\boldsymbol{q}}}+\frac{b_{\boldsymbol{p}+\boldsymbol{q}}C^ia_{\boldsymbol{p}}}{E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{q}}}\right)\Psi_0, \tag{26}$$

and $\Psi^{i(+)}_{\boldsymbol{p}+\boldsymbol{c},\boldsymbol{p}}$ is the scattering solution derived hereinafter. (We add the superscript (+) to indicate the outgoing wave solution.)

At first to obtain the expansion coefficient $C_\pi(\boldsymbol{q})$, we take the scalar product of Eq. (25) with $\Psi^i_\pi(q)$ and get

$$C_\pi(q)=\frac{-4}{\sqrt{2\omega_{\boldsymbol{q}}}}\int d\boldsymbol{p}\left(\frac{1}{E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}-\omega_{\boldsymbol{q}}}+\frac{1}{E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{q}}}\right). \tag{27}$$

Next in order to have the scattering solution, we put

$$\Psi^{i(+)}_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}}=-a^*_{\boldsymbol{p}+\boldsymbol{q}}B^ib^*_{\boldsymbol{q}}\Psi_0+\chi^{i(+)}_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}}. \tag{28}$$

Substituting this in Eq. (19), we have

$$\chi^{i(+)}_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}}=\frac{1}{H-E_0-E_{\boldsymbol{p}+\boldsymbol{q}}-E_{\boldsymbol{p}}-i\varepsilon}\frac{8}{(2\pi)^3}\frac{g}{M^2}\int d\boldsymbol{p}'(-a^*_{\boldsymbol{p}'+\boldsymbol{q}}B^ib^*_{\boldsymbol{p}'}+b_{\boldsymbol{p}'+\boldsymbol{q}}C^ia_{\boldsymbol{p}'})\Psi_0. \tag{29}$$

We get also

$$0=b_{\boldsymbol{p}+\boldsymbol{q}}C^ia_{\boldsymbol{p}}\Psi_0-\frac{1}{H-E_0+E_{\boldsymbol{p}+\boldsymbol{q}}+E_{\boldsymbol{p}}}\frac{8}{(2\pi)^3}\frac{g}{M^2}\int d\boldsymbol{p}'(-a^*_{\boldsymbol{p}'+\boldsymbol{q}}B^ib^*_{\boldsymbol{p}'}+b_{\boldsymbol{p}'+\boldsymbol{q}}C^ia_{\boldsymbol{p}'})\Psi_0, \tag{30}$$

from an identity $b_{\boldsymbol{p}+\boldsymbol{q}}C^ia_{\boldsymbol{p}}(H-E_0)\Psi_0=0$. Adding Eq. (28) and Eq. (30), we obtain

$$\Psi^{i(+)}_{p+q,p} = (-a^*_{p+q}B^i b^*_p + b_{p+q}C^i a_p)\Psi_0$$
$$-\left(\frac{1}{E_{p+q}+E_p+E_0-H+i\varepsilon}+\frac{1}{E_{p+q}+E_p+H-E_0}\right)\frac{8}{(2\pi)^3}\frac{g}{M^2}\times$$
$$\times\int dp'(-a^*_{p'+q}B^i b^*_{p'}+b_{p'+q}C^i a_{p'})\Psi_0, \tag{31}$$

and so from this is derived the equation

$$\int dp\,\Psi^{i(+)}_{p+q,p} = \left(1-\frac{8}{(2\pi)^3}\frac{g}{M^2}\int dp\left(\frac{1}{E_{p+q}+E_p+E_0-H+i\varepsilon}+\frac{1}{E_{p+q}+E_p+H-E_0}\right)\right)$$
$$\times\int dp'\,(-a^*_{p'+q}B^i b^*_{p'}+b_{p'+q}C^i a_{p'})\Psi_0. \tag{32}$$

Furthermore we can prove

$$(\Psi^{i(+)}_{p+q,p},\ \Psi^{j(+)}_{p'+q',p'}) = \delta_{ij}\delta_{pp'}\delta_{qq'}, \tag{33}$$

after troublesome calculation by making use of Eq. (28). Then we can calculate $C_s(p)$ and it is determined as follows,

$$C_s(p) = \left(1-\frac{8}{(2\pi)^3}\frac{g}{M^2}\int dp'\left(\frac{1}{E_{p'+q}+E_{p'}+E_0-E_{p+q}-E_p+i\varepsilon}\right.\right.$$
$$\left.\left.+\frac{1}{E_{p'+q}+E_{p'}+E_{p+q}+E_p-E_0}\right)\right)^{-1}. \tag{34}$$

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