On the Coupled Free Vibrations of a Suspension Bridge (II)

By

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Following the previous investigation on the free vibrations of a suspension bridge, this paper presents the specific characteristic associated with the second class of (torsional) modes. Numerical works are carried out by use of the Rayleigh-Ritz method under the lower order of approximations.

1. Introduction

For the study of free vibrations of a suspension bridge, the fundamental differential equations of motions have been obtained by applying Hamilton's principle to the structural system Fig. 1 considered here.[1] This results in the nine differential equations with two constraint conditions assuming the completely inextensible hangers. The modes of free vibrations are thus classified into two classes, deflectional and torsional modes. Previously the deflectional modes were investigated theoretically and from structural points of view. The installment

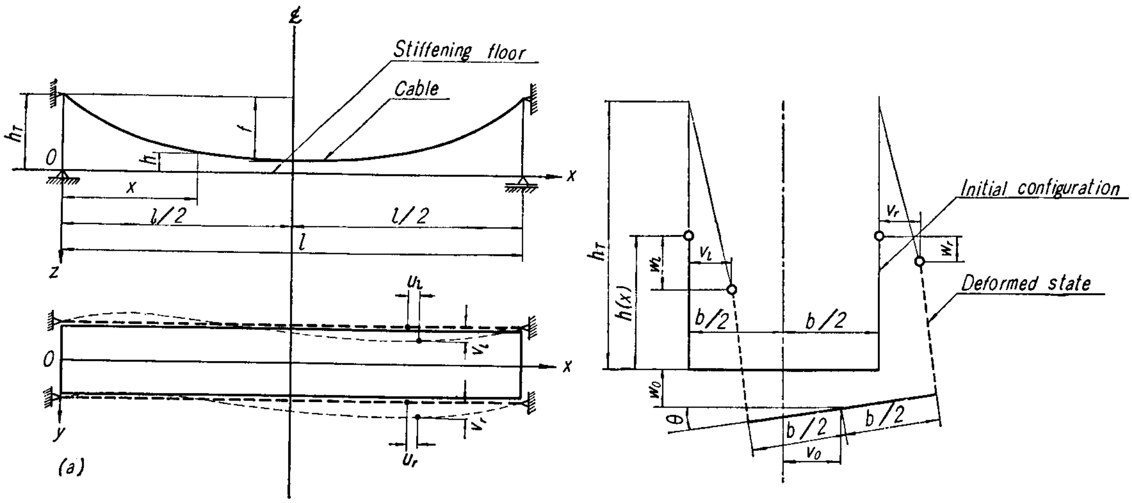

Fig. 1. Configuration of a suspension bridge.

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of center-diagonal stays is found to correspond to the existence of singularity of differential equations.

In this paper the torsional modes of free vibrations are investigated in the same manner as before, which are expressed by the simultaneous differential equations defined previously. Since it seems rather difficult to find the exact solutions for this problem, note the fact that the lateral horizontal displacement of stiffening floor and cables do not contain explicitly any other modes of displacements in the expressions. It may be considered that practically the lateral modes are treated separately from the other displacements. In **2.** the analytical solutions for these lateral modes are sought by the Ritz method, and in **3.** the torsional modes of stiffening floor are considered.

2. Lateral Horizontal Modes of Free Vibrations

The differential equations for the second class of modes of free vibrations of a suspension bridge are of the coupled form of 9 simultaneous equations including torsional, lateral horizontal, vertical displacements of stiffening floor and all displacement components of cables. However, restricting our attention to the case for lateral horizontal modes of stiffening floor and cables, one may find the expressions in which the above two modes of displacements only appear explicitly under the assumption that the longitudinal displacements vanish identically throughout the spanlength. Thus we have

$$\frac{w_f}{g}\ddot{v}_0+\frac{\partial^2}{\partial x^2}\left\{EI_z\frac{\partial^2 v_0}{\partial x^2}\right\}+2\lambda_1(v_0-v_l)+2\lambda_2(v_0-v_r)=0 \tag{1}$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{v}_l-H_w\frac{\partial^2 v_l}{\partial x^2}-2\lambda_1(v_0-v_l)=0 \tag{2}$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{v}_r-H_w\frac{\partial^2 v_r}{\partial x^2}-2\lambda_2(v_0-v_r)=0 \tag{3}$$

for which let us assume that

$$v_r=v_l\equiv v$$

and

$$\lambda_1=\lambda_2=\frac{H_wh''}{2h}=\frac{w_f+2w_c(1+h'^2)^{1/2}}{4h}$$

Then we have

$$\frac{w_f}{g}\ddot{v}_0+\frac{\partial^2}{\partial x^2}\left\{EI_z\frac{\partial^2 v_0}{\partial x^2}\right\}+\frac{2H_wh''}{h}(v_0-v)=0 \tag{4}$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{v}-H_w\frac{\partial^2 v}{\partial x^2}-\frac{H_wh''}{h}(v_0-v)=0 \tag{5}$$

From eq. (4) it is written that

$$\frac{2w_c}{g}(1+h'^2)^{1/2}\left\{v_0+\frac{h}{2H_w h''}\left[\frac{w_f}{g}v_0+\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 v_0}{\partial x^2}\right)\right]\right\}$$
$$-2H_w\frac{\partial^2}{\partial x^2}\left[v_0+\frac{h}{2H_w h''}\left\{\frac{w_f}{g}v_0+\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 v_0}{\partial x^2}\right)\right\}\right]+\frac{w_f}{g}v_0+\frac{\partial^2}{\partial^2 x}\left(EI_z\frac{\partial^2 v_0}{\partial x^2}\right)=0$$

If $h'=\text{const.}\ll 1$, then the above expressions are reduced to

$$\frac{w_c w_f}{H_w h'' g^2}h\ddot{v}_0+\frac{w_c h}{H_w h'' g}\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 \ddot{v}_0}{\partial x^2}\right)+\frac{2w_c}{g}\ddot{v}_0-\frac{w_f}{h'' g}\frac{\partial^2}{\partial x^2}(h\ddot{v}_0)$$
$$-\frac{1}{h''}\frac{\partial^2}{\partial x^2}\left\{h\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 \ddot{v}_0}{\partial x^2}\right)\right\}-2H_w\frac{\partial^2 v_0}{\partial x^2}+\frac{w_f}{g}\ddot{v}_0+\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 v_0}{\partial x^2}\right)=0$$

Let $v_0=V_0(x)e^{i\omega t}$

then

$$-\frac{EI_z}{h''}\{hV_0''''\}''-\frac{EI_z w_c}{H_w h'' g}h\omega^2 V_0''''+EI_z V_0''''+\frac{w_f\omega^2}{h'' g}(hV_0)''-2H_w V_0''$$
$$+\left\{2\omega^4\left(\frac{w_c w_f h}{2H_w h'' g^2}\right)-\frac{2w_c}{g}\omega^2-\frac{\omega^2 w_f}{g}\right\}V_0=0$$

for $EI_z=\text{const.}$

which may be rewritten as

$$(hV_0'''')''+(a_1h+a_2)V_0''''+\{(a_3h+a_4)V_0\}''+(a_5h+a_6)V_0=0 \qquad (6)$$

This is the fundamental differential equation for the free vibrations of lateral horizontal modes of a suspension bridge which is of the sixth order with variable coefficients of quadratic form.

Taking v instead of v_0, the fundamental equation is written as alternatively

$$(hV'')''''+\{(a_1h+a_2)V\}''''+(a_3h+a_4)V''+(a_5h+a_6)V=0 \qquad (7)$$

The relation between eq. (6) and eq. (7) is merely the adjoint differential equation with each other.

By virtue of the general theory of differential equations eq. (6) or eq. (7) can have analytic solutions associated with appropriate boundary conditions provided that h does not vanish in the range of consideration. If h, the length of hangers, vanishes at certain points, then we should pay attention to the original form of eq's (4) and (5) which indicate that the v_0-v term also vanishes at the same degree as h vanishes and that there exists an analytic solution for this problem. This corresponds to the fact that we assume easily an analytic solution physically for the case of free vibrations of a bridge stiffened by a center diagonal stay, which mathematically constitute the singularity of differential equations for the problem.

Before proceeding to formulate this problem it should be noticed that the laterally horizontal vibrations of a suspension bridge are classified into two types depending on singularity of differential equations in exactly the same manner as the previous case of the deflectional oscillations. Additionally the modes in question may also be subclassified into symmetric and antisymmetric modes.

Because of complexity in the form of differential equations under consideration the approximate solutions are now sought by use of the Rayleight-Ritz method. For this purpose let us assume the modes as following,

$$\begin{aligned} V_0 &= \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \\ V &= \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l} \end{aligned} \tag{8}$$

where n takes the odd numbers for symmetric and the even numbers for antisymmetric modes for the case without center diagonal stays.

For the case with center diagonal stays we have

$$\begin{aligned} V_0 &= \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \\ V &= \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} + D_n\frac{h}{l}\sin\frac{n\pi x}{l} \end{aligned} \tag{9}$$

where n takes the odd numbers for symmetric and the even numbers for antisymmetric modes.

From eq's (4) and (5) the difference of total kinetic and total potential energies are given as

$$\begin{aligned} T-U = \int_0^l \Big[\omega^2\Big\{\frac{w_f}{g}V_0^2+\frac{2w_c}{g}\sqrt{1+h'^2}V^2\Big\} - \Big\{EI_z\Big(\frac{\partial^2 V_0}{\partial x^2}\Big)^2 + \frac{2H_w h''}{h}(V_0-V)^2 \\ + 2H_w\Big(\frac{\partial V}{\partial x}\Big)^2\Big\}\Big]dx \end{aligned} \tag{10}$$

Substituting eq's (8) into eq. (10) one obtains

$$\delta(T-U) = \delta\sum_n\sum_m\{H_{nm}^{(1)}C_nC_m + 2H_{nm}^{(2)}C_nD_m + H_{nm}^{(3)}D_nD_m\} = 0$$

$$\therefore \sum_n (2H_{nm}^{(1)}C_n + 2H_{nm}^{(2)}D_n) = 0 \qquad \text{for } m = 1, 2, \cdots\cdots$$

$$\sum_n (2H_{nm}^{(2)}C_n + 2H_{nm}^{(3)}D_n) = 0 \qquad \text{for } m = 1, 2, \cdots\cdots$$

Thus the characteristic equation is obtained as

$$\begin{vmatrix} H_{11}^{(1)}, & H_{21}^{(1)}, & \cdots\cdots, & H_{11}^{(2)}, & H_{21}^{(2)}, & \cdots\cdots \\ H_{12}^{(2)}, & H_{22}^{(1)}, & \cdots\cdots, & H_{12}^{(2)}, & H_{22}^{(2)}, & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots & \cdots\cdots & \\ H_{11}^{(2)}, & H_{12}^{(2)}, & \cdots\cdots, & H_{11}^{(3)}, & H_{21}^{(3)}, & \cdots\cdots \\ H_{21}^{(2)}, & H_{22}^{(2)}, & \cdots\cdots, & H_{22}^{(3)}, & H_{22}^{(3)}, & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots & \cdots\cdots & \end{vmatrix} = 0$$

where $H_{jk}^{(i)}$ are explicitly specified in the appendix I.

In the same manner substituting eq's (9) into eq. (10) and taking variation of modes result in

$$\delta(T-U)=\delta\sum_{n}\sum_{m}\{C_nC_m\tilde{H}^{(1)}_{nm}+C_nD_m\tilde{H}^{(2)}_{nm}+C_mD_n\tilde{H}^{(2)}_{mn}+D_nD_m\tilde{H}^{(3)}_{nm}\}=0$$

for which the characteristic equation forms as

$$\begin{vmatrix} 2\tilde{H}^{(1)}_{11}, & \tilde{H}^{(1)}_{12}+H^{(1)}_{21}, & \cdots\cdots, & 2\tilde{H}^{(2)}_{11}, & \tilde{H}^{(2)}_{12}+\tilde{H}^{(2)}_{21}, & \cdots\cdots \\ \tilde{H}^{(1)}_{21}+\tilde{H}^{(1)}_{12}, & 2\tilde{H}^{(1)}_{21}, & \cdots\cdots, & \tilde{H}^{(2)}_{21}+\tilde{H}^{(2)}_{12}, & 2\tilde{H}^{(2)}_{22}, & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ 2\tilde{H}^{(2)}_{11}, & \tilde{H}^{(2)}_{12}+\tilde{H}^{(2)}_{21}, & \cdots\cdots, & \tilde{H}2^{(3)}_{11}, & \tilde{H}^{(3)}_{12}+\tilde{H}^{(3)}_{21}, & \cdots\cdots \\ \tilde{H}^{(2)}_{21}+\tilde{H}^{(2)}_{12}, & 2\tilde{H}^{(2)}_{22}, & \cdots\cdots, & \tilde{H}^{(3)}_{21}+\tilde{H}^{(3)}_{12}, & 2\tilde{H}^{(3)}_{22}, & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots & \cdots\cdots & & \cdots\cdots \end{vmatrix}=0$$

where $\tilde{H}^{(i)}_{jk}$ are explicitly specified in the appendix II.

The numerical results are shown in the appendix III which indicate that

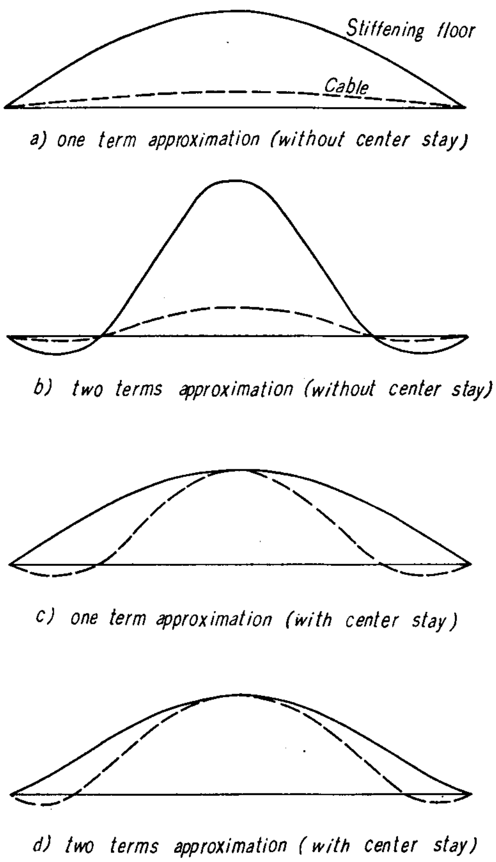

Fig. 2. Principal symmetric modes of lateral horizontal vibrations.

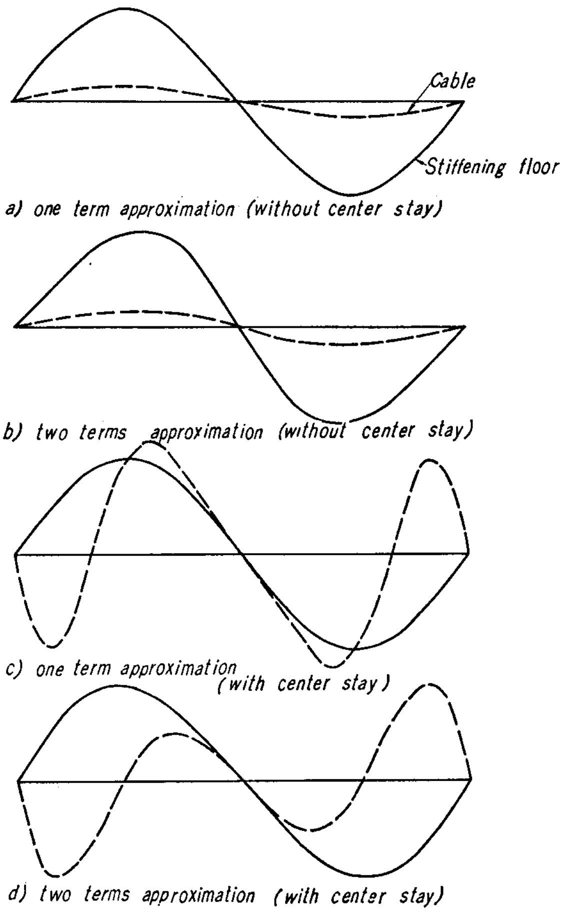

Fig. 3. Principal antisymmetric modes of lateral horizontal vibrations.

the difference of natural frequencies of a suspension bridge with center diagonal stays and one without center diagonal stays is more remarkable for the symmetric modes than the case for the antisymmetric modes. Less remarkable differences of the natural frequencies of suspension bridges with and without center stays are partly due to the assumption that the horizontal longitudinal displacements of cables are neglected here. (Fig. 2 & Fig. 3)

3. Torisional Modes of Free Vibrations

In connection with the former paragraph the torsional vibrational mode are considered here from the structural point of view. As already mentioned the second class of vibrational modes of a suspension bridge is termed as torsional, which comprises all components of displacements of stiffening floor as well as cables. This envisages to some extent the complexity of torsional behavior mathematically.

The torsional motions of a suspension bridge are particulary characterized by the following expression,

$$\frac{w_f}{gA_f}(I_y+I_z)\ddot{\theta}-\frac{\partial}{\partial x}\left\{\mu(I_y+I_z)\frac{\partial\theta}{\partial x}\right\}+b\lambda_1\left(h+w_0+\frac{b\theta}{2}-w_l\right)$$
$$-b\lambda_2\left(h+w_0-\frac{b\theta}{2}-w_r\right)=0$$

which shows dominate coupling of torsional modes of stiffening floor and vertical modes of displacements of floor and cables. Thus adding the bending resistance of main girders of stiffening floor to the above equation[2] and assuming that

$$\lambda_1=\lambda_2\equiv\lambda\,,\qquad v_r=v_l=u_r=u_l=0$$

then it is written as

$$\frac{w_f}{gA_f}(I_y+I_z)\ddot{\theta}+\frac{\partial^2}{\partial x^2}\left\{EI_{ym}\frac{\partial^2(b\theta)}{\partial x^2}\right\}-\frac{\partial}{\partial x}\left\{G(I_y+I_z)\frac{\partial\theta}{\partial x}\right\}+b\lambda(\theta-w_l-w_r)=0 \qquad (11)$$

where A_f signifies the true cross sectional area of stiffening floor and I_{ym} signifies the moment of inertia of a main girder of stiffening floor. And relating expressions for the vertical modes are of the form of:

$$\frac{w_f}{g}w_0+\frac{\partial^2}{\partial x^2}\left(EI_y\frac{\partial^2 w_0}{\partial x^2}\right)+2\lambda(w_0-w_l-w_r)=0 \qquad (12)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{w}_l-\frac{\partial}{\partial x}\left\{E_cA_c\frac{h'^2}{1+h'^2}\frac{\partial w_l}{\partial x}\right\}-H_w\frac{\partial}{\partial x}\left\{\frac{1}{1+h'^2}\frac{\partial w_l}{\partial x}\right\}-2\lambda\left(w_0+\frac{b\theta}{2}-w_l\right)=0 \qquad (13)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{w}_r-\frac{\partial}{\partial x}\left\{E_cA_c\frac{h'^2}{1+h'^2}\frac{\partial w_r}{\partial x}\right\}-H_w\frac{\partial}{\partial x}\left(\frac{1}{1+h'^2}\frac{\partial w_r}{\partial x}\right)-2\lambda\left(w_0-\frac{b\theta}{2}-w_r\right)=0 \qquad (14)$$

By the similar assumption for the reaction of hangers as before, namely $\lambda=w_f/4h$ we have the four coupled simulaneous equations for the torsional vibrational problem of a suspension bridge, which is in the same way subclassified into two cases by singularity of the equations.

For the lowest order of approximations one finds that

$$w_l = w_0 + \frac{b\theta}{2}$$

$$w_r = w_0 - \frac{b\theta}{2}$$

under which relations of dispacements the kinetic and potential energies become

$$\begin{aligned} T = &\frac{1}{2}\int_0^l \frac{w_f}{gA_f}\left\{(I_y+I_z)\left(\frac{\partial\theta}{\partial t}\right)^2 + A_f\left(\frac{\partial w_0}{\partial t}\right)^2\right\}dx \\ &+\frac{1}{2}\int_0^l \frac{w_c}{g}\sqrt{1+h'^2}\left\{\left(\dot{w}_0+\frac{b\dot{\theta}}{2}\right)^2+\left(w_0-\frac{b\dot{\theta}}{2}\right)^2\right\}dx \\ U = &\frac{1}{2}\int_0^l\left\{EI_y\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2+\mu(I_y+I_z)\left(\frac{\partial\theta}{\partial x}\right)^2\right\}dx \\ &+\frac{1}{2}\int_0^l E_cA_c\left(\frac{h'^2}{1+h'^2}\right)^2\left\{\left(w_0'+\frac{b\theta'}{2}\right)^2+\left(w_0'-\frac{b\theta'}{2}\right)^2\right\}dx \\ &+\frac{1}{2}\int_0^l H_w\left\{-h'\left(w_0'+\frac{b\theta'}{2}\right)+\frac{1}{2(1+h'^2)}\left(w_0'+\frac{b\theta'}{2}\right)^2\right\}dx \\ &+\frac{1}{2}\int_0^l H_w\left\{-h'\left(w_0'-\frac{b\theta'}{2}\right)+\frac{1}{2(1+h'^2)}\left(w_0'-\frac{b\theta'}{2}\right)^2\right\}dx \end{aligned}$$

This indicates elimination of coupling of the deflectional modes of stiffening floor and the torsional modes. It is therefore thought that different magnitudes of displacements of stiffening floor and cables characterize the complex coupling of modes in suspension bridge structures. For the lowest approximation we obtain by virtue of the above energy expressions

$$\begin{aligned} &\frac{w_f}{gA_f}(I_y+I_z)\frac{\partial^2\theta}{\partial t^2}+\frac{w_c}{2g}b^2\sqrt{1+h'^2}\frac{\partial^2\theta}{\partial t^2}+EI_{ym}b\frac{\partial^4\theta}{\partial x^4}-\mu(I_y+I_z)\frac{\partial^2\theta}{\partial x^2} \\ &\qquad -\frac{\partial}{\partial x}\left\{\frac{E_cA_c}{2}\left(\frac{b^2h'^2}{1+h'^2}\right)\frac{\partial\theta}{\partial x}\right\}-\frac{\partial}{\partial x}\left\{\frac{H_wb^2}{2(1+h'^2)}\frac{\partial\theta}{\partial x}\right\}=0 \qquad (15) \end{aligned}$$

Expanding the torsional modes into Fourier's series and applying the Ritz method we find the approximate frequencies and the corresponding modes as shown in the appendix IV.

4. Conclusions

The free vibrations of a suspension bridge are investigated from theoretical points of view. The fundamental differential equations are obtained by applying the variational principle for this problem, which is thus classified into two modes, termed as deflectional and torsional respectively.

In this paper we consider the lowest order of approximations for the torsional modes, subdividing into two typical cases, namely the lateral horizontal modes and the purely torsional modes of stiffening floor. For both cases the numerical results are obtained by use of the Ritz method for dimensions of the proposed Akashi Strait Bridge in Japan.[3] Theoretically the characteristics of torsional modes exist in coupling of various displacements, which possibly results in complex behaviors of a suspension bridge under wind stream. Though, because of complexity in the form of differential equations, the torsional problem is here subdivided into the lateral horizontal modes and the purely torsional modes of stiffening floor, the theoretical foundation is established for the results obtained by various investigatiors.[4),5]

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References

1) N. Shiraishi, the Memoirs of the Faculty of Eng'g, Kyoto Univ. Vol. XXV Part 2, 1963
2) H. Bleich and others, The mathematical theory of vibrations in suspension bridge, Dep't of Commerce, U.S.A. 1950
3) I. Konishi and others, The report to Kobe City on the experimental investigation on the proposed Akashi Strait Bridge, Dep't of Civil Eng'g, Kyoto Univ. 1959 (in Japanese)
4) M. Ito, Trans. of JSCE, No. 81, May, 1962
5) Y. Yamada, On the horizontal lateral free vibrations of a suspension bridge, Annual Meeting of the Kansai Branch of JSCE, 1962

Appendix I

$$H^{(1)}_{nm} = \omega^2 \frac{w_f}{g}\frac{l}{2}\delta_{nm} - EI_z\left(\frac{n\pi}{l}\right)^2\left(\frac{m\pi}{l}\right)^2\frac{l}{2}\delta_{nm} - \int_0^l \frac{2H_w h''}{h}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx$$

$$H^{(2)}_{nm} = \int_0^l \frac{4H_w h''}{h}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx$$

$$H^{(3)}_{nm} = \omega^2\int_0^l \frac{w_c}{g}\sqrt{1+h'^2}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx - \int_0^l \frac{2H_w h''}{h}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx$$
$$-2H_w\left(\frac{nm\pi^2}{2l}\right)\delta_{nm}$$

Appendix II

$$\bar{H}^{(1)}_{nm} = \omega^2\left\{\frac{w_f}{g}\frac{l}{2}\delta_{nm} + \int_0^l \frac{2w_c}{g}\sqrt{1+h'^2}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx\right\}$$
$$-EI_z\left(\frac{n\pi}{l}\right)^2\left(\frac{m\pi}{l}\right)^2\frac{l}{2}\delta_{nm} - \int_0^l 2H_w\left(\frac{n\pi}{l}\right)\left(\frac{m\pi}{l}\right)\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx$$

$$\bar{H}^{(2)}_{nm} = \omega^2\int_0^l \frac{2w_c h}{gl}\sqrt{1+h'^2}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx$$
$$-2H_w\int_0^l\left\{\frac{n\pi}{l}\cos\frac{n\pi x}{l}\left(\frac{h'}{l}\sin\frac{m\pi x}{l} + \frac{m\pi h}{l^2}\cos\frac{m\pi x}{l}\right)\right\}dx$$

$$\bar{H}^{(3)}_{nm} = \omega^2 \int_0^l \frac{2w_c}{g} \frac{h^2}{l^2} \sqrt{1+h'^2} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx$$
$$- \int_0^l \frac{2H_w h'' h}{l^2} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} dx$$
$$- \int_0^l 2H_w \left(\frac{h'}{l} \sin\frac{n\pi x}{l} + \frac{n\pi h}{l^2} \cos\frac{n\pi x}{l}\right)\left(\frac{h'}{l} \sin\frac{m\pi x}{l} + \frac{m\pi h}{l^2} \cos\frac{m\pi x}{l}\right) dx$$

Appendix III. Numerical illustrations

Symmetric Modes of the lateral horizontal vibrations

	without center-stay	with center-stay
one term approximation	$\omega_1 = 2.1690$ $v_0 = \sin \pi x/l$ $v = 0.160 \sin \pi x/l$	$\omega_1 = 2.5106$ $v_0 = \sin \pi x/l$ $v = \sin \pi x/l$ $-33.176 \frac{h}{l} \sin \frac{\pi x}{l}$
two terms approximation	$\omega_1 = 1.8789$ $v_0 = \sin \frac{\pi x}{l} - 0.629 \sin \frac{3\pi x}{l}$ $v = 0.1955 \sin \frac{\pi x}{l}$ $-0.1140 \sin \frac{3\pi x}{l}$	$\omega_1 = 2.3950$ $v_0 = \sin \frac{\pi x}{l} - 0.0558 \sin \frac{3\pi x}{l}$ $v = \sin \frac{\pi x}{l} - 0.0558 \sin \frac{3\pi x}{l}$ $-12.941 \frac{h}{l} \sin \frac{3\pi x}{l}$ $-3.364 \frac{h}{l} \sin \frac{3\pi x}{l}$

Anti-symmetric Modes of the lateral horizontal vibrations

	without center-stay	with centers-stay
one term approximation	$\omega_1 = 1.8605$ $v_0 = \sin \frac{2\pi x}{l}$ $v = 0.1546 \sin \frac{2\pi x}{l}$	$\omega_1 = 1.9426$ $v_0 = \sin \frac{2\pi x}{l}$ $v = \sin \frac{2\pi x}{l}$ $-29.852 \frac{h}{l} \sin \frac{4\pi x}{l}$
two terms approximation	$\omega_1 = 1.8067$ $v_0 = \sin \frac{2\pi x}{l} - 0.1385 \sin \frac{4\pi x}{l}$ $v = 0.1627 \sin \frac{2\pi x}{l}$ $-0.0292 \sin \frac{4\pi x}{l}$	$\omega_1 = 1.8875$ $v_0 = \sin \frac{2\pi x}{l} + 0.0802 \sin \frac{4\pi x}{l}$ $v = \sin \frac{2\pi x}{l} + 0.0802 \sin \frac{4\pi x}{l}$ $-35.617 \frac{h}{l} \sin \frac{2\pi x}{l}$ $-6.5579 \frac{h}{l} \sin \frac{4\pi x}{l}$

Appendix IV. Numerical Illustrations

Natural Frequencies of the torsional vibrations (symmetric modes), ω_1

$I(m^4)$	5	10	15	20	25	30
(I)	2.949	3.061	3.168	3.279	3.372	3.468
(II)	2.546	2.720	2.874	3.012	3.143	3.264

(I); one term approximation
(II); two terms approximation

Torsional Modes for the two terms approximation

$$I = 5 \qquad \theta = \sin\frac{\pi x}{1} - 0.218 \sin\frac{3\pi x}{1}$$

$$I = 10 \qquad \theta = \sin\frac{\pi x}{1} - 0.195 \sin\frac{3\pi x}{1}$$

$$I = 15 \qquad \theta = \sin\frac{\pi x}{1} - 0.1765 \sin\frac{3\pi x}{1}$$

Natural Frequnecies of the trosional vibrations (anti-symmetric modes), ω_1

$I(m^4)$	5	10	15	20	25	30
(I)	4.931	5.313	5.558	5.793	6.017	6.232
(II)	4.082	4.678	4.994	5.303	5.586	5.850

(I); one term approximation
(II); two terms approximation

Torsional Modes for the two terms approximation

$$I = 5 \qquad \theta = \sin\frac{2\pi x}{1} - 0.301 \sin\frac{4\pi x}{1}$$

$$I = 10 \qquad \theta = \sin\frac{2\pi x}{1} - 0.298 \sin\frac{4\pi x}{1}$$

$$I = 15 \qquad \theta = \sin\frac{2\pi x}{1} - 0.271 \sin\frac{4\pi x}{1}$$