A Statistical Study on the Response of Non-linear Control Systems Subjected to a Non-stationary Gaussian Random Input

By

Yoshikazu SAWARAGI*, Yoshifumi SUNAHARA* and Takashi SOEDA**

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In our previous studies, the authors have described the statistical studies on the response of non-linear control systems subjected to an ergodic stationary Gaussian random input.

In this paper, an analytical method of the statistical evaluation of non-stationary random responses which are considered to arise quite often for non-linear control systems in practice is described. Firstly, a non-stationary white Gaussian random process and a non-stationary Orenstein-Uhlenbeck one considered in this paper are explained. Secondly, a statistical approach to find the non-stationary response of non-linear control systems subjected to such a random signal is described. Finally, illustrations of the analytical procedure described here are shown by several examples in detail.

1. Introduction

Various analytical studies on the evaluation of the response of non-linear control systems subjected to Gaussian random inputs have been developed in our previous papers[1]. However, we have assumed that the random signal considered in our previous works is represented by an ergodic stationary ensemble. Strictly speaking, sometimes the random input to the system is non-stationary, that is, the statistical characteristics of it vary with time. From the practical viewpoint, we must, therefore, consider the case where the characteristics of input signals applied to the system are non-ergodic and non-stationary random signals[2]. A typical case of non-stationary random process is a white noise or a Markov process whose spectral density changes with time respectively. In this paper, we treat the following two examples. One is a non-stationary white Gaussian noise and the other is a non-stationary Markov process. We perform the evaluation of the mean squared value of the response of non-linear control systems subjected to such an input.

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* Department of Applied Mathematics and Physics

** Department of Mechanical Engineering, Tokushima University

List of Principal Symbols

t : time variable

τ : time duration between the time instants t_1 and t_2

ω and f: angular frequency and frequency respectively

s : operator with respect to the Laplace transformation

$v(t)$: desired signal to the system

$u(t)$: Gaussian random disturbance

$x(t)$: system output

$z(t)$ and $y(t)$: input and output of a non-linear element respectively

$f[z(t)]$: transfer characteristic of a non-linear element of zero-memory type

A_i, k and T: circuit parameters

$R_u(t_1;\tau)$: the auto-correlation function of the signal $u(t)$ depending on both the time instant t_1 and the time duration τ

$S_u(t_1;f)$: the spectral density of the signal $u(t)$ depending on both the time instant t_1 and the frequency f

$\psi_z(t)$: the mean squared value varying with the time t of $z(t)$

$\langle\cdot\rangle_{av.}$: the symbol representing ensemble average of $\cdot$

$p(z, t)$: the time-dependent probability density function of $z(t)$

2. Statistical Characteristics of a Non-stationary Random Input

Let us consider the correlation function and the spectral density of a non-stationary random signal $u(t)$ applied to the control system. Since the auto-correlation function, $R_u(t_1; t_2)$, of $u(t)$ with respect to $u(t_1)$ and $u(t_2)$ depending on the time instants t_1 and t_2 is written by

$$R_u(t_1;t_2) = \langle u(t_1)\,u(t_2)\rangle_{av.} = \langle u(t_2)\,u(t_1)\rangle_{av.}, \quad (1)$$

then, by using $\tau=t_1-t_2>0$, we have

$$\begin{aligned} R_u(t_1;t_2) &= \langle u(t_1)\,u(t_1-\tau)\rangle_{av.} \\ &= \langle u(t_2)\,u(t_2+\tau)\rangle_{av.} \\ &= R_u(t_1;\tau) \\ &= R_u(t_2;-\tau). \quad (2) \end{aligned}$$

The non-stationary auto-correlation function is, thus, characterized by using the time duration, τ and the time t_1 or t_2[3),4)].

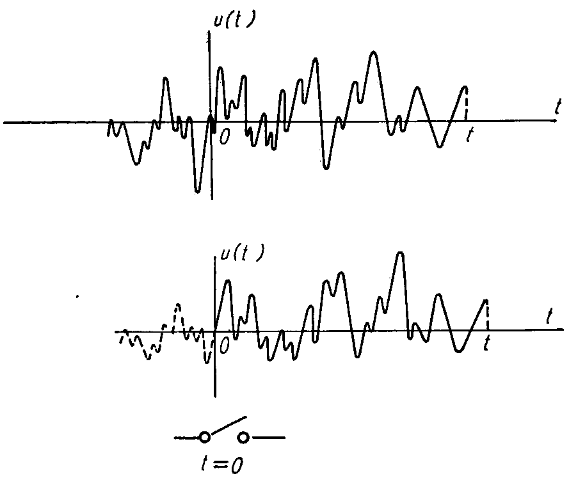

Fig. 1. Illustrations of non-stationary random signals

We consider the non-stationary spectral density corresponding to Eq. (2). The random signal considered here is divided into two cases. One is a non-stationary random signal, $u(t)$, as shown in Fig. 1(a) and the other is a non-stationary random signal, which suddenly appeared by the action of switching-on at the time instant $t=0$, as shown in Fig. 1(b). Therefore, the spectral density, $S_u(t\,;f)$, is given as follows;

(1) The case where $u(t')$ is defined in the range $-\infty<t'<t$ ($u(t')=0$ for $t<t'$) (Fig. 1(a))

$$S_u(t\,;f)=4\int_0^\infty R_u(t\,;\tau)\cos 2\pi f\tau d\tau\,. \tag{3}$$

(2) The case where $u(t')$ is defined in the range $0<t'<t$ ($u(t')=0$ for $t<t'$) (Fig. 1(b))

$$S_u(t\,;f)=4\int_0^t R_u(t\,;\tau)\cos 2\pi f\tau d\tau\,. \tag{4}$$

When a non-stationary spectral density is known, the auto-correlation function can be obtained by applying the Inverse Fourier transformation with the help of Appendix A.

3. Typical Examples of Non-stationary Random Signals

(a) Non-stationary White Noise[5]

The auto-correlation function and the spectral density of non-stationary random signal considered here are respectively given as follows;

$$R_u(t_1\,;t_2)=N(t_1\,;t_2)\delta(|t_1-t_2|)=N(t_1\,;t_1-\tau)\delta(\tau) \tag{5}$$

and

$$S_u(t\,;f)=4N(t_1\,;t_2)=k_0(t)\,,\quad(-\infty<f<\infty) \tag{6}$$

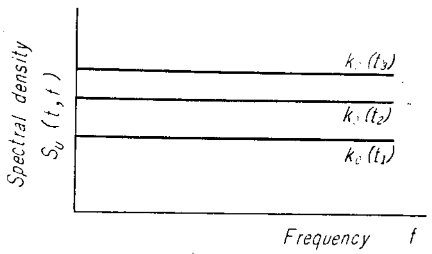

Fig. 2. An example of the spectral density of a non-stationary white noise

where δ is Dirac's Delta function and $k_0(t_1)$ is the amplitude of the noise depending on the time t_1. The value of the spectral density is uniformly distributed over the all range of the frequency as shown in Fig. 2. When the system is excited by a white noise whose spectral density varies with

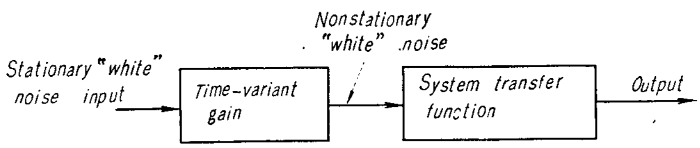

Fig. 3. Equivalent structure of non-stationary white noise

time, the situation can be considered as shown in Fig. 3.

(b) Non-stationary Orenstein-Uhlenbeck Processes[6]

The auto-correlation function and the spectral density for the non-stationary Orenstein-Uhlenbeck process are respectively given as follows;

$$R_u(t;\tau) = \frac{\sigma}{2}\exp(-\beta|\tau|)\left\{1-\exp\left(-\frac{2t}{\sigma}\right)\right\} \tag{7}$$

and

$$S_u(t;f) = \sigma\left\{1-\exp\left\{-\frac{2t}{\sigma}\right\}\right\}\left[\frac{2\beta}{\beta^2+(2\pi f)^2}+\frac{2}{\beta^2+(2\pi f)^2}\exp(-\beta t)\right.$$
$$\left.\times(2\pi f\sin 2\pi ft-\beta\cos 2\pi ft)\right], \tag{8}$$

where σ and β are positive constants (See Appendix B). The random process

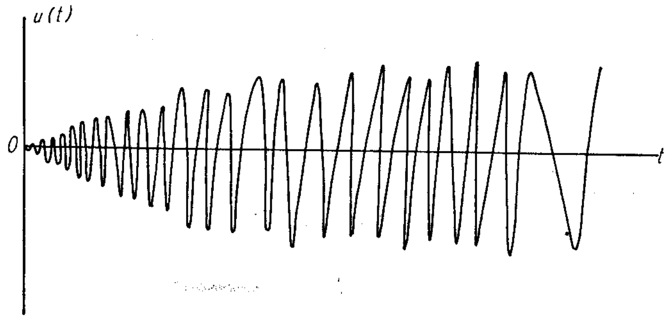

Fig. 4. An example of a non-stationary Orenstein-Uhlenbeck process

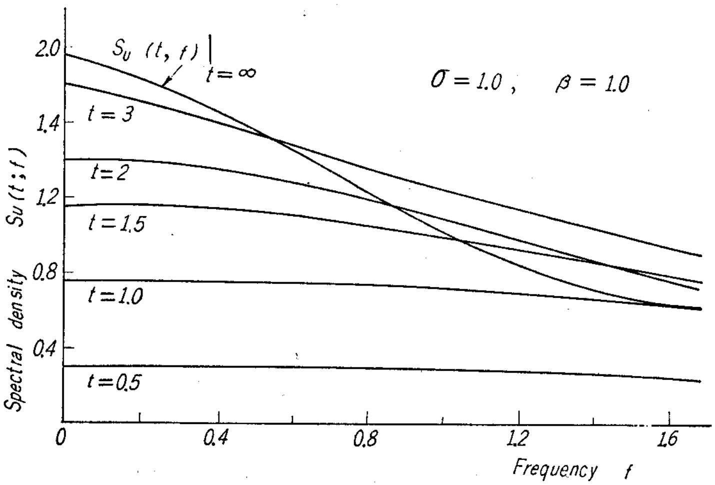

Fig. 5. An example of the non-stationary spectral density depending on time

with the auto-correlation function as shown in Eq. (7) can easily be regarded as the Orenstein-Uhlenbeck process characterized by

$$R_u(t\,;\tau)|_{t=\infty} = \frac{\sigma}{2}\exp(-\beta|\tau|), \tag{9}$$

as the time t tends to infinity[7]. An example is illustrated in Fig. 4. The form of the spectral density varying with time is shown in Fig. 5.

4. Calculations of the Mean Sqared Value, $\psi_z(t)$, of the Response $z(t)$

We consider a typical control system as shown in Fig. 6 containing a symmetric non-linear element whose instantaneous output, $y(t)$, is related with instantaneous input, $z(t)$, as $y=f(z)$. In Fig. 6,

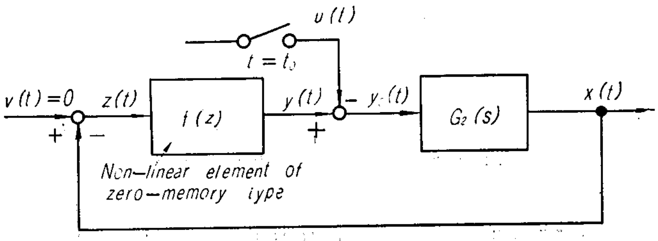

Fig. 6. Block diagram of a non-linear control system with a suddenly applied non-stationary random input

$$G_2(s) = k\Big/\sum_{i=0}^{N} A_i s^i\,, \tag{10}$$

where A_i and k are constant coefficients respectively. Then we have

$$\sum_{i=0}^{\infty} A_i \frac{d^i z(t)}{dt^i} + kf[z(t)] = k\,u(t). \tag{11}$$

In the control system under the consideration, by applying the statistical linearization technique to the non-linear element in the system, we may replace this non-linear element by a linear one with the equivalent gain, κ. The relation between the disturbance $u(t)$ and the response $z(t)$ can therefore be expressed as

$$\sum_{i=0}^{\infty} A_i \frac{d^i z(t)}{dt^i} + k\kappa z(t) = ku(t). \tag{12}$$

In Eq. (12), the equivalent gain, κ, is defined by

$$\kappa[\psi_z(t)] = \int_{-\infty}^{\infty} zf(z)p(z,\,t)\,dz\Big/\int_{-\infty}^{\infty} z^2 p(z,\,t)\,dz\,, \tag{13}$$

where $p(z,\,t)$ is the Gaussian probability density function of $z(t)$ at the time t[4]. If we formally express the unit-impulse-response function of the equivalent linearized control system by $W_1(t,\,\tau_1)$, the response $z(t)$ at the time t may be expressed as

$$z(t)=\int_{t_0}^{t} W_1(t,\tau_1)u(\tau_1)d\tau_1\,. \tag{14}$$

To calculate the auto-correlation function for the signal $z(t)$, we proceed by writing

$$z(t_1)z(t_2)=\int_{t_0}^{t_1}\int_{t_0}^{t_2} W_1(t_1,\tau_1)W_1(t_2,\tau_2)u(\tau_1)u(\tau_2)d\tau_1 d\tau_2\,. \tag{15}$$

Averaging both sides of Eq. (15) over the ensemble of random signal $u(t)$ and assuming that the averaging process can be carried out under the integral sign, we have

$$\langle z(t_1)z(t_2)\rangle_{\text{av.}}=\int_{t_0}^{t_1}\int_{t_0}^{t_2} W_1(t_1,\tau_1)W_2(t_2,\tau_2)\langle u(\tau_1)u(\tau_2)\rangle_{\text{av.}}d\tau_1 d\tau_2\,. \tag{16}$$

Therefore, setting $t_1=t_2=t$ in Eq. (16), the mean squared value, $\psi_z(t)$, is given by

$$\psi_z(t)=\int_{t_0}^{t}\int_{t_0}^{t} W_1(t,\tau_1)W_2(t,\tau_2)\langle u(\tau_1)u(\tau_2)\rangle_{\text{av.}}d\tau_1 d\tau_2\,. \tag{17}$$

In Eq. (17), since the function $W(t,\tau_i)$ $(i=1,2)$ involves an unknown function such as the equivalent gain, $\kappa[\psi_z(t)]$, it is impossible to directly evaluate $\psi_z(t)$. Therefore, as shown in the previous paper[4], we consider that $t_{j+1}-t_j=\Delta_j$ $(j=0,1,2,\cdots)$ and the value of equivalent gain corresponding to Δ_j is constant. The mean squared value, $\psi_z(t_n)$, of the response $z(t)$ at the time $t=t_n$ is thus given by

$$\begin{aligned}\psi_z(t)|_{t=t_n}&\equiv\psi_z(t_n)\\&=\sum_{i,j=0}^{N-1}\int_{t_j}^{t_{j+1}}\int_{t_i}^{t_{i+1}} W_1(t_n-\tau_1)|_{\kappa=\kappa_j}W_1(t_n-\tau_2)|_{\kappa=\kappa_i}\langle u(\tau_1)u(\tau_2)\rangle_{\text{av.}}d\tau_1 d\tau_2\,,\end{aligned} \tag{18}$$

where $W_1(t_n-\tau_l)$ $(l=1,2)$ means the response of the system at the time $t=t_n$ to the impulsive signal appied at the time $t=\tau$ under the assumption that $\kappa=$constant. The equivalent gain κ_m $(m=i,j)$ in Eq. (18) is given by

$$\kappa_m=\kappa_m[\psi_z(t_m)]\,,\qquad (m=i,j)\,. \tag{19}$$

By considering the initial condition, the mean squared value, $\psi_z(t)$, can successively be calculated from Eqs. (18) and (19). When the difference between the values of equivalent gains κ_{j+1} and κ_j changes very slowly with respect to the variation of time, Eq. (18) may approximately be written as[4]

$$\psi_z(t_n)=\int_{t_0}^{t_n}\int_{t_0}^{t_n} W_1(t_n-\tau_1)W_1(t_n-\tau_2)|_{\kappa=\kappa_{n-1}}\langle u(\tau_1)u(\tau_2)\rangle_{\text{av.}}d\tau_1 d\tau_2\,. \tag{20}$$

5. Examples

Example 1. We consider a non-linear control systems given by Eq. (11). Let $N=1$, $A_0=1$ and $A_1=T$, where T is the time constant of the system. We assume that the statistical characteristic of a random input applied to the system at the time $t=0$ is given by Eq. (5). We calculate the mean squared value, $\psi_z(t)$, of the response $z(t)$ by using Eq. (20).

In this case, the relation between the input signal $u(t)$ and the response $z(t)$ of the equivalent linearized control system can be expressed as

$$T\frac{dz}{dt}+z(t)+k\kappa z(t)=ku(t), \tag{21}$$

where κ is defined by Eq. (13). Since the weighting function for the equivalent linearized control system under the assumption that $\kappa=$constant is given by

$$W_1(t-\tau_1)=\frac{k}{T}\exp\left\{1-\frac{1+k\kappa}{T}(t-\tau_1)\right\}, \qquad (t>\tau_1). \tag{22}$$

then, by substituting Eqs. (5) and (22) into Eq. (20), we have

$$\psi_z(t_n)=\frac{k^2}{T^2}\int_0^{t_n}\exp\left\{-2\frac{1+k\kappa}{T}(t_n-\tau_1)\right\}\Big|_{\kappa=\kappa_{n-1}}k_0(\tau_1)d\tau_1. \tag{23}$$

If the form $k_0(\tau_1)$ is given by

$$k_0(\tau_1)=\{1-\exp(-\gamma\tau_1)\}^2 \qquad (\gamma>0), \tag{24}$$

then Eq. (23) becomes

$$\psi_z(t_n)=\left[\frac{k^2}{2T(1+k\kappa)}-\frac{k^2}{2}\cdot\frac{T\gamma^2}{(1+k\kappa)(1+k\kappa-\gamma T)\{2(1+k\kappa)-\gamma T\}}\exp\left\{-2(1+k\kappa)\frac{t_n}{T}\right\}\right.$$
$$\left.+\frac{k^2}{2T(1+k\kappa-\gamma T)}\exp(-2\gamma t_n)-\frac{2k^2}{T\{2(1+k\kappa)-\gamma T\}}\exp(-\gamma t_n)\right]_{\kappa=\kappa_{n-1}}. \tag{25}$$

We evaluate the response of the control system containing a non-linear element with the saturated characteristic; viz.

$$f(z)=\begin{cases}-a & (z<-a)\\ z & (|z|<a)\\ a & (z>a)\end{cases} \tag{26}$$

where a is the clipping level of the non-linear characteristic. The equivalent gain, κ, corresponding to Eq. (26) yields[4]

$$\kappa=2\Phi(a/\sqrt{\psi_z(t)}), \tag{27}$$

where

$$\Phi(\zeta)=\frac{1}{\sqrt{2\pi}}\int_0^{\zeta}\exp\left(-\frac{\xi^2}{2}\right)d\xi. \tag{28}$$

Therefore, by using Eqs. (25) and (27), we can successively evaluate the mean squared value under the condition, $\kappa_0=1$. The numerical result is shown in Fig. 7.

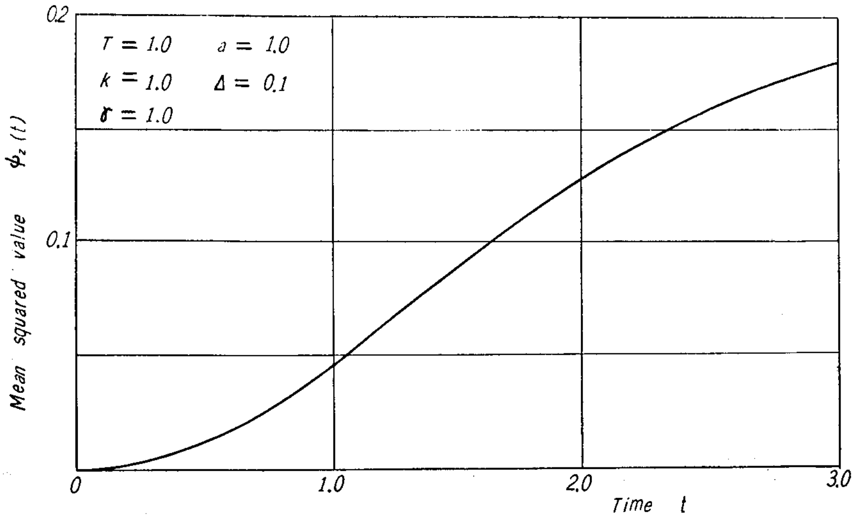

Fig. 7. The mean squared value of the response of the non-linear control system shown in example 1

Example 2. We consider that the statistical characteristic of $u(t)$ is given by Eq. (7) and the relation between the input signal $u(t)$ and the response $z(t)$ of the system is shown as Eq. (11). By a similar method as mentioned in Fig. 1, $\psi_z(t_n)$ is calculated as follows;

$$\frac{1}{\sigma}\psi_z(t_n) = k^2\Bigg[\frac{1}{(1+k\kappa)(1+k\kappa+\beta T)} + \frac{4T/\sigma}{(1+k\kappa-\beta T)(1+k\kappa+\beta T)(1+k\kappa+\beta T-2T/\sigma)} \times\exp\left\{-\beta\left(1+\frac{1+k\kappa}{T}\right)t_n\right\} - \frac{T/\sigma}{(1+k\kappa)(1+k\kappa-\beta T)(1+k\kappa-T/\sigma)}\exp\left\{-2(1+k\kappa)\frac{t_n}{T}\right\} - \frac{T/\sigma}{(1+k\kappa-T/\sigma)(1+k\kappa+\beta T-2T/\sigma)}\exp\left(-\frac{2t_n}{\sigma}\right)\Bigg]\Bigg|_{\kappa=\kappa_{n-1}}. \qquad (29)$$

Fig. 8 shows the mean squared value of the response of the control system containing the non-linear element with the saturation characteristic as shown in Eq. (26).

Example 3. The block diagram is shown in Fig. 9. The auto-correlation function of the desired signal $v(t)$ is given

$$R_v(t\,;\tau) = \frac{\sigma_v}{2}\exp(-\beta|\tau|)\left\{1-\exp\left(-\frac{2t}{\sigma_v}\right)\right\}, \qquad (\beta>0). \qquad (30)$$

Under the assumption, κ=constant, since the relation between the desired signal $v(t)$ and the response $z(t)$ of the equivalent linearized control system can be expressed as

$$T\frac{dz(t)}{dt}+z(t)+k\kappa z(t) = T\frac{dv(t)}{dt}+v(t), \qquad (31)$$

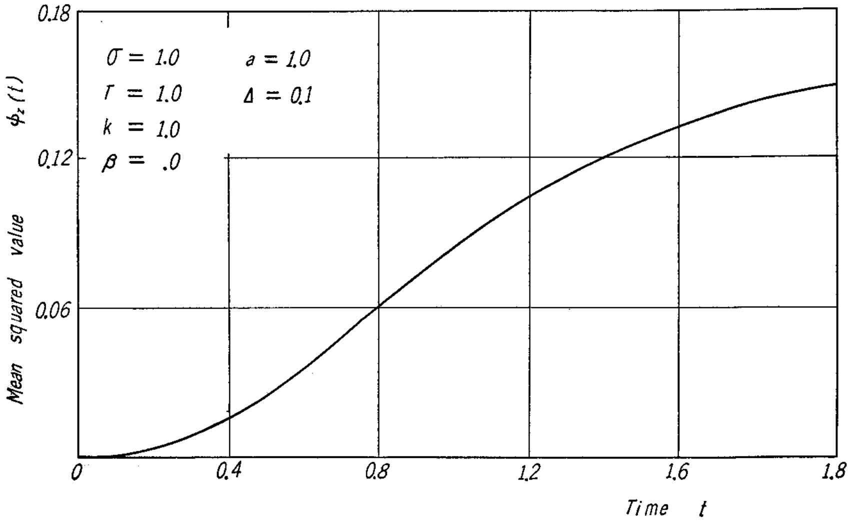

Fig. 8. The mean squared value of the response of the non-linear control system shown in example 2

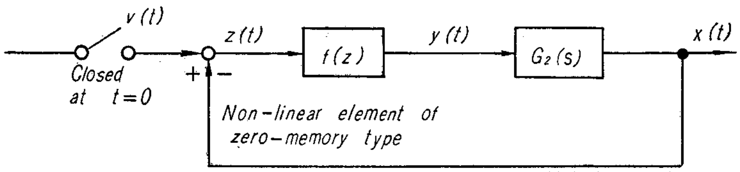

Fig. 9. Block diagram of the non-linear control system shown in example 3

then the weighting function, $W_1(t-\tau_1)$, of the system becomes

$$W_1(t-\tau_1) = \delta(t-\tau_1)+\frac{1+k\kappa}{T}\exp\left\{-(1+k\kappa)\frac{t-\tau_1}{T}\right\}, \quad (t>\tau_1). \tag{32}$$

Therefore, $\psi_z(t_n)$ becomes as follows;

$$\begin{aligned}\frac{1}{\sigma_v}\psi_z(t_n) = \Bigg[&\frac{1+\beta T(1+k\kappa)}{(1+k\kappa)(1+k\kappa+\beta T)}+\frac{4k^2\kappa^2T/\sigma_v}{(1+k\kappa+\beta T)(1+k\kappa-\beta T)(1+k\kappa+\beta T-2T/\sigma_v)}\\ &\times\exp\left\{-\left(\beta+\frac{1+k\kappa}{T}\right)t\right\}+\frac{2k\kappa}{1+k\kappa+\beta T}\left\{1-\exp\left(-\frac{2t}{\sigma_v}\right)\right\}\exp\left\{-\left(\beta+\frac{1+k\kappa}{T}\right)t\right\}\\ &+k^2\kappa^2\left\{\frac{2}{(1+k\kappa-\beta T)(1+k\kappa+\beta T-2T/\sigma_v)}\exp\left\{-2(1+k\kappa)\frac{t}{T}\right\}\right.\\ &\left.-\frac{1-k\kappa+\beta T}{1+k\kappa+\beta T}\exp(-2t/\sigma_v)\right\}\Bigg]_{\kappa=\kappa_{n-1},\, t=t_n}. \end{aligned} \tag{33}$$

It is easily shown that the first term of Eq. (33) gives the value of the steady state. Fig. 10 shows the mean squared value of the response of the control

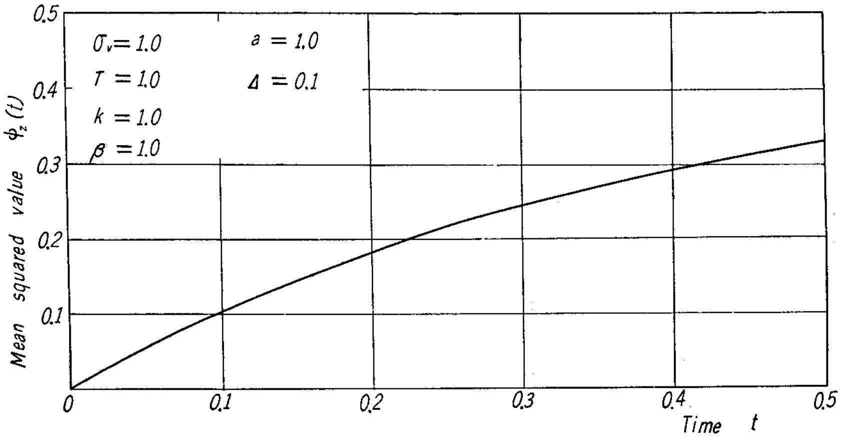

Fig. 10. The mean squared value of the response of the non-linear control system shown in example 3

system containing a non-linear element with the saturation characteristic as shown in Eq. (26).

6. Conclusions

In this paper, statistical characteristics of non-stationary white Gaussian noise and non-stationary Markov process, and the statistical study on the non-stationary random response of the system subjected to such inputs have been described. Use is made of an extension of the equivalent gain of a non-linear element to stationary random input to the case of non-stationary random input. It is emphasized that the present analysis can be applied to time-variant non-linear control systems[8].

Appendix A. Calculations of the Non-stationary Spectral Density and the Auto-correlation Function of a Random Signal $u(t)$

(a) The case where $u(t')$ is defined in the range $-\infty < t' < t$ ($u(t')=0$ for $t'>t$)

If we denote the Fourier transformation of $u(t')$ by $A_u(t;f)$, we have

$$A_u(t;f) = \int_{-\infty}^{t} u(t')\exp(-j2\pi ft')dt'. \tag{A-1}$$

The conjugate type corresponding to Eq. (A-1) becomes

$$A_u^*(t;f) = \int_{-\infty}^{t} u(t_1')\exp(j2\pi ft_1')dt_1', \tag{A-2}$$

where the star notation for the complex conjugate is used. The joint energy spectrum $E(t;f)$[2], is given by

$$E(t;f)=\int_{-\infty}^{t}\int_{-\infty}^{t}u(t')u(t_1')\exp\{j2\pi f(t_1'-t')\}dt'dt_1'. \tag{A-3}$$

Taking the ensemble average of both sides of Eq. (A-3), we get

$$\langle E(t;f)\rangle_{\text{av.}}=\int_{-\infty}^{t}\int_{-\infty}^{t}\langle u(t')u(t_1')\rangle_{\text{av.}}\exp\{j2\pi f(t_1'-t')\}dt'dt_1'. \tag{A-4}$$

Let us change the region of integration in Eq. (A-4). Letting $t_1'-t'=\tau$, it follows that[3]

$$\int_{-\infty}^{t}\int_{-\infty}^{t}dt'dt_1'=\int_{-\infty}^{t}\int_{0}^{\infty}dt_1'd\tau+\int_{-\infty}^{t}\int_{-\infty}^{0}dt'd\tau. \tag{A-5}$$

By applying this relation to Eq. (A-4) and noting the following relations;

$$\begin{aligned}\langle u(t')u(t_1')\rangle_{\text{av.}}&=\langle u(t_1')u(t_1'-\tau)\rangle_{\text{av.}}=R_u(t_1';\tau)\\&=\langle u(t')u(t'+\tau)\rangle_{\text{av.}}=R_u(t';-\tau)=R_u(t_1'-\tau;-\tau),\end{aligned} \tag{A-6}$$

we have

$$\langle E(t;f)\rangle_{\text{av.}}=2\int_{-\infty}^{t}\int_{0}^{\infty}R_u(t_1';\tau)\cos 2\pi f\tau dt_1'd\tau. \tag{A-7}$$

Therefore, the spetral denity of the input signal $u(t)$ can also be obtained by

$$S_u(t;f)=2\frac{\partial}{\partial t}[\langle E(t;f)\rangle_{\text{av.}}]=4\int_{0}^{\infty}R_u(t;\tau)\cos 2\pi f\tau d\tau. \tag{A-8}$$

under the assumption that the initial value is zero. Multiplying $\cos 2\pi f\rho$ on the both sides of Eq. (A-8) and integrating with respect to f, we have

$$R_u(t;\rho)=\int_{0}^{\infty}S_u(t;f)\cos 2\pi f\rho df. \tag{A-9}$$

(b) The case where $u(t')$ is defined in the range $t>t'>0$ ($u(t')=0$ for $t'>t$)

By noting that the value of the lower bound of the integral formula given by Eqs. (A-1) and (A-2) is zero, the joint energy spectrum corresponding to Eq. (A-4) becomes

$$\langle E(t;f)\rangle_{\text{av.}}=\int_{0}^{t}\int_{0}^{t}\langle u(t_1')u(t')\rangle_{\text{av.}}\exp\{j2\pi f(t_1'-t')\}dt_1'dt'. \tag{A-10}$$

Changing the region of integration in Eq. (A-10) by using the following formula;

$$\int_{0}^{t}\int_{0}^{t}dt_1'dt'=\int_{0}^{t}\int_{0}^{t-\tau}dt_1'd\tau+\int_{-t}^{0}\int_{0}^{t+\tau}dt'd\tau, \tag{A-11}$$

we have

$$\langle E(t;f\rangle_{\text{av.}}=2\int_{0}^{t}\int_{0}^{t-\tau}R_u(t_1';-\tau)\cos 2\pi f\tau dt_1'd\tau. \tag{A-12}$$

The spectral density of the signal $u(t)$ can also be obtained by

$$S_u(t;f)=4\int_{0}^{t}R_u(t;\tau)\cos 2\pi f\tau d\tau. \tag{A-13}$$

It is easily shown that the auto-correlation function becomes

$$R_u(t;\rho)=\int_0^\infty S_u(t;f)\cos 2\pi f\rho df. \tag{A-14}$$

Appendix B. Calculation of the Auto-correlation Function of Non-stationary Orenstein-Uhlenbeck Process

We assume that the random signal $u(t)$ is the one dimensional Gaussian Markov process whose probability density function satisfies a Fokker-Planck equation

$$\left\{L^*-\frac{\partial}{\partial t}\right\}p(u,t;u_0,t_0)=0, \tag{B-1}$$

where the symbol L^* represents

$$L^*=\frac{1}{2}\frac{\partial^2}{\partial u^2}+\{b_0(t)+b_1(t)u\}\frac{\partial}{\partial u}+b_1(t) \tag{B-2}$$

and $p(u, t; u_0, t_0)$ is the transition probability density function of the signal $u(t)$ being in the state u at the time t under the condition that at time t_0 it was in the state u_0. The probability density function for the Orenstein-Uhlenbeck process results from the values $b_0=0$ and $b_1=1/\sigma=$constant. Since the probability density function obtained by considering the condition mentioned above is given by

$$p(u,t)=\frac{1}{\sqrt{2\pi\sigma\{1-\exp(-2t/\sigma)\}/2}}\exp\left[-\frac{2u^2}{\sigma\{1-\exp(-2t/\sigma)\}}\right], \tag{B-3}$$

then

$$\psi_u(t)=\frac{\sigma}{2}\left\{1-\exp\left(-\frac{2t}{\sigma}\right)\right\}, \tag{B-4}$$

where we assume that the values of the initial conditions, u_0 and t_0, are all zero. On the other hand, the Orenstein-Uhlenbeck process is defined by the random process with

$$R_u(t,\tau)|_{t=\infty}\equiv R_u(\tau)=\exp(-\beta|\tau|)\times\text{const.},\qquad (\beta>0), \tag{B-5}$$

as its auto-correlation function. Therefore it can easily be shown that the auto-correlation function of the random signal considered here is given by

$$R_u(t,\tau)=\frac{\sigma}{2}\exp(-\beta|\tau|)\{1-\exp(-2t/\sigma)\}. \tag{B-6}$$

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