Statistical Analysis of Chopper-Modulated Circuits

By

Shigenori HAYASHI and Kōichi MIZUKAMI*

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This paper describes an approach to the statistical analysis of chopper-modulated circuits excited by random inputs.

In general the output response of those networks containing time variable elements such as a periodically operated switch, to say, the chopper, becomes the nonstationary random function, even though the input signal of the random function to networks is stationary, therefore the analytical method has to be considered for the nonstationary random process.

In this paper we introduce the available technique for such process according to Zadeh's methods which are probably the powerful tools for the analysis of linear variable networks.

The two types of practical interest are treated, that is, the transformer coupled chopper-modulated circuit and the resistance-capacitance coupled chopper-modulated circuit which are frequently used to conventional chopper amplifiers. The results obtained in this paper make clear commonly the fundamental statistical characteristics of these circuits.

1. Introduction

The chopper amplifier consists essentially of the chopper-modulated circuit, the a.c conventional amplifier and the synchronizing rectifier circuit.

In recent years this amplifier has been developed in many fields and frequently used in several types of d.c and a.c amplifiers as well as in the control process.

The chopper-modulated circuit containing a periodically operated switch plays an important role in the system of chopper amplifiers to convert input signals to signals which could be amplified in the a.c conventional amplifier.

There has been a significant advance in several studies on the characteristic of the chopper-modulated circuit for regular inputs such as a sine wave signal, but the response of the chopper-modulated circuit to random inputs has been not considered practical on the whole.

It is necessary to make clear the performance of the chopper-modulated

* Departement of Electrical Engineering, II.

circuit to random inputs, because it becomes an availiable element in the control process at the present time.

In the control field the statistical method of the analysis and synthesis of the system has remarkably progressed, but if variable elements are contained in the control system, then it is well known that of which the statistical analysis probably becomes the most difficult problem.

However it should be noted that an attempt to the statistical analysis of linear variable networks has been made in the past notably by L. A. Zadeh who established the theorem[1], that is, the correlation function of the input and output of a variable network N may formally be regarded as the input and output of a variable network N^* whose system function is the correlation function of the system function N.

The present paper will basically develop an approach[2,3] to the analysis of chopper-modulated circuits excited by random inputs by the application of Zadeh's method.

2. Statistical Method of the System Analysis

First we introduce the statistical method of the analysis of variable networks developed by L. A. Zadeh in summary.

Briefly the system function of a variable network N is defined as a function $H(j\omega\,;t)$ such that $H(j\omega\,;t)\varepsilon^{j\omega t}$ is the response of N to the exponential input $\varepsilon^{j\omega t}$. Thus denoting the input and output of N by the symbols $e_1(t)$ and $e_2(t)$, respectively. The definition of $H(j\omega\,;t)$ read

$$H(j\omega\,;t)=\frac{e_2(t)}{e_1(t)}\Bigg]_{e_1(t)=\varepsilon^{j\omega t}}.$$

The most important property of the system function of a variable network N is that it related the output and input signal of N in much the same manner as the system function of a fixed network relates its output and input signals.

Now it will be recalled that the correlation function of a signal $e(t)$ is defined by the relation

$$\Psi(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T e(t)e(t+\tau)dt\,. \tag{1}$$

This definition applies to any random-periodic signal, that is, a signal whose amplitude distribution at any one instant is a periodic function of them. In particular, it applies to the two extreme forms of random-periodic signals, namely, periodic signals and stationary signals. For stationary signals the correlation function may be written as

$$\Psi(\tau)=\overline{e(t)e(t+\tau)}$$

where the bar represents the operation of averaging over the ensemble.

As is well known, the correlation function and the power spectrum of a signal are Fourier transforms of each other. Thus, denoting the power spectrum of a signal by the symbol $S(\omega)$, the relation between $S(\omega)$ and $\Psi(\tau)$ reads

$$S(\omega)=\int_{-\infty}^{\infty}\Psi(\tau)\varepsilon^{-j\omega\tau}d\tau \tag{2}$$

and

$$\Psi(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)\varepsilon^{j\omega\tau}d\omega\,. \tag{3}$$

It is evident that $S(\omega)$ may be found from the knowledge of $\Psi(\tau)$ and vice versa.

Turning to the problems under consideration, it will be assumed that the transmission characteristics of a network N vary with time in a random-periodic manner. In terms of the system function of N this means that $H(j\omega\,;t)$ is a random-periodic function of time involving $j\omega$ as a parameter. The input to N is assumed to be a random-periodic function of time which is uncorrelated with $H(j\omega\,;t)$. The correlation functions of the input and output of N will be denoted by the symbol $\Psi_1(\tau)$ and $\Psi_2(\tau)$, respectively.

The problem is to establish a relation between $\Psi_1(\tau)$, $\Psi_2(\tau)$, and a function describing the pertinent statistical properties of $H(j\omega\,;t)$.

Here, without the proof of the theorem, it may be found that the pertient statistical properties of N are represented by a function which in view of its form might appropriately be called "the correlation function of the system function of the network" or, more simply, the correlation function of N. This function is denoted by the symbol $\Psi_H(\tau\,;\omega)$ and is defined by the relation

$$\Psi_H(\tau\,;\omega)=\lim_{T\to\infty}\frac{1}{T}\int_0^T H(j\omega\,;t)H(-j\omega\,;t+\tau)dt \tag{4}$$

which in the case of stationary system functions equivalent to

$$\Psi_H(\tau\,;\omega)=\overline{H(j\omega\,;t)H(-j\omega\,;t+\tau)}\,. \tag{5}$$

$\Psi_H(\tau\,;\omega)$ has a remarkable property which is that $\Psi_H(\tau\,;\omega)$ may be regarded as the system function of a variable network N^* such that the correlation function $\Psi_1(\tau)$ and $\Psi_2(\tau)$ are the input and output of this network, respectively. Hence it will be recognized, of course, that N^* is not a physical system since its system function $\Psi_H(\tau\,;\omega)$ is an even function of ω.

This result is expressed by the following relation

$$\Psi_2(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_H(\tau\,;\omega)S_1(\omega)\varepsilon^{j\omega\tau}d\omega \tag{6}$$

where $S_1(\omega)$ is the power spectrum of the input

$$S_1(\omega) = \int_{-\infty}^{\infty} \Psi_1(\tau)\varepsilon^{-j\omega\tau} d\tau \,, \tag{7}$$

and $\Psi_H(\tau\,;\omega)$ is the correlation function of $H(j\omega\,;t)$.

Comparison of (6) with the relation connecting the output and input of a variable network

$$e_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega\,;\,t)E_1(j\omega)\varepsilon^{j\omega t} d\omega \tag{8}$$

leads to the interpretation of (6) stated above, where

$$E_1(j\omega) = \int_{-\infty}^{\infty} e_1(t)\varepsilon^{-j\omega\tau} d\tau \,. \tag{9}$$

In addition it is shown that the Fourier transforms of the input and output of a variable network are related to each other by

$$E_2(j\omega) = \int_{-\infty}^{\infty} \Gamma(j\omega'\,;\,j\omega)E_1(j\omega')d\omega' \tag{10}$$

where the so-called "bifrequency system function", $\Gamma(j\omega'\,;\,j\omega)$ is the Fourier transform of $H(j\omega'\,;\,\tau)\varepsilon^{j\omega'\tau}$

$$\Gamma(j\omega'\,;\,j\omega) = \int_{-\infty}^{\infty} H(j\omega'\,;\,\tau)\varepsilon^{j\omega'\tau}\varepsilon^{-j\omega\tau} d\tau \,. \tag{11}$$

Now, since the power spectra $S_1(\omega)$ and $S_2(\omega)$ are the Fourier transforms of $\Psi_1(\tau)$ and $\Psi_2(\tau)$, it follows from (10) that

$$S_2(\omega) = \int_{-\infty}^{\infty} \Gamma^*(j\omega'\,;\,j\omega)S_1(\omega')d\omega' \tag{12}$$

where

$$\Gamma^*(j\omega'\,;\,j\omega) = \int_{-\infty}^{\infty} \Psi_H(\tau\,;\,\omega')\varepsilon^{j\omega'\tau}\varepsilon^{-j\omega\tau} d\tau \,. \tag{13}$$

This expression provides the desired relation between the power spectra of the input and output of N.

The theorem established above regarding the relation connecting $\Psi_1(\tau)$, $\Psi_2(\tau)$, and $\Psi_H(\tau\,;\,\omega)$, has many practical application.

The use of this theorem will be illustrated below by its application to the problems under consideration, namely, the transformer coupled chopper-modulated circuit and the resistance-capacitance coupled chopper-modulated circuit.

3. The Transformer Coupled Chopper-Modulated Circuit

The configuration of this system is shown in Fig. 1. Hence it may be assumed that the equivalent circuit to this system is found as shown in Fig. 2

and the chopper s operates ideally as shown in Fig. 3, which is called the ideal full-wave modulator.

It is now convenient to consider that the time variable system illustrated in Fig. 4 composed of $H(j\omega\,;t)$ and $G(j\omega)$ equivalent to the transformer coupled chopper-modulated circuit, where $H(j\omega\,;t)$ is the system function of the chopper s as defined below and $G(j\omega)$ is the transfer function of the equivalent circuit shown in Fig. 2.

In this case the system function $H(j\omega\,;t)$ is independent of frequency, i.e., it is of the form

$$H(j\omega\,;t)=H(t) \tag{14}$$

and the modulation process as shown in Fig. 3 can be best expressed by the Fourier series as

$$H(t)=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin(2m+1)\omega_0 t \tag{15}$$

where ω_0 is the frequency of the chopper s.

Now it is assumed that the correlation function of the input signal is

$$\Psi_1(\tau)=\varepsilon^{-\alpha|\tau|} \tag{16}$$

and correspondingly its power spectrum is

$$S_1(\omega)=\frac{2a}{\omega^2+a^2} \tag{17}$$

which is shown in Fig. 5.

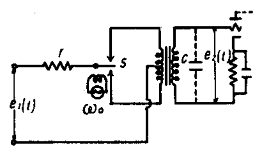

Fig. 1. Transformer coupled chopper-modulated circuit.

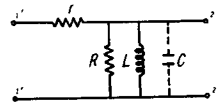

Fig. 2. Equivalent circuit.

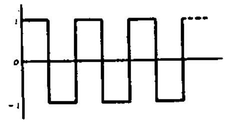

Fig. 3. Operation of chopper.

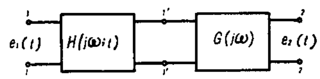

Fig. 4. Time variable system equivalent to Fig. 1.

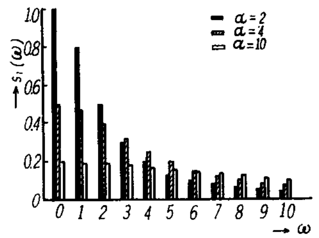

Fig. 5. Power spectrum of the input random signal.

The correlation function of $H(t)$ in this case becomes

$$\Psi_H(\tau\,;\omega)=\Psi_H(\tau)=\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\omega_0\tau\,. \tag{18}$$

Consequently the expression for the correlation function of the system at points 1′-1′ in Fig. 4 is given by

$$\begin{aligned}\Psi_{1'}(\tau)&=\Psi_H(\tau)\Psi_1(\tau)\\&=\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\omega_0\tau\cdot\varepsilon^{-\alpha|\tau|}\,.\end{aligned} \tag{19}$$

The corresponding power spectrum is readily obtained by applying the

Fourier transforms of $\Psi_1'(\tau)$ as

$$S_1'(\omega)=\int_{-\infty}^{\infty}\Psi_1'(\tau)\varepsilon^{-j\omega\tau}d\tau\,. \tag{20}$$

Therefore the power spectrum of the output of this system is written as

$$S_2(\omega)=|G(j\omega)|^2S_1'(\omega)\,, \tag{21}$$

which is the well-known relation connecting the power spectra of the output and input of a fixed network, in this case, $G(j\omega)$.

From (21), we have

$$S_2(\omega)=\frac{8}{\pi^2}|G(j\omega)|^2\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\left\{\frac{a}{a^2+(\omega-\overline{2m+1}\omega_0)^2}+\frac{a}{a^2+(\omega+\overline{2m+1}\omega_0)^2}\right\}. \tag{22}$$

Although the corresponding correlation function $\Psi_2(\tau)$ may be obtained by application of the inverse Fourier transforms of $S_2(\omega)$, or the techniques of the convolution integral, the following method, however, is simpler than the direct methods above for the evaluation of $\Psi_2(\tau)$.

Now assumed that $S_2(\omega)$ can be expressed in the form

$$S_2(\omega)=\frac{P(j\omega)}{Q(j\omega)}, \tag{23}$$

then the evaluation of $\Psi_2(\tau)$ results in

$$\Psi_2(\tau)=\sum_{\mu}\frac{P(j\omega_\mu)}{\dfrac{\partial}{\partial j\omega_\mu}Q(j\omega_\mu)}\varepsilon^{j\omega_\mu t} \tag{24}$$

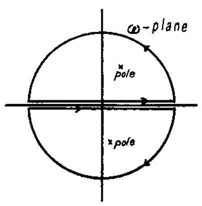

Fig. 6. Integral pass.

where $j\omega_\mu$'s is poles of $S_2(\omega)$, which is the well-known method of Heaviside's expansion.

In applying this techniques to (22), it should be remembered that since the integral pass in ω-plane must be taken as illustrated in Fig. 6, which is different in the upper plane and in the lower plane of ω, the sign of the results for poles in the lower plane should be changed.

The expression for the correlation function of the output of the system in this case reduces finally to

$$\Psi_2(\tau)=\sum_{m=0}^{\infty}\{k_{1,m}\varepsilon^{-a|\tau|}+k_{2,m}\varepsilon^{-\alpha|\tau|}\cos\omega_m\tau+k_{3,m}\varepsilon^{-\alpha|\tau|}\sin\omega_m\tau\} \tag{25}$$

where

$$\left.\begin{aligned}k_{1,m}&=\frac{16}{\pi^2}\cdot\frac{1}{(2m+1)^2}\cdot\frac{K_1a(\omega_m^2-b_1b_2)}{(\omega_m^2+b_1^2)(\omega_m^2+b_2^2)}\\ k_{2,m}&=\frac{8}{\pi^2}\cdot\frac{1}{(2m+1)^2}\cdot\left\{K+\frac{2aK_1(\omega_m^2+b_1b_2)}{(\omega_m^2+b_1^2)(\omega_m^2+b_2^2)}\right.\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}
k_{3,m} &= \frac{16}{\pi^2}\cdot\frac{1}{(2m+1)^2}\cdot\frac{K_1 a\omega_m(b_1-b_2)}{(\omega_m^2+b_1^2)(\omega_m^2+b_2^2)} \\
\omega_m &= (2m+1)\omega_0\,,\quad b_1 = a+\alpha\,,\quad b_2 = a-\alpha\,,\quad a = Rr/L(R+r)\,, \\
K &= R^2/(R+r)^2\,,\quad K_1 = -aK/2\,.
\end{aligned}\right\}$$

It is the important results in this paper that the statistical response of the system under consideration may be readily determined from Eqs. (22) and (25). The numerical results are illustrated in Fig. 7, which are the normalized correlation function of the output when

$$|G(j\omega)|^2 = \frac{K\omega^2}{a^2+\omega^2}, \tag{27}$$

namely, where the capacitance C is removed in the transformer coupled chopper-modulated circuit.

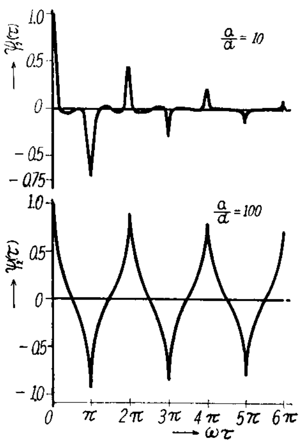

Fig. 7. Correlation function of the output in the case where $R=1000\,\Omega$, $r=50\,\Omega$, $L=1\,H$, and $\omega_0=120$.

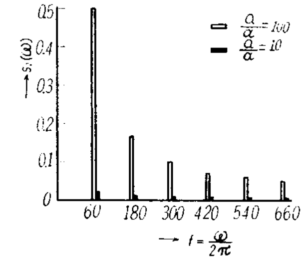

Fig. 8. Power spectrum corresponding to Fig. 7.

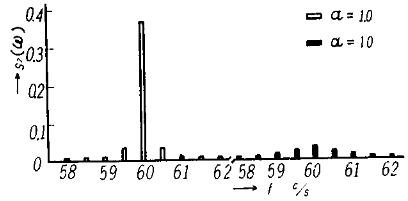

Fig. 9. Power spectra in the vicinity of the chopper frequency in the same case as Fig. 7.

The corresponding power spectrum is shown in Fig. 8 compared with each other, and Fig. 9 illustrates the spectra in the vicinity of the chopper frequency.

While on the other hand we take account of the capacitance C, the transfer function $G(j\omega)$ becomes

$$G(j\omega) = \frac{j\omega LR}{R_0+j\omega LR_1} \tag{28}$$

where

$$\begin{aligned} R_0 &= r(R-\omega^2 LRC) \\ R_1 &= R+r\,, \end{aligned}$$

Hence assume that the following resonant condition is satisfied in the chopper-modulated circuit

$$\omega^2 = \frac{1}{LC}, \tag{29}$$

then we have

$$G(j\omega) = \frac{R}{R+r} \tag{30}$$

Substituting (30) into Eq. (22), the power spectrum $S_2(\omega)$ in this case is obtained as shown in Figs. 10 and 11, where the resonant frequency $\omega/2\pi$ is chosen to be 60 c/s which is equal to the chopper frequency, so that the capacitance C must be

$$C = \frac{1}{\omega^2 L} = \frac{10^{-4}}{1.44\pi^2} F.$$

The numerical results presented in this paper were obtained by making use of the digital computer (KDC-1) for calculating of Eqs. (22) and (25).

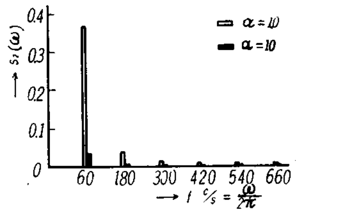

Fig. 10. Power spectrum in the case where $R=1000\Omega$, $r=500\Omega$, $\omega_0=120\pi$, $L=1\,H$ and $C=10^{-4}/1.44\pi^2 F$.

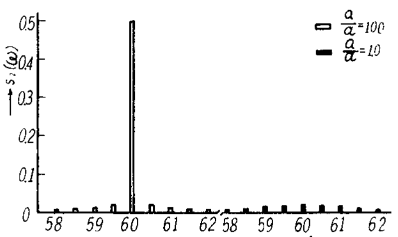

Fig. 11. Power spectra in the vicinity of the chopper frequency in the same case as Fig. 10.

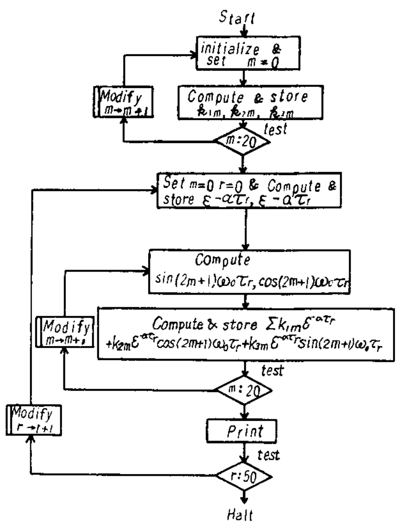

Fig. 12. Digital computer flowchart for the evaluation of the correlation function $\Psi_2(\tau)$.

The flowchart of the evaluation of, for example, the correlation function $\Psi_2(\tau)$ is shown in Fig. 12.

4. The Resistance-Capacitance Coupled Chopper-Modulated Circuit

Now consider that the configuration of this system consists of R.C elements as shown in Fig. 13.

In the same manner as the previous case, we can obtain the same form of the correlation function $\Psi_2(\tau)$ and power spectrum $S_2(\omega)$ of the output as the case of Eqs. (22) and (25) with some different constants, assuming the same random input as the form in the previous case.

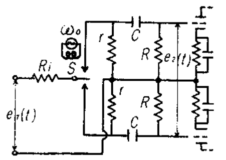

Fig. 13. R.C coupled chopper - modulated circuit.

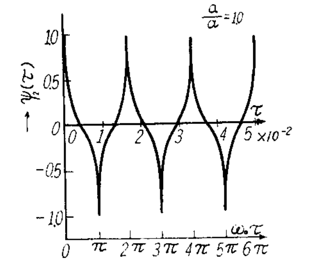

Fig. 14. Correlation function of the output of R.C coupled chopper-modulated circuit.

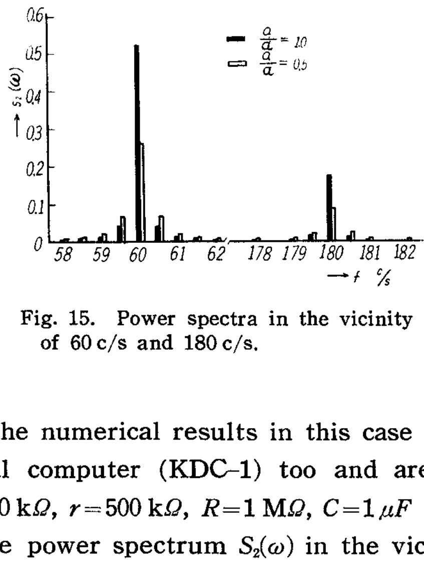

Fig. 15. Power spectra in the vicinity of 60 c/s and 180 c/s.

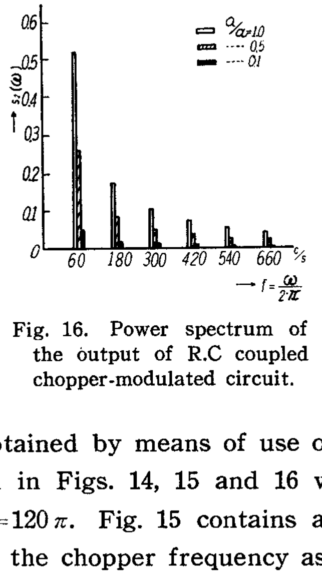

Fig. 16. Power spectrum of the output of R.C coupled chopper-modulated circuit.

The numerical results in this case were obtained by means of use of the digital computer (KDC-1) too and are shown in Figs. 14, 15 and 16 where $R=100\,\mathrm{k}\Omega$, $r=500\,\mathrm{k}\Omega$, $R=1\,\mathrm{M}\Omega$, $C=1\,\mu F$ and $\omega_0=120\,\pi$. Fig. 15 contains a plot of the power spectrum $S_2(\omega)$ in the vicinity of the chopper frequency as well as three times of one.

5. Conclusion

It has been the purpose of this paper to examine the output response of the chopper-modulated circuits to random inputs. First we have introduced, in general, the statistical method of the analysis of variable networks for input signals of the random function, On the basis of these techniques the transmission characteristics of the chopper-modulated circuits of two types have been clarified by observing the properties of the correlation function and power spectrum.

It is evident that this approach offers the advantages of mathematical simplicity, ease of interpretation, and the fact it is readily calcurable by making use of the digital computer.

The results given in this paper may serve as a reference to the circuit designer as well as to other fields of the random process, for instance, the control process and circuit theory.

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