Approximate Solution of Optimal Control Problem by Using Linear Programming Technique

By

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This paper treats an approximate solution of optimal control problem by means of the linear programming technique. Let the system be linear, then the solution of a set of differential equations which governs the system is given by the variation-of-constants formula. The state variables of the system at a fixed time are described by the definite integral, the integrand of which is a linear form in control variables. Upon use of a suitable integration formula, the integrals are approximately represented by a weighted sum of a finite number of values of the integrand.

By introducing auxiliary variables, the performance index which is required to be minimum is expressed as a linear function of the variables subject to constraints. Thus, the minimization of a functional is approximately reduced to the minimization of a linear function of many variables subject to linear constraints. This problem is a linear programming problem, and can be solved by using the simplex method. A feasible basic solution to the linear program is shown also.

1. Introduction

The structure of optimal control processes is the subject of a great deal of current research. The maximum principle[1] formulated by Pontryagin has been one of the fundamental results. Since the maximum principle does not prescribe the initial conditions for the auxiliary differential equation, a difficulty arises if one wishes actually to compute the optimal control.

This problem was solved by Neustadt[2] for linear control system. Namely, he obtained an iterative procedure for computing the initial values of the auxiliary adjoint system of differential equations, in the case where the control system is linear. Ho[3] presented a successive approximation technique for determining an optimal control for the linear time-invariant system.

This paper presents an approximate method for determining an optimal

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control by means of the linear programming technique. Zadeh and Whalen[4] showed that some of the problems of optimal control of discrete-time systems can be reduced to linear programming problems.

This paper essentially follows the results of Zadeh and Whalen[4], but treats linear continuous-time systems. The solution of a linear system of differential equations which governs the control system is given by the variation-of-constants formula.[5] The state variables of the system at a fixed time are described by the definite integral, the integrand of which is a linear form in control variables. Upon use of a suitable integration formula, the integrals are approximately represented by a weighted sum of a finite number of values of the integrand. By introducing auxiliary variables, the performance index which is required to be minimum is expressed as a linear function of the variables subject to constraints. Thus, the minimization of a functional is approximately reduced to the minimization of a linear function of many variables subject to linear constraints. This problem is a linear programming problem, and can be solved by using the simplex method.

A feasible basic solution to the linear program is shown also.

2. Problem Formulation

Consider a linear dynamic system governed by the system of differential equations,

$$\left.\begin{aligned} \frac{dx_i(t)}{dt} &= \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{r} b_{ik}(t)u_k(t), \\ i &= 1,\cdots,n. \end{aligned}\right\} \tag{1}$$

Equation (1) may be rewritten in vector form as

$$\frac{d\boldsymbol{x}(t)}{dt} = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \tag{2}$$

where $\boldsymbol{x}(t)$ is the state vector of dimension n, $\boldsymbol{u}(t)$ is the control vector of dimension r, and $A(t)$ and $B(t)$ are $n\times n$ and $n\times r$ matrices which are continuous in the time t.

Each component of the control vector $\boldsymbol{u}(t)$ is assumed to be subject to the constraint

$$\left.\begin{aligned} -\beta_i \leq u_i &\leq \gamma_i, \\ i &= 1,\cdots,r, \end{aligned}\right\} \tag{3}$$

where β_i and γ_i $(i=1,\cdots,r)$ are positive constants.

In this paper, we consider two classes of optimal control problem, i.e., (P-1) Given initial states $x(0)=x^0$ and a time T, determine the control vector $u(t)$, constrained as Eqs. (3), which minimizes the Euclidean norm defined by

$$\|x(T)\| = \sum_{i=1}^{n} |x_i(T)| . \tag{4}$$

(P-2) Given initial states $x(0)=x^0$ and a small positive number ε, determine the control vector $u(t)$ constrained as Eqs. (3), and a minimal time T such that

$$\|x(T)\| \leq \varepsilon . \tag{5}$$

Problem (P-2) is a class of the well-known time optimal problem. If the problem (P-1) can be solved, then the problem (P-2) will also be solved by solving (P-1) repeatedly for various values of T, and obtaining a dependence of $\|x(T)\|$ on T. For the problem (P-1), it is obvious from physical view-point that a minimum exists for any $A(t)$, $B(t)$, and x^0. The same does not hold true for the problem (P-2). In fact, it is known that the matrices $A(t)$ and $B(t)$ must satisfy certain conditions before (P-2) can be solved for all x^0. A system satisfying such conditions is called controllable.[6,7]

Assuming an optimum exists for (P-1) and (P-2), the engineer is interested in obtaining a numerical solution to the problems for any given initial condition. In the following, we shall show how the problem (P-1) can be reduced approximately to a linear programming problem.

3. Solution of the Fundamental Equation

Let $\Phi(t)$ be the fundamental matrix for the homogeneous system

$$\frac{dx(t)}{dt} = A(t)x(t), \tag{6}$$

i.e., $\Phi(t)$ be the $n\times n$ matrix function satisfying the equations[5]

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t), \quad \Phi(0) = E, \tag{7}$$

where E is an unit matrix.

The solution of Eq. (2) with initial values $x(0)=x^0$ is given by[5]

$$x(t) = \Phi(t)x^0 + \Phi(t)\int_0^t \Phi^{-1}(s)B(s)u(s)ds . \tag{8}$$

Let

$$\Phi(t)=\begin{pmatrix}\varphi_{11}(t)\varphi_{12}(t)\cdots\varphi_{1n}(t)\\ \varphi_{21}(t)\varphi_{22}(t)\quad\varphi_{2n}(t)\\ \vdots \qquad\qquad \vdots\\ \varphi_{n1}(t)\varphi_{n2}(t)\cdots\varphi_{nn}(t)\end{pmatrix}=\Big[\boldsymbol{\varphi}_1(t)\boldsymbol{\varphi}_2(t)\cdots\boldsymbol{\varphi}_n(t)\Big], \tag{9}$$

and

$$\left.\begin{aligned}\Phi(t)\Phi^{-1}(s)=\Psi(t,s)&=\begin{pmatrix}\psi_{11}(t,s)\psi_{12}(t,s)\cdots\psi_{1n}(t,s)\\ \psi_{21}(t,s)\psi_{22}(t,s)\quad\psi_{2n}(t,s)\\ \vdots \qquad\qquad \vdots\\ \psi_{n1}(t,s)\psi_{n2}(t,s)\cdots\psi_{nn}(t,s)\end{pmatrix}\\ &=\Big[\boldsymbol{\psi}_1(t,s)\ \boldsymbol{\psi}_2(t,s)\cdots\boldsymbol{\psi}_n(t,s)\Big],\end{aligned}\right\} \tag{10}$$

then Eq. (8) can be rewritten as

$$\left.\begin{aligned}\boldsymbol{x}(t)&=\sum_{i=1}^{n}\boldsymbol{\varphi}_i(t)x_i^0+\int_0^t\sum_{i=1}^{n}\boldsymbol{\psi}_i(t,s)\sum_{j=1}^{r}b_{ij}(s)u_j(s)ds,\\ &=\sum_{i=1}^{n}\boldsymbol{\varphi}_i(t)x_i^0+\sum_{j=1}^{r}\int_0^t\left\{\sum_{i=1}^{n}b_{ij}(s)\boldsymbol{\psi}_i(t,s)\right\}u_j(s)ds.\end{aligned}\right\} \tag{11}$$

The values of the state variables at a fixed time T are given by

$$\left.\begin{aligned}x_k(T)&=\sum_{i=1}^{n}\varphi_{ki}(T)x_i^0\\ &\quad+\sum_{j=1}^{r}\int_0^T\left\{\sum_{i=1}^{n}b_{ij}(s)\psi_{ki}(T,s)\right\}u_j(s)ds,\\ &\qquad k=1,\cdots,n.\end{aligned}\right\} \tag{12}$$

Putting

$$u_j(t)=V_j(t)-\beta_j,\qquad j=1,\cdots,r, \tag{13}$$

and substituting Eqs. (13) into Eqs. (12) yields

$$\left.\begin{aligned}x_k(T)&=\sum_{i=1}^{n}\varphi_{ki}(T)x_i^0-\sum_{j=1}^{r}\beta_j\int_0^T\left\{\sum_{i=1}^{n}b_{ij}(s)\psi_{ki}(T,s)\right\}ds\\ &\quad+\sum_{j=1}^{r}\int_0^T\left\{\sum_{i=1}^{n}b_{ij}(s)\psi_{ki}(T,s)\right\}V_j(s)ds,\\ &\qquad k=1,\cdots,n.\end{aligned}\right\} \tag{14}$$

The constraints (3) are rewritten, in terms of the variables V_j's, as

$$0\leq V_j\leq(\beta_j+\gamma_j),\qquad j=1,\cdots,r. \tag{15}$$

Introducing new notations, we define as follows:

$$\left.\begin{aligned}\sum_{i=1}^{n}\varphi_{ki}(T)x_i^0-\sum_{j=1}^{r}\beta_j\int_0^T\left\{\sum_{i=1}^{n}b_{ij}(s)\psi_{ki}(T,s)\right\}ds&=g_k(T),\\ \sum_{i=1}^{n}b_{ij}(s)\psi_{ki}(T,s)&=f_{kj}(T,s),\\ k=1,\cdots,n,\qquad j=1,\cdots,r.\end{aligned}\right\} \tag{16}$$

Since the functions $\varphi_{ki}(t)$ and $\psi_{ki}(t,s)$ $(k,i=1,\cdots,n)$ are known, just defined functions $g_k(T)$ and $f_{kj}(T, s)$ are known functions respectively. Upon use of the just defined new notations, Eqs. (14) can be rewritten as

$$\left.\begin{array}{c} x_k(T) = g_k(T) + \sum_{j=1}^{r} \int_0^T f_{kj}(T, s) V_j(s) ds \,, \\ k = 1, \cdots, n \,. \end{array}\right\} \tag{17}$$

In the case where A is a constant matrix, the fundamental matrix for Eq. (6) is given by[5]

$$\Phi(t) = e^{tA} \,. \tag{18}$$

We can determine the elements $\varphi_{ij}(t)$ $(i, j = 1, \cdots, n)$ of the matrix $\Phi(t)$ by using Sylvester expansion theorem.[8] The theorem states that if the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix A are distinct, then

$$e^{tA} = \sum_{i=1}^{n} e^{\lambda_i t} \prod_{j \neq i}^{j=1, \cdots, n} \frac{\lambda_j E - A}{\lambda_j - \lambda_i} \,, \tag{19}$$

where E is the unit matrix. Under the same assumption that A is a constant matrix, the matrix $\Psi(t, s)$ defined by Eq. (10) is expressed as

$$\Psi(t, s) = e^{(t-s)A} = \Phi(t-s) \,. \tag{20}$$

Hence,

$$\psi_{ij}(t, s) = \varphi_{ij}(t-s) \,, \qquad i, j = 1, \cdots, n \,. \tag{21}$$

4. Reduction to a Linear Programming Problem

As stated in (P-1), the performance index which is required to be minimum is taken as

$$\| x(T) \| = \sum_{i=1}^{n} | x_i(T) | \,.$$

This is a functional of the functions $V_j(t)$ $(0 \leq t \leq T, \ j = 1, \cdots, r)$. Hence, the problem consists in the minimization of the functional. We can replace approximately the minimization of the functional by the minimization of a function of many variables as we shall state in the following.

The approximate integration formula using a finite number of values of the integrand is applied to the definite integral in the right-hand side of Eqs. (17). It may be the trapezoidal formula, or Simpson's composite formula, or Gaussian integration formula.[9] After applying a numerical integration formula, Eqs. (17) are expressed as

$$\left.\begin{aligned} x_k(T) \cong g_k(T) + T\sum_{j=1}^{r}\sum_{\nu=1}^{N} c_\nu f_{kj}(T, s_\nu)V_j(s_\nu), \\ k = 1,\cdots,n. \end{aligned}\right\} \tag{22}$$

where c_ν's are the weights assigned to the values of the integrand at the points s_ν, and N is a number of terms. The values of s_ν's and the weights c_ν's are known in each integration formula. For example, in the trapezoidal formula we have

$$\left.\begin{aligned} s_\nu = \frac{\nu-1}{N-1}T, \qquad \nu = 1,\cdots,N, \\ c_1 = c_N = 1/2(N-1), \\ c_2 = c_3 = \cdots = c_{N-1} = 1/(N-1). \end{aligned}\right\} \tag{23}$$

In the Simpson's composite formula, we have

$$\left.\begin{aligned} s_\nu = \frac{\nu-1}{N-1}T, \qquad \nu = 1,\cdots,N, \\ c_1 = c_N = 1/3(N-1), \\ c_2 = c_4 = \cdots = c_{N-1} = 4/3(N-1), \\ c_3 = c_5 = \cdots = c_{N-2} = 2/3(N-1), \end{aligned}\right\} \tag{23$'$}$$

where N must be an odd number.

In Eqs. (22), we further define the following notations:

$$\left.\begin{aligned} Tc_\nu f_{kj}(T, s_\nu) = a_{k,(j-1)N+\nu}(T), \\ V_j(s_\nu) = v_{(j-1)N+\nu}. \end{aligned}\right\} \tag{24}$$

Then Eqs. (22) can be rewritten as

$$\left.\begin{aligned} x_k(T) \cong g_k(T) + \sum_{i=1}^{rN} a_{ki}(T)v_i, \\ k = 1,\cdots,n, \end{aligned}\right\} \tag{25}$$

where $g_k(T)$ and $a_{ki}(T)$ are the known constants for a given T. Equations (25) are linear forms in the variables v_i's $(i=1,\cdots,rN)$. The constraints (15) are rewritten as

$$\left.\begin{aligned} 0 \leq v_{(j-1)N+\nu} \leq (\beta_j + \gamma_j), \\ j = 1,\cdots,r, \qquad \nu = 1,\cdots,N. \end{aligned}\right\} \tag{26}$$

In view of Eqs. (4) and (25), our purpose is to determine the variables v_i $(i=1,\cdots,rN)$ which minimize the performance index

$$J(v, T) = \sum_{k=1}^{n}\left| g_k(T) + \sum_{i=1}^{rN} a_{ki}(T)v_i \right|. \tag{27}$$

This problem can be reduced to a linear programming problem as follows.[4]

Let introduce $2n$ non-negative auxiliary variables w_k and z_k $(k=1,\cdots,n)$ such that

$$\left.\begin{aligned} & g_k+\sum_{i=1}^{rN} a_{ki}v_i = \delta_k(z_k-w_k), \\ \text{where} \quad & \\ & \delta_k = \begin{cases} +1, & \text{if} \quad g_k \geq 0, \\ -1, & \text{if} \quad g_k < 0, \end{cases} \\ & z_k \geq 0, \quad w_k \geq 0, \quad k=1,\cdots,n. \end{aligned}\right\} \tag{28}$$

The minimization of Eq. (27) under the constraints (26) is equivalent to the minimization of

$$Q = \sum_{k=1}^{n}(z_k+w_k), \tag{29}$$

under the constraints (26) and (28). Because for any v_i's the minimum value of Eq. (29) is attained by setting,

$$\left.\begin{aligned} & \left.\begin{aligned} w_k &= 0, \quad \text{if} \quad g_k \geq 0 \\ z_k &= 0, \quad \text{if} \quad g_k < 0 \end{aligned}\right\} \quad \text{and} \quad g_k+\sum_{i=1}^{rN} a_{ki}v_i \geq 0, \\ & \left.\begin{aligned} z_k &= 0, \quad \text{if} \quad g_k \geq 0 \\ w_k &= 0, \quad \text{if} \quad g_k < 0 \end{aligned}\right\} \quad \text{and} \quad g_k+\sum_{i=1}^{rN} a_{ki}v_i < 0, \\ & \qquad k=1,\cdots,n. \end{aligned}\right\} \tag{30}$$

Namely, according to the signs of g_k and $g_k+\sum_{i=1}^{rN} a_{ki}v_i$, one of the variables z_k and w_k vanishes. Thus, we have

$$\min_v J(v, T) = J_m(T) = \min Q. \tag{31}$$

By introducing slack variables[10] y_j's $(j=1,\cdots,rN)$, the constraints (26) are expressed as

$$\left.\begin{aligned} & v_j+y_j = \kappa_j, \quad v_j \geq 0, \quad y_j \geq 0, \\ & \qquad j=1,\cdots,rN, \\ \text{where} \quad & \kappa_j = \beta_1+\gamma_1 \quad (j=1,\cdots,N), \\ & \kappa_j = \beta_2+\gamma_2 \quad (j=N+1,\cdots,2N), \\ & \quad \vdots \\ & \kappa_j = \beta_r+\gamma_r \quad (j=(r-1)N+1,\cdots,rN). \end{aligned}\right\} \tag{32}$$

Thus, the minimization of Eq. (27) is reduced to the minimization of Eq. (29) under the constraints:

$$\left.\begin{aligned} z_k \quad - \sum_{i=1}^{rN} \delta_k a_{ki} v_i - w_k &= |g_k| , \\ y_j \ + \ v_j \qquad &= \kappa_j \\ z_k,\ w_k,\ y_j,\ v_i &\geq 0 , \\ k = 1, \cdots, n,\ \ j &= 1, \cdots, rN . \end{aligned}\right\} \tag{33}$$

This is a linear programming problem. In view of Eqs. (33), an initial basic feasible solution[10] to the linear program can be chosen as

$$\left.\begin{aligned} v_j = w_k &= 0 , \\ z_k = |g_k| ,\ \ y_j &= \kappa_j , \\ k = 1, \cdots, n,\ \ j &= 1, \cdots, rN . \end{aligned}\right\} \tag{34}$$

Using the simplex method for linear programming and starting with the basic feasible solution (34), we can attain the optimal feasible solution in a finite number of iterations.

Expressing the performance index Q in terms of the non-basic variables[10] v_j and w_k $(j=1, \cdots, rN,\ k=1, \cdots, n)$ yields

$$Q = \sum_{k=1}^{n} |g_k| + \sum_{i=1}^{rN} \Big(\sum_{k=1}^{n} \delta_k a_{ki} \Big) v_i + 2 \sum_{k=1}^{n} w_k . \tag{35}$$

Equations (33) and (35) are the canonical form[10] in the linear program, where the simplex method can be applied directly.

5. Solution of Time Optimal Problem (P-2)

If we solve the problem (P-1) repeatedly for various values of T and obtain a dependence of $\|x(T)\|$ on T, then we can solve the problem (P-2) for an arbitrary value of ε, provided that the solution exists.

Let $\varDelta T$ be a small positive number, then

$$\left.\begin{aligned} J_m(T+\varDelta T) = \min_{v} J(v,\ T+\varDelta T) &\simeq \min \sum_{i=1}^{n} |x_i(T+\varDelta T)| \\ &\simeq \min \sum_{i=1}^{n} |x_i(T) + (dx_i/dt)_T \varDelta T | . \end{aligned}\right\} \tag{36}$$

If we can choose a control vector $u(T)$, subject to the constrains (3), such that $x_i(T)$ and $(dx_i/dt)_T$ take an opposite sign to each other for all i's $(i=1, \cdots, n)$ and for arbitrary $T>0$, i.e., such that

$$\left.\begin{aligned} x_i(T) \Big\{ \sum_{j=1}^{n} a_{ij}(T) x_j(T) + \sum_{k=1}^{r} b_{ik}(T) u_k(T) \Big\} &\leq 0 , \\ i = 1, \cdots, n,\ \ T &> 0 , \end{aligned}\right\} \tag{37}$$

then in view of Eqs. (36) we obtain

$$J_m(T+\Delta T) \leq \min \sum_{i=1}^{n} |x_i(T)| \cong J_m(T). \tag{38}$$

Eq. (38) means that $J_m(T)$ is a monotone decreasing function of T. When the values of x_i's $(i=1,\cdots,n)$ are comparatively small, Eqs. (37) will hold in usual cases.

Let the matrix A be constant, and all eigenvalues λ_i's $(i=1,\cdots,n)$ of the matrix A be real, non-positive, and distinct. Then upon use of a suitable linear transformation (for example Lurie's transformation[11]), Eqs. (1) can be transformed into the canonical form

$$\left.\begin{aligned} dx_i/dt &= \lambda_i x_i + \sum_{k=1}^{r} b'_{ik} u_k, \\ i &= 1,\cdots,n. \end{aligned}\right\} \tag{39}$$

Since λ_i's are the non-negative values, setting $u_k=0$ $(k=1,\cdots,r)$ in Eqs. (39) yields

$$\left.\begin{aligned} x_i(dx_i/dt) &= \lambda_i x_i^2 \leq 0, \\ i &= 1,\cdots,n. \end{aligned}\right\} \tag{40}$$

Therefore, in this case $J_m(T)$ is a monotone decreasing function.

Assuming that $J_m(T)$ is a monotone decreasing function, we can obtain the minimum value T_l of the time which satisfies

$$J_m(T_l) \leq \varepsilon,$$

by starting with a suitable value $T_1(<T_l)$ and iterating the computations of the linear program for the successive values of T given by

$$\left.\begin{aligned} T_{\nu+1} &= T_\nu + J_m(T_\nu)\frac{\Delta T}{J_m(T_\nu)-J_m(T_\nu+\Delta T)}, \\ \nu &= 1, 2,\cdots, \end{aligned}\right\} \tag{41}$$

until T_l is found. The above-mentioned procedure is due to the Newton's iterative method. Thus, we can obtain the approximate minimum time T_l and the sampled data v_i $(i=1,\cdots,rN)$ of the optimal control functions which transfer x^0 to the ε-neighbourhood of the origin of the state space.

6. Conclusions

We showed that the optimal control problem in linear system can be reduced to the linear programming problem. An initial basic feasible solution of the linear program is shown also. We can obtain an accurate solution to the problem if we increase the number N of the terms in the numerical intergration formula.

The simplex method for the solution of linear programming problems is a powerful algorithm which is fit for machine computation. The iterative procedure for solving the time optimal problem (P-2) is suitable for high speed digital computers.

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