Analysis of Sampled-Data Feedback Control Systems with Finite Pulse Width

By

Shigenori HAYASHI* and Kōichi MIZUKAMI*

(Received June 3, 1964)

A method for analysis of sampled-data feedback control systems with finite pulse width is presented in this work.

The analysis is based on introducing a new technique with some approximations by the theories of the Periodically Interrupted Electric Circuits.

The results make it possible to obtain the transient and steady-state response of such systems containing a sampler with any pulse width. Furthermore, by utilizing this method, the stability criterion is readily developed and its application to systems under consideration make clear the effects of the sampler with finite pulse width on the performance of sampled-data feedback control systems, which are of considerable importance in view of its engineering applications.

Some illustrative examples are given to clarify the method involved and its numerical results are presented to show the performance of the system dynamics. Moreover the results in this investigation are examined by the simulation of the control system in question by means of the analog computer.

1. Introduction

Analysis and design of sampled-data control systems have been developed after considerable investigation during the past decade. Especially, the development of the z-transform and modified z-transform in recent years has become an important tool in the analysis and synthesis of sampled-data systems.

However, in almost all cases the sampled version of a given signal has continued to be regarded idealistically as a train of impulses of infinitesimal duration.

More recently, efforts have been made to deal more realistically with the sampling problem by recognizing that in practice the duration of a pulse formed by sampling a function of time is not only not infinitesimal, but often appreciable.

Farmanfarma[1] introduced the p-transform to allow consideration of pulse

* Department of Electrical Engineering, II.

width, and Tou[2] introduced the τ-transform for the same purpose. These efforts led to the establishment of methods of analysis and synthesis of open-loop sampled-data systems with appreciable pulse width.

Furthermore, Farmanfarma[3] has published an exact method for the analysis and synthesis of closed-loop sampled-data systems with finite pulse width. However, the exact method is very tedious. Murphy and Kennedy[4] developed an approximate method for such systems from a different point of view.

The purpose of this paper is to present a method which allows the treatment of sampled-data systems with pulse width ranging from zero to the sampling period[5,6]. The technique presented here, which is based on the theories of the Periodically Interrupted Electric Circuits, yields approximately the correct value of the system response at all instants and also develops the valuable stability criterion for systems under consideration.

Investigating the stability of sampled-data feedback control systems with finite pulse width is of considerable importance to examine the effects of pulse width on the system dynamics.

Numerical examples are worked out to show the application of the method presented in this paper, and also the results are examined by means of analog computer simulation.

2. General Analytical Method Based on the Theories of the Periodically Interrupted Electric Circuits

The sampled-data system with finite pulse width under consideration is shown in Fig. 1.

As is well-known, the basic difference between a continuous system and a finite pulsed sampled-data system is due to the presence of the sampler in the latter. Thus any general investigation of the pulsed system must start with the study of the sampler, and the relation between its input and output time functions.

Fig. 2 illustrates the input and output of a sampler with pulse width h and sampling period T.

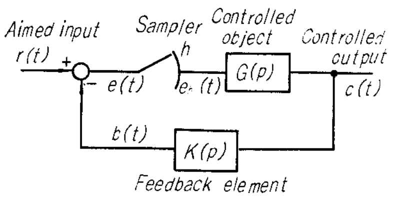


Fig. 1. Sampled-data feedback control system with finite pulse with,

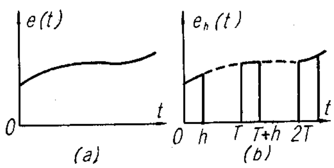


Fig. 2. (a)-Input signal to sampler. (b)-Sampler output,

Hence, the relation between input $e(t)$ and output $e_h(t)$ in the time domain may be written as:

$$e_h(t) = e(t)\{u(t)-u(t-h)\}+e(t)\{u(t-T) -u(t-T-h)\}+\cdots = e(t)\sum_{n=0}^{\infty}\{u(t-nT)-u(t-nT-h)\} \quad (2.1)$$

where $u(t)$ is a unit step function.

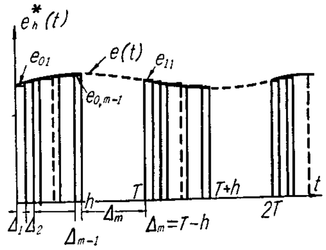


Fig. 3. Pulse width h of sampler output is subdivided into $m-1$ narrow pulses with each different width Δi $(i=1, 2, \cdots, m-1)$.

Now if pulse width h of sampler output is subdivided into $m-1$ narrow pulses with each different width Δ_i, $(i=1, 2, \cdots, m-1)$ as shown in Fig. 3, then the output $e_h(t)$ is expressed in the form, too

$$\begin{aligned} e_h(t) &= e(t)\{u(t)-u(t-\Delta_1)\}+e(t)\{u(t-\Delta_1)-u(t-\Delta_1-\Delta_2)\}+\cdots \\ &\quad +e(t)\{u(t-\sum_{i=1}^{m-2}\Delta_i)-u(t-\sum_{i=1}^{m-1}\Delta_i)\}+e(t)\{u(t-T)-u(t-T-\Delta_1)\}+\cdots \\ &\quad +e(t)\{u(t-nT)-u(t-nT-\Delta_1)\}+\cdots \\ &\quad +e(t)\{u(t-nT-\sum_{i=1}^{m-2}\Delta_i)-u(t-nT-\sum_{i=1}^{m-1}\Delta_i)\}+\cdots \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{m-2}e(t)\{u(t-nT-\sum_{i=0}^{j}\Delta_i)-u(t-nT-\sum_{i=0}^{j+1}\Delta_i)\} \end{aligned} \quad (2.2)$$

where $\Delta_0=0$.

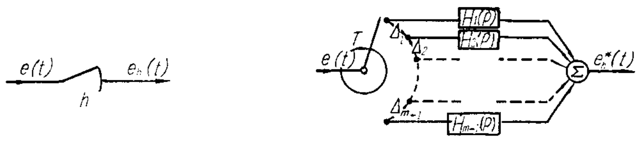
For the application of our new technique to the system under discussion, we first assume that the output pulse $e_h(t)$ may be expressed approximately by staircase functions as indicated in Fig. 3, in other words, that it is flat-topped pulses during each different width Δ_i, then we have

$$\begin{aligned} e_h^*(t) &= e(0)\{u(t)-u(t-\Delta_1)\}+e(\Delta_1)\{u(t-\Delta_1)-u(t-\Delta_1-\Delta_2)\}+\cdots \\ &\quad +e(nT)\{u(t-nT)-u(t-nT-\Delta_1)\}+\cdots \\ &\quad +e(nT+\sum_{i=1}^{m-2}\Delta_i)\{u(t-nT-\sum_{i=1}^{m-2}\Delta_i)-u(t-nT-\sum_{i=1}^{m-1}\Delta_i)\}+\cdots \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{m-2}e(nT+\sum_{i=0}^{j}\Delta_i)\{u(t-nT-\sum_{i=0}^{j}\Delta_i)-u(t-nT-\sum_{i=1}^{j+1}\Delta_i)\} \end{aligned} \quad (2.3)$$

Accordingly, it may be obviously said, from inspection of Eq. (2.3), that the sampler with finite pulse width h in practice is replaced by an idealized cyclic sampler approximately equivalent to it. This ideal periodic sampler with sampling period T is illustrated in Fig. 4 (B), which is followed by zero order hold networks with different holding time Δ_i, that is

$$H_i(p) = \frac{1-\varepsilon^{-p\Delta_i}}{p}, \qquad i = 1, 2, \cdots, m-1. \quad (2.4)$$

Thus the system under consideration as shown in Fig. 1 can be evidently



(A) Sampler with finite pulse width h in practice.

(B) An idealized cyclic sampler approximately equivalent to the sampler in Fig. 4 (A).

Fig. 4. Cyclic ideal sampler with sampling period T followed by zero order hold networks $H_i(p)=(1-\varepsilon^{-p\Delta_i})/p$.

rewritten as illustrated in Fig. 5 according to replacement of the cyclic ideal sampler and zero order hold networks. Hereafter, we consider this approximate equivalent system in the following analysis by means of the present method in this paper.

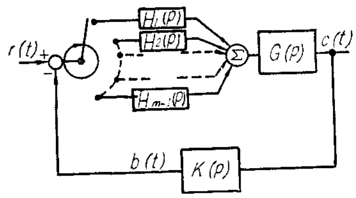


Fig. 5. Equivalent block diagram of the sampled-data feedback control system with finite pulse width h.

Now assume that a linear controlled object with transfer function $G(p)$, as shown in Fig. 5, is generally an α-th order element and subjects to the sampled output $e_h^*(t)$, and that a feedback element with transfer function $K(p)$ is generally a β-th order element.

For brevity, we introduce a new symbol such that

$$e_h^*(nT)\equiv e_{n1},\quad e_h^*(nT+\Delta_1)\equiv e_{n2},\quad e^*(nT+\sum_{i=1}^{r-1}\Delta_i)\equiv e_{nr} \tag{2.5}$$

where $n=0,1,2,\cdots$ and $r=1,2,\cdots,m$.

During from $t=nT$ to $t=nT+\Delta_1$, that is, its interval is said so as the 1st circuit mode on the n-th stage, the controlled output $c_n(t)$ may be obtained by solving the differential equation of the form

$$\frac{d^\alpha}{d\tau^\alpha}c_n(\tau)+a_1\frac{d^{\alpha-1}}{d\tau^{\alpha-1}}c_n(\tau)+\cdots+a_{\alpha-1}\frac{d}{d\tau}c_n(\tau)+a_\alpha c_n(\tau)=e_{n1}\qquad 0\leq\tau\leq\Delta_1 \tag{2.6}$$

where $a_1,\cdots,a_\alpha$ are constants of the controlled object.

Here it is convenient to express Eq. (2.6) in the matrix notation as

$$\begin{pmatrix} D & a_1D & a_2D\cdots\cdots a_{\alpha-1} & D+a_\alpha \\ -1 & D & 0\cdots\cdots\cdots & 0 \\ 0 & -1 & D\cdots\cdots\cdots & 0 \\ \vdots & \cdots\cdots & \cdots\cdots & \vdots \\ \vdots & \cdots\cdots & \cdots\cdots & \vdots \\ 0 & \cdots\cdots 0 & -1 & D \end{pmatrix}\begin{pmatrix} c_n^{\alpha-1}(\tau) \\ \vdots \\ \vdots \\ \vdots \\ c_n'(\tau) \\ c_n(\tau)\end{pmatrix}=\begin{pmatrix} e_{n1} \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0\end{pmatrix}\qquad 0\leq\tau\leq\Delta_1 \tag{2.7}$$

where $D=\frac{d}{d\tau}$, and $c_n^i(\tau)=\frac{d^i}{d\tau^i}c_n(\tau)$, $(1=1,2,\cdots,\alpha-1)$, In feedback loop, the differential equation is written in the form

$$\frac{d^{\beta}}{d\tau^{\beta}}b_n(\tau)+d_1\frac{d^{\beta-1}}{d\tau^{\beta-1}}b_n(\tau)+\cdots+d_{\beta-1}\frac{d}{d\tau}b_n(t)+d_{\beta}b_n(\tau)=c_n(\tau) \quad 0\leq\tau\leq\Delta_1 \tag{2.8}$$

where $b_n(\tau)$ is the output of the feedback element in the 1st circuit mode on the n-th stage, and $d_1, \cdots, d_{\beta}$ are constants of the feedback element.

Similarly Eq. (2.8) is expressed in matrix notation

$$\begin{pmatrix} D & d_1D & d_2D\cdots\cdots d_{\beta-1} & D+d_{\beta} \\ -1 & D & 0 \cdots\cdots & 0 \\ 0 & -1 & \cdots\cdots & \vdots \\ \vdots & \cdots\cdots & & \vdots \\ \vdots & \cdots\cdots & & 0 \\ 0 & \cdots\cdots 0 & -1 & D \end{pmatrix} \begin{pmatrix} b_n^{\beta-1}(\tau) \\ \vdots \\ \vdots \\ \vdots \\ b_n'(\tau) \\ b_n(\tau) \end{pmatrix} = \begin{pmatrix} c_n(\tau) \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad 0\leq\tau\leq\Delta_1 \tag{2.9}$$

where $D=\frac{d}{d\tau}$, and $b_n^i(\tau)=\frac{d^i}{d\tau^i}b_n(\tau)$, $(i=1, 2, \cdots, \beta-1)$.

While the relation between $e(t)$ and $b(t)$ is represented as follows, clearly by inspection of the system under discussion

$$\begin{aligned} e_{n1} &= \{r(t)-b(t)\}_{t=nT} \\ &\vdots \\ e_{nr} &= \{r(t)-b(t)\}_{t=nT+\sum_{i=1}^{r-1}\Delta_i}. \end{aligned} \tag{2.10}$$

Now, for simplicity, Eqs. (2.7) and (2.9) are rewritten in the abbreviate matrix forms, respectively

$$[z_G(D)][y_{n1}(\tau)]=[w_{n1}] \tag{2.11}$$

and

$$[z_K(D)][x_{n1}(\tau)]=[T'][y_{n1}(\tau)] \tag{2.12}$$

where $[T']$ is a transformation matrix whose elements consist of 0 or ±1.

While, $[w_{n1}]$ is expressed as

$$\begin{aligned} [w_{n1}] &= \{[R(t)]-[T'][x(t)]\}_{t=nT} \\ &= [R_{n1}]-[T'][x_{n1}] \end{aligned} \tag{2.13}$$

where $[x_{n1}]=[x_{n-1,m}(\Delta_m)]$, and $[R_{n1}]=[R_{n-1,m}(\Delta_m)]$ consists of the aimed input $r(t)$.

Then substituting Eq. (2.13) into Eq. (2.11) yields

$$[z_G(D)][y_{n1}(\tau)]=[R_{n1}]-[T'][x_{n1}]. \tag{2.14}$$

Hence it is first necessary to obtain $[x_{n-1,m}(\tau)]$ from Eq. (2.12) for determination of $[x_{n1}]=[x_{n-1,m}(\Delta m)]$.

By the use of the Laplace transformation*, Eq. (2.12) becomes

* $F(p)=p\int_0^{\infty}f(t)\,\varepsilon^{-pt}\,dt$,

$$[Z_K(p)][X_{n-1,m}(p)] = [T'][Y_{n-1,m}(p)] + p[L_K][x_{n-1,m}^{+0}] \tag{2.15}$$

where elements of $[L_K]$ consist of elements l_{ij} when elements of $[Z_K(p)]$ have the form of $l_{ij}p + r_{ij}$.

Premultiplying each term on the both-hand side of Eq. (2.15) by $[Z_K(p)]^{-1}$, we have

$$[X_{n-1,m}(p)] = [Z_K(p)]^{-1}[T'][Y_{n-1,m}(p)] + p[Z_K(p)]^{-1}[L_K][x_{n-1,m}^{+0}]. \tag{2.16}$$

Transforming Eq. (2.16) into time function by the use of the inverse Laplace transformation**, we get

$$\begin{aligned}[x_{n-1,m}(\tau)] &= [\varphi_K(\tau)]*[T'][y_{n-1,m}(\tau)] + [\chi_K(\tau)][x_{n-1,m}^{+0}] \\ &= [\varphi_K\cdot y_{n-1,m}(\tau)] + [\chi_K(\tau)][x_{n-1,m-1}(\varDelta_{m-1})] \qquad 0 \leq \tau \leq \varDelta_m\end{aligned} \tag{2.17}$$

where a symbol "*" means the convolution integral, and its result is written of the form $[\varphi_K\cdot y_{n-1,m}(\tau)]$, for brevity, while

$$\left.\begin{aligned}[\varphi_K(\tau)] &= \mathfrak{H}[Z_K(p)]^{-1}, \\ [\chi_K(\tau)] &= \mathfrak{H}p[Z_K(p)]^{-1}[L_K].\end{aligned}\right\} \tag{2.18}$$

Similarly, Eq. (2.14) is expressed in the form of p-function as

$$[Z_G(p)][Y_{n1}(p)] = [R_{n1}] - [T'][x_{n1}] + p[L_G][y_{n1}^{+0}]. \tag{2.19}$$

By a similar way, Eq. (2.19) becomes

$$[Y_{n1}(p)] = [Z_G(p)]^{-1}\{[R_{n1}] - [T'][x_{n1}]\} + p[Z_G(p)]^{-1}[L_G][y_{n1}^{+0}] \tag{2.20}$$

then, the solution in time domain is given by

$$\begin{aligned}[y_{n1}(\tau)] &= [\varphi_G(\tau)]\{[R_{n1}] - [T'][x_{n1}]\} + [\chi_G(\tau)][y_{n1}^{+0}] \\ &= [\varphi_G(\tau)][R_{n1}] + [\chi_G(\tau)][y_{n-1,m}(\varDelta_m)] - [\varphi_G(\tau)][T'][x_{n-1,m}(\varDelta_m)] \\ &\qquad 0 \leq \tau \leq \varDelta_1\end{aligned} \tag{2.21}$$

where

$$[\varphi_G(\tau)] = \mathfrak{H}[Z_G(p)]^{-1} \tag{2.22}$$

$$[\chi_G(\tau)] = \mathfrak{H}p[Z_G(p)]^{-1}[L_G] \tag{2.23}$$

and the unknown matrices $[x_{n-1,m}(\varDelta_m)]$ and $[y_{n-1,m}(\varDelta_m)]$ are readily determined by the following recurrence formulae in matrix notation

$$[x_{n-1,m}(\varDelta_m)] = [\varphi_K\cdot y_{n-1,m}(\varDelta_m)] + [\chi_K(\varDelta_m)][x_{n-1,m-1}(\varDelta_{m-1})] \tag{2.24}$$

and

$$\begin{aligned}[y_{n-1,m}(\varDelta_m)] &= [\varphi_G(\varDelta_m)][R_{n-1,m}(\varDelta_m)] + [\chi_G(\varDelta_m)][y_{n-1,m-1}(\varDelta_{m-1})] \\ &\quad - [\varphi_G(\varDelta_m)][T'][x_{n-1,m-1}(\varDelta_{m-1})].\end{aligned} \tag{2.25}$$

** $f(t) = \mathfrak{H}F(p) = \dfrac{1}{2\pi i}\lim_{\beta\to\infty}\int_{r-i\beta}^{r+i\beta}\dfrac{F(p)}{p}\varepsilon^{pt}\,dp.$

It is evident in the end from these recurrence formulae that these two values on the left-hand side of Eqs. (2.24) and (2.25) are given by the given initial value matrices $[y_{11}^{+0}]$ and $[x_{11}^{+0}]$ respectively.

Therefore, Eq. (2.21) is the general solution of the system under discussion in the 1st circuit mode on the n-th stage, in other words, during from $t=nT$ to $t=nT+\Delta_1$.

Consequently, with above relations in mind, the solution in the i-th circuit mode on the n-th stage can be written as

$$[y_{ni}(\tau)] = [\varphi_G(\tau)][R_{ni}]+[\chi_G(\tau)][y_{n,i-1}(\Delta_{i-1})]-[\varphi_G(\tau)][T'][x_{n,i-1}(\Delta_{i-1})] \quad 0\leq\tau\leq\Delta_i \quad (i=1,2,\cdots,m.\quad n=0,1,\cdots) \tag{2.26}$$

while

$$[x_{n,i-1}(\Delta_{i-1})] = [\varphi_K\cdot y_{n,i-1}(\Delta_{i-1})]+[\chi_K(\Delta_{i-1})][x_{n,i-2}(\Delta_{i-2})] \tag{2.27}$$

and

$$[y_{n,i-1}(\Delta_{i-1})] = [\varphi_G(\Delta_{i-1})][R_{n,i-1}]+[\chi_G(\Delta_{i-1})][y_{n,i-2}(\Delta_{i-2})] -[\varphi_G(\Delta_{i-1})][T'][x_{n,i-1}(\Delta_{i-2})]. \tag{2.28}$$

Since the matrices on the left-hand side of Eqs. (2.27) and (2.28) can be finally determined by the given initial matrices $[y_{11}^{+0}]$ and $[x_{11}^{+0}]$ according to the recurrence formulae, Eq. (2.26) results in the general solution in any circuit mode on any stage.

3. Numerical Examples and The Effects of Pulse Width on the Stability of the Control System

First we consider the control system under discussion with unity feedback $K(p)=1$.

In this case, the solution is reduced to the more simple form as follows, putting $[x_{n,i-1}(\Delta_{i-1})]$ on the right-hand side of Eq. (2.26) into $[y_{n,i-1}(\Delta_{i-1})]$,

$$[y_{ni}(\tau)] = [\varphi_G(\tau)][R_{ni}]+[\chi(\tau)][y_{n,i-1}(\Delta_{i-1})] \qquad 0\leq\tau\leq\Delta_i \tag{3.1}$$

where $[\chi(\tau)]=[\chi_G(\tau)]-[\varphi_G(\tau][T']$

and the initial matrix $[y_{n,i-1}(\Delta_{i-1})]$ is written of the recurrence formula, but in a more compact form, as

$$\begin{aligned}[y_{n,i-1}(\Delta_{i-1})] = {} & [\varphi_G(\Delta_{i-1})][R_{n,i-1}]+[\chi(\Delta_{i-1})][\varphi_G(\Delta_{i-2})][R_{n,i-2}] \\ & +[\chi(\Delta_{i-1})][\chi(\Delta_{i-2})][\varphi_G(\Delta_{i-3})][R_{n,i-3}]+\cdots \\ & +[\chi(\Delta_{i-1})][\chi(\Delta_{i-2})]\cdots[\chi(\Delta_1)][y_{n-1,m}(\Delta_m)]\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}[y_{n-1,m}(\Delta_m)] = {} & [\Phi_{n-1,m}]+[B_m][\Phi_{n-2,m}]+[B_m]^2[\Phi_{n-3,m}]+\cdots \\ & +[B_m]^{n-2}[\Phi_{1m}]+[B_m]^{n-1}[y_{11}^{+0}]\end{aligned} \tag{3.3}$$

where

$$[B_m] = [\chi(\Delta_m)][\chi(\Delta_{m-1})]\cdots[\chi_1(\Delta_1)] \tag{3.4}$$

$$[\Phi_{rm}] = [\varphi_G(\Delta_m)][R_{rm}] + [\chi(\Delta_m)][\varphi_G(\Delta_{m-1})][R_{r,m-1}] + \cdots + [\chi(\Delta_m)][\chi(\Delta_{m-1})]\cdots[\chi(\Delta_2)][\varphi_G(\Delta_1)][R_{r1}]. \tag{3.5}$$

Therefore, substituting Eq. (3.3) into Eq. (3.2) yields

$$[y_{n,i-1}(\Delta_{i-1})] = [\Phi_{n,i-1}] + [B_{i-1}]\{[\Phi_{n-1,m}] + [B_m][\Phi_{n-2,m}] + \cdots + [B_m]^{n-2}[\Phi_{1m}] + [B_m]^{n-1}[y_{11}^{+0}]\}. \tag{3.6}$$

Furtheremore, when the pulse width h is subdivided into $m-1$ narrow pulses with each equal width Δ, that is,

$$\Delta_i = \Delta = h/(m-1), \quad (i = 1, 2, \cdots, m-1) \quad \text{and} \quad \Delta_m = T-h. \tag{3.7}$$

$[B_m]$ and $[\Phi_{rm}]$ become, respectively

$$[B_m] = [\chi(\Delta_m)][\chi(\Delta)]^{m-1} \tag{3.8}$$

and

$$[\Phi_{rm}] = [\varphi_G(\Delta_m)][R_{rm}] + [\chi(\Delta_m)][\varphi_G(\Delta)][R_{r,m-1}] + \cdots + [\chi(\Delta_m)][\chi(\Delta)]^{m-2}[\varphi_G(\Delta)][R_{r1}]. \tag{3.9}$$

Consequently, Eq. (3.6) is substituted into Eq. (3.1), then we get finally the solution of the system with unity feedback in question in the i-th circuit mode on the n-th stage.

Now let us apply the above established method to an example of a third-order system as shown in Fig. 6.

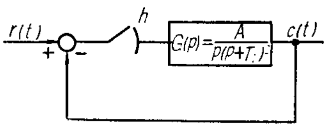


Fig. 6. An example of a third-order system with unity feedback.

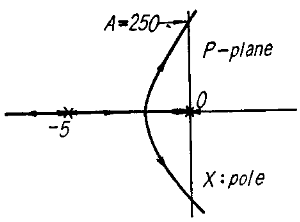


Fig. 7. Root loci of $G(p)$ with $T_1 = 5.0$.

Here assume that the controlled object is of the form

$$G(p) = \frac{A}{p(p+T_1)^2} \tag{3.10}$$

whose root loci are illustrated in Fig. 7, and that the pulse width h of the sampler is subdivided into $m-1$ narrow pulses with each equal width Δ as shown in Eq. (3.7).

The response to a unit step input for the system illustrated in Fig. 6 can be readily evaluated by the use of Eqs. (3.1), (3.8), (3.9) and following relations of Eqs. (3.12) and (3.13), since in this case it is satisfied that

$$[\Phi_{rm}] = [\Phi_{r-1,m}] = \cdots = [\Phi_{1m}] = [\Phi_{r+1,m}] \equiv [\Phi_m], \tag{3.11}$$

then

$$\begin{aligned}[y_{n-1,m}(\Delta_m)] &= \{[U]+[B_m]+[B_m]^2+\cdots+[B_m]^{n-2}\}[\Phi_m]+[B_m]^{n-1}[y_{11}^{+0}] \\ &= \{[U]-[B_m]\}^{-1}\{[U]-[B_m]^{n-1}\}[\Phi_m]+[B_m]^{n-1}[y_{11}^{+0}] \end{aligned} \tag{3.12}$$

and

$$[y_{n,i-1}(\Delta_{i-1})] = [\Phi_{i-1}]+[B_{i-1}](\{[U]-[B_m]\}^{-1}\{[U]-[B_m]^{n-1}\}[\Phi_m]+[B_m]^{n-1}[y_{11}^{+0}]) \tag{3.13}$$

where $[U]$ is a unit matrix and $[B_m]^{n-1}$ is calculable by the use of the Sylvester expansion theorem of the form

$$[B_m]^{n-1} = \sum_{r=1}^{l} \alpha_r^{n-1} \frac{\prod_{\substack{s=1,2,\cdots,l \\ r\neq s}} \{\alpha_s[U]-[B_m]\}}{\prod_{\substack{s=1,2,\cdots,l \\ r\neq s}} (\alpha_s-\alpha_r)}$$

where $\alpha_1, \cdots, \alpha_l$ are the distinct latent roots of $[B_m]$.

In the case where the controlled object is given by Eq. (3.10), the several fundamental matrices in the solution of the example shown in Fig. 6 are expressed as follows,

$$[y_{ni}(\tau)] = \begin{pmatrix} \ddot{c}_{ni}(\tau) \\ \dot{c}_{ni}(\tau) \\ c_{ni}(\tau) \end{pmatrix} \tag{3.14}$$

where $\ddot{c}_{ni}(\tau)=\frac{d^2}{d\tau^2}c_{ni}(\tau)$ and $\dot{c}_{ni}(\tau)=\frac{d}{d\tau}c_{ni}(\tau)$.

$$[R_{ni}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.15}$$

$$[T'] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

$$[\varphi_G(\tau)] = \begin{pmatrix} \varphi_{11}(\tau) & \varphi_{12}(\tau) & \varphi_{13}(\tau) \\ \varphi_{21}(\tau) & \varphi_{22}(\tau) & \varphi_{23}(\tau) \\ \varphi_{31}(\tau) & \varphi_{32}(\tau) & \varphi_{33}(\tau) \end{pmatrix} \tag{3.17}$$

where

$$\left.\begin{aligned} &\varphi_{11}(\tau) = A\tau\varepsilon^{-T_1\tau}, \quad \varphi_{12}(\tau) = -1+\varepsilon^{-T_1\tau}-T_1\tau\varepsilon^{-T_1\tau}, \quad \varphi_{13}(\tau) = -T_1^2\tau\varepsilon^{-T_1\tau}, \\ &\varphi_{21}(\tau) = \frac{A}{T_1^2}\{1-(1+T_1\tau)\varepsilon^{-T_1\tau}\}, \quad \varphi_{22}(\tau) = \tau\varepsilon^{-T_1\tau}, \quad \varphi_{23}(\tau) = -\{1-(1+T_1\tau)\varepsilon^{-T_1\tau}\}, \end{aligned}\right|$$

$$\left.\begin{aligned} \varphi_{31}(\tau) &= \frac{2A}{T_1^3}(\varepsilon^{-T_1\tau}-1)+\frac{A}{T_1^2}(\tau+\tau\varepsilon^{-T_1\tau}), \\ \varphi_{32}(\tau) &= \frac{1}{T_1^2}\{1-(1+T_1\tau)\varepsilon^{-T_1\tau}\}, \quad \varphi_{33}(\tau)=\frac{2A}{T_1}(1-\varepsilon^{-T_1\tau})-\tau\varepsilon^{-T_1\tau}. \end{aligned}\right\} \tag{3.18}$$

$$[\chi_G(\tau)] = \begin{pmatrix} \chi_{11}(\tau) & \chi_{12}(\tau) & \chi_{13}(\tau) \\ \chi_{21}(\tau) & \chi_{22}(\tau) & \chi_{23}(\tau) \\ \chi_{31}(\tau) & \chi_{32}(\tau) & \chi_{33}(\tau) \end{pmatrix} \tag{3.19}$$

where

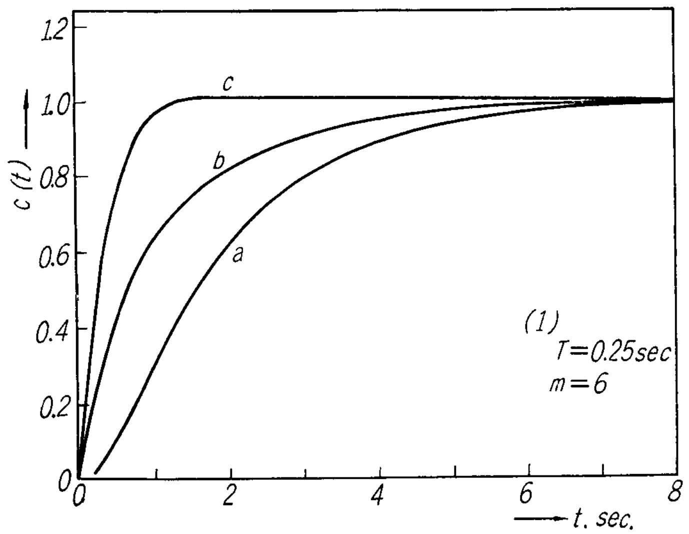
$$\left.\begin{aligned} &\chi_{11}(\tau)=(1+T_1\tau)\varepsilon^{-T_1\tau}, \quad \chi_{12}(\tau)=-T_1^2\tau\varepsilon^{-T_1\tau}, \quad \chi_{13}(\tau)=0, \\ &\chi_{21}(\tau)=\tau\varepsilon^{-T_1\tau}, \quad \chi_{22}(\tau)=(1+T_1\tau)\varepsilon^{-T_1\tau}, \quad \chi_{23}(\tau)=0, \\ &\chi_{31}(\tau)=\frac{1}{T_1^2}\{1-(1+T_1\tau)\varepsilon^{-T_1\tau}\}, \quad \chi_{32}(\tau)=\tau\varepsilon^{-T_1\tau}+\frac{2}{T_1}\{1-(1+T_1\tau)\varepsilon^{-T_1\tau}\}, \\ &\chi_{33}(\tau)=1. \end{aligned}\right\} \tag{3.20}$$

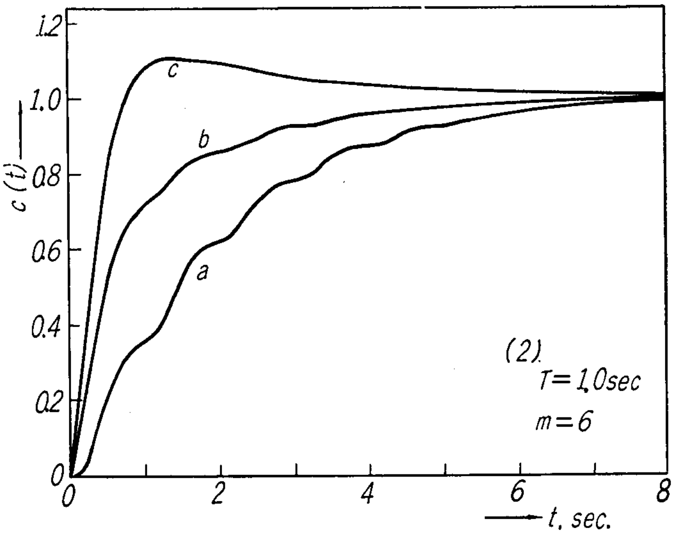
and $[\chi(\tau)]=[\chi_G(\tau)]-[\varphi_G(\tau)][T']$ is of the form, from (3.17) and (3.19),

$$[\chi(\tau)] = \begin{pmatrix} \chi_{11}(\tau), & \chi_{12}(\tau), & \chi_{13}(\tau)-\varphi_{11}(\tau) \\ \chi_{21}(\tau), & \chi_{22}(\tau), & \chi_{23}(\tau)-\varphi_{21}(\tau) \\ \chi_{31}(\tau), & \chi_{32}(\tau), & \chi_{33}(\tau)-\varphi_{31}(\tau) \end{pmatrix} \tag{3.21}$$

whose elements are given by Figs. (3.18) and (3.20).
The results based on Eq. (3.1) are easily calculated by means of the Digital Computer (KDC-1) as illustrated in Fig. 8, where $G(p)=100/p(p+5)^2$ ($A=100$, $T_1=5.0$) and $h/T=0.25$, on three cases of initial conditions and sampling periods.

Next we consider the stability criterion for the sampled-data feedback





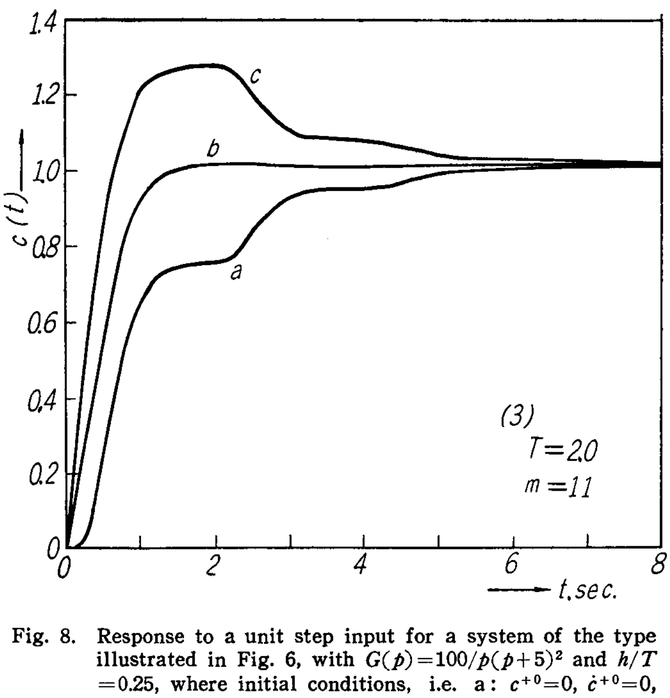


Fig. 8. Response to a unit step input for a system of the type illustrated in Fig. 6, with $G(p)=100/p(p+5)^2$ and $h/T=0.25$, where initial conditions, i.e. a : $c^{+0}=0$, $\dot{c}^{+0}=0$, $\ddot{c}^{+0}=0$, b : 0, 1.0, 0, c : 0, 2.0, 0.

control system with finite pulse width in question and examine the effects of finite sampling on the performance of the system.

The criterion is established under the presumption that the system should be supposed stable so long as the factors containing the initial values of the system variable vanish away gradually from the solution as time goes.

Accordingly, in our case, the stability criterion for the system may be achieved by examining the initial matrices of Eqs. (3.12) or (3.13), whose values must be limited one for the stable system.

As was well-defined in References (7), by the use of the Sylvester expansion theorem, it is evident that the necessary and sufficient condition that the system should be stable is that the absolute values of all the latent roots of $[B_m]$ should be less than unity, or in other words, that all the latent roots should lie inside the unit circle on the complex domain with its center at the origin.

In general, assuming $[B_m]$ to be the n-th order matrix, the characteristic equation of $[B_m]$ is of the form

$$\delta\{\alpha[U]-[B_m]\}=0 \qquad (3.22)$$

and this determinant reduces to an algebraic equation of the form

$$a_0\alpha^n+a_1\alpha^{n-1}+\cdots+a_{n-1}\alpha+a_n=0 \qquad (3.23)$$

where $a_0, a_1, \cdots, a_n$ are constants.

The solutions $\alpha_1, \alpha_2, \cdots, \alpha_n$ of this equation result in the latent roots of $[B_m]$.

Here, it is convenient to do the bilinear transformation such as $\alpha=(\lambda+1)/(\lambda-1)$ in Eq. (3.23), and its results become

$$b_0\lambda^n+b_0\lambda^{n-1}+\cdots+b_{n-1}\lambda+b_n=0 \qquad (3.24)$$

where $b_0, b_1, \cdots, b_n$ are constants determined by constants $a_0, a_1, \cdots, a_n$.

Then, we can immediately apply a stability criterion of the Hurwitz type to Eq. (3.24), because that the points inside the unit circle on the α-plane can be transformed to those lying on the left half region of the complex λ-plane by this bilinear transformation.

Now based on these stability criterion, we examine the stability of the example system shown in Fig. 6.

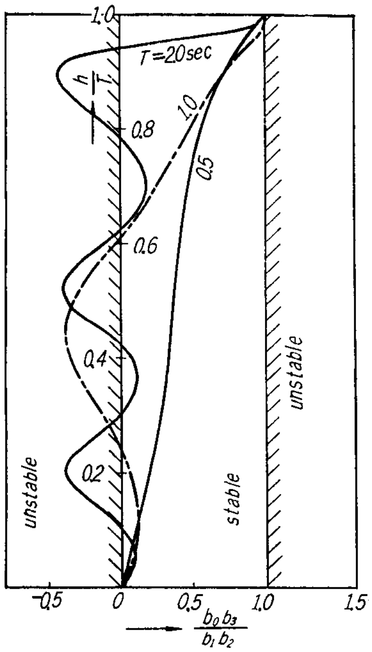


Fig. 9. Region for stability.

The numerical results illustrated in Fig. 9 make clear the relations between the finite pulse width h with respect to the sampling period T and the stability of the control system under discussion, where the controlled system $G(p)=250/p(p+5)^2$, ($A=250$, $T_1=5.0$), that is, a value of $A=250$ is a critical point between the stable and unstable regions of the system in linear continuous cases as shown in Fig. 7.

In this case, the analogue form to Eq. (3.24) is written as

$$b_0\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0 \tag{3.25}$$

therefore, the following conditions are the necessary and sufficient one for the stable system by the Hurwitz's criterion

$$\begin{aligned} &b_0,\ b_1,\ b_2,\ b_3>0 \\ &\begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix} = b_1b_2-b_0b_3>0 \end{aligned} \tag{3.26}$$

These numerical evalutions are done by means of the Digital Computer (KDC-1), whose flowchart is shown in Fig. 10.

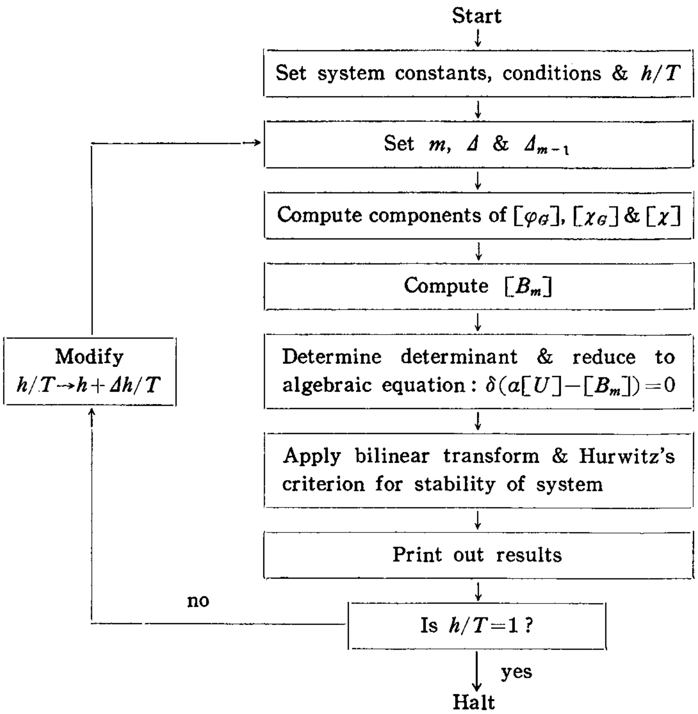


Fig. 10. Digital computer flowchart for calculating the stability of the system.

From inspection of Fig. 9, when the sampling period $T=0.5$, the system is stable for any value of h/T, while on the other hand, when the sampling period becomes large such as $T=1.0$ or $T=2.0$, the stability of the system is variable with respect to the change of h/T.

It is now of interest to clarify the performance of the system under consideration due to the change of pulse width of the sampler when the control system is naturally unstable in continuous systems.

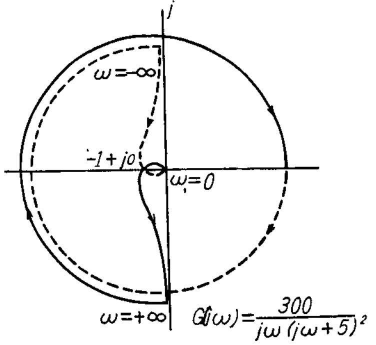


Fig. 11. Inverse transfer function of an unstable control system.

For instance, we take $A=300$, $T_1=5.0$ in our example system of Fig. 6 which is unstable as its inverse transfer function is plotted in Fig. 11.

The numerical results in this example are illustrated in Fig. 12 which demonstrate that the pulse width of the sampler affects remarkably the stability of the

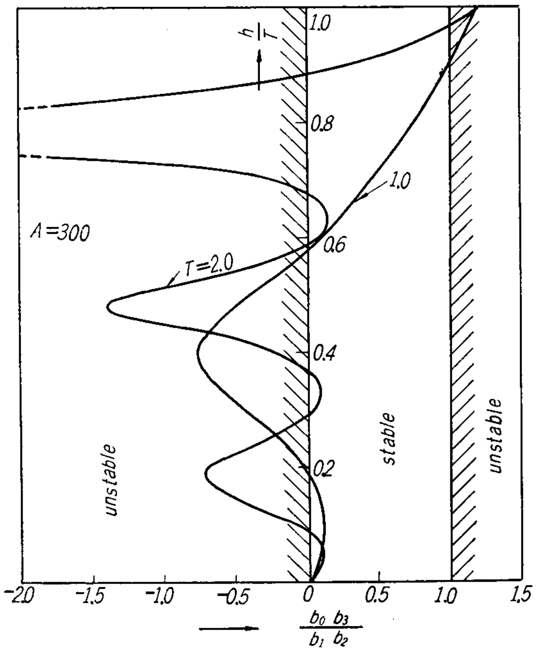


Fig. 12. Region for stability.

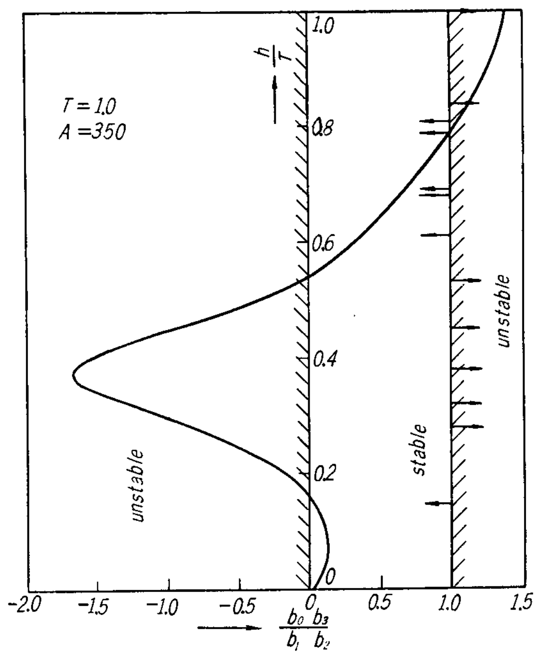


Fig. 13. Region for stability where arrow symbols "← and →" show results by the analog computer.

control system, and that the unstable controlled object becomes stable by the use of the sampler with finite pulse width.

Another result when $A=350$, $T_1=5.0$ is shown in Fig. 13 comparing the theoretical result with the simulated one by means of an analog computer.

Both results indicate a close coincidence as might be expected, and one result by the analog computer is illustrated in Fig. 14 when the sampling period $T=1$ sec and $h/T=0.45$, which shows an unstable response.

The analogue simulation of this example may take the form shown in Fig. 15 where the sampler with finite pulse width is simulated by electronic elements as shown in Fig. 16.

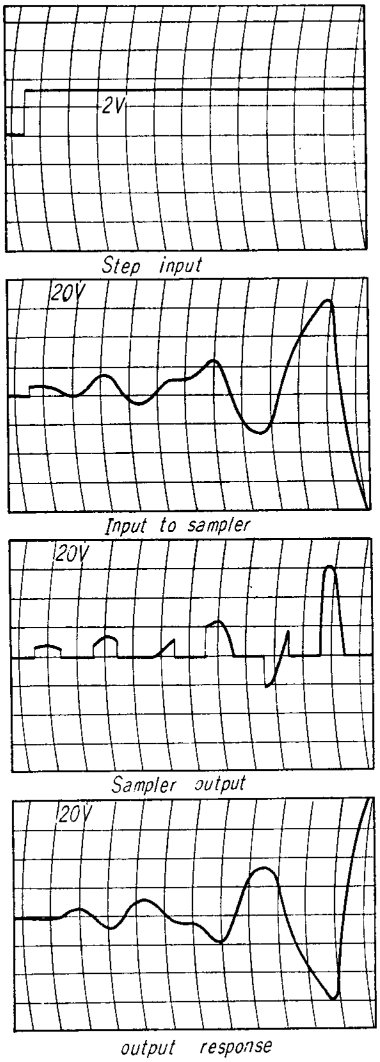


Fig. 14. A simulated result by the analog computer when $G(p)=350/p(p+5)^2$, $T=1$ sec and $h/T=0.45$.

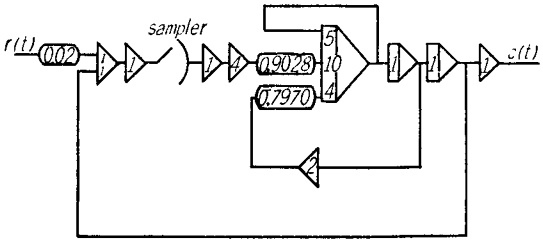


Fig. 15. Analogue simulation diagram.

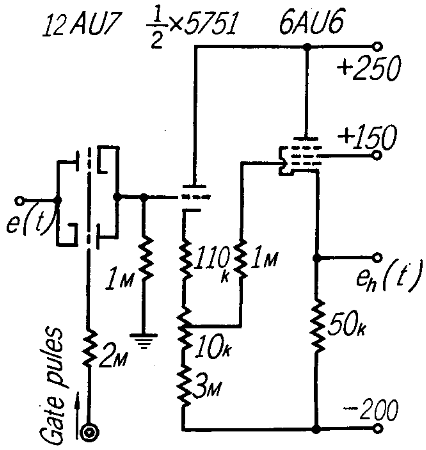


Fig. 16. Representation of a sampler with finite pulse width.

4. Conclusion

The response of a sampled-data feedback control system with finite pulse width can be predicted with good accuracy by the use of the theories of the Periodically Interrupted Electric Circuits described here.

Based on this new technique, the stability criterion of the sampled-data feedback control system with finite pulse width is easily derived and relatively is valuable and accuracy as was indicated in numerical results in the present work.

In an effort to examine the stability of the finite sampling control system, it is evident that the effects of the sampler with finite pulse width is a considerable important for the performance of the system, and that even if the controlled object is originally unstable, its system can be made stable by making use of a sampler with finite pulse width.

Acknowledgements

The authors with to thank Assistant Prof. S. Hoshino and Dr. S. Iwai of the Department of Electronics for their valuable advices and discussions in this work. Thanks are also due to Bachelor of Eng. S. Matsuo of Mitsubishi Electric Company at present and to staff members of the center of the Digital Computer (KDC–1), University of Kyoto, for their helps with our numerical computations.

References

1) G. Farmanfarma: Analysis of Linear Sampled-Data Systems with Finite Pulse Width; Open Loop; AIEE. Transactions, Vol. 75, pt. 1, pp. 808~819, Jan. (1957).
2) J. T. Tou: Digital and Sampled-Data Control System; McGraw-Hill Book Company, (1959).
3) G. Farmanfarma: General Analysis and Stability Study of Finite Pulsed Feedback Systems; AIEE. Transactions, Vol. 77, pt. 2, pp. 148~158, July (1958).
4) G. J. Murphy and H. B. Kennedy: Closed-Loop Analysis of Sampled-Data Systems with Appreciable Pulse Width; AIEE. Transactions, Vol. 77, pt. 2, pp. 659~665, Jan. (1959).
5) S. Hayashi and K. Mizukami: Analysis of Sampled-Data Feedback Control Systems with Finite Pulse Width; Convention Records at the General Meeting on Automatic Control in Japan, No. 124, pp. 41~42, Nov. (1962) and No. 130, pp. 59~60, Nov. (1963).
6) S. Hayashi and K. Mizukami: Analysis of Sampled-Data Feedback Control Systems with Finite Pulse Width II; Convention Records at the Annual Meeting of the Inst. of Elect. Eng. of Japan, No. 316, April (1963).
7) S. Hayashi: Periodically Interrupted Electric Circuits; Denki-Shoin (1961).