A Method to Optimize the Stability of a Linear Dynamic System

By

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(Received June 25, 1964)

A systematic and numerical procedure is described for optimizing the stability of a linear dynamic system. The process is an application of the so-called "steepest-descent method", and since the stability of a linear dynamic system is determined by the real part of the roots of the characteristic equation, the practical procedure is suggested in this paper. A numerical example is presented for maximizing the damping of the perturbed motion of a earth satellite by the use of the gravity-oriented principle.

1. Introduction

The method of gradients or "the steepest-descent method" as it is sometimes called, is an elementary concept for the solution of optimum problems, both in the ordinary calculus and the calculus of variations. Since the stability of a linear dynamic system is determined by the real part of the roots of the characteristic equation, it is necessary to solve the equation to discuss the stability of the perturbed motion. However, in general, the characteristic equation is an algebraic equation of higher degree, and the coefficients of the equation are influenced by several variables. It is, therefore, rather difficult to find the best combination of control variables to optimize the stability in general. The so-called "steepest-descent method" is an useful tool for solving such problems. The practical procedure requires solving the characteristic equation with a high-speed digital computer, finding the least damped mode or the root having the largest real part and then attempting to improve the damping of this mode. This process must be repeated over and over until the damping of the least damped mode is maximized. The numerical computation is not so tedious, but sometimes it may be sensitive to the initial conditions and the hunting phenomenon may occur. When it happens, the

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initial conditions should be changed and the computation must be continued again. The optimum value will then obtained.

2. Application of the Steepest-Descent Method to Optimize the Stability Problems

For the optimization problem of the stability of a linear system, it is necessary to solve the characteristic equation and find the real part of the roots. The characteristic equation is in general expressed by

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{1}$$

where the coefficients a_i are given by

$$a_i = f(\beta_k) \qquad \begin{array}{l} i = 0, 1, 2, \cdots, n-1 \\ k = 1, 2, \cdots, m \end{array} \tag{2}$$

and β_k are the "control variables" which are free to choose.

Roots of the characteristic equation are generally expressed by

$$\lambda_j = n_j \pm i\omega_j \tag{3}$$

if the roots are real, $\omega_j = 0$.

The stability criterion is the value of the real part of the roots, i.e. n_j in Eq. (3), and to optimize the stability, $|n_j|$ should be maximized or n_j should be minimized because n_j is negative for the stable state.

At the starting point the control variables are the initial values, i.e.

$$\beta_k = \beta_k{}^* \tag{4}$$

and the roots of the characteristic equation are

$$\lambda_j{}^* = n_j{}^* \pm i\omega_j{}^* \tag{5}$$

Consider small perturbations of the control variables about the starting point, i.e.

$$\beta_k = \beta_k{}^* + \Delta\beta_k \tag{6}$$

Those perturbations cause small changes of the roots,

$$\lambda_j{}^* + \Delta_j = (n_j{}^* + \Delta n_j) \pm i(\omega_j{}^* + \Delta\omega_j) \tag{7}$$

Since λ or n is a function of coefficients a_i, then from Eq. (2)

$$n_j = F(\beta_k)$$

$$\therefore\ dn_j = \sum_{k=1}^{n} \left(\frac{\partial n_j}{\partial \beta_k}\right)^* \cdot d\beta_k \tag{8}$$

When the largest of n_j is assumed n_1, or $|n_1|$ is the smallest, n_1 should be optimized (i.e. minimized) at the first step.

In order to apply the steepest-descent method, we define

$$(dp)^2 = \sum_k \alpha_k (d\beta_k)^2 = \text{const.} \tag{9}$$

where α_k are the positive weighting numbers. dp coresponds to a step in the direction of maximum gradients in the m-hypersurface, and hence the value of dp must be chosen small enough to insure the linearization to be resonable. To maximize $|dn|$ for a small perturbation $d\beta$ under a constraint condition given by Eq. (9), consider the quantity

$$\sum_k \left(\frac{\partial n}{\partial \beta_k}\right)^* d\beta_k + \mu \left[(dp)^2 - \sum_k \alpha_k (d\beta_k)^2\right] \tag{10}$$

where μ is a Lagrange Multiplier. The maximum $|dn|$ is obtained when

$$\left(\frac{\partial n}{\partial \beta_k}\right)^* - 2\mu\cdot\alpha_k(d\beta_k) = 0$$

$$\therefore \quad d\beta_k = \frac{1}{2\mu\cdot\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^* \tag{11}$$

Substituting Eq. (11) into Eq. (9),

$$(dp)^2 = \left(\frac{1}{2\mu}\right)^2 \sum_k \frac{1}{\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^{*2}$$

$$\therefore \quad \frac{1}{2\mu} = \frac{dp}{\left\{\sum_k \frac{1}{\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^{*2}\right\}^{1/2}} \tag{12}$$

Substituting Eq. (12) into Eq. (11),

$$d\beta_k = dp\cdot \frac{\frac{1}{\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^*}{\left\{\sum_k \frac{1}{\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^{*2}\right\}^{1/2}} \tag{13}$$

Since dn should be negative, $d\beta_k$ must be chosen so that $(\partial n/\partial \beta_k)^* d\beta_k$ is negative from Eq. (8), i.e. when

$$\left(\frac{\partial n}{\partial \beta_k}\right)^* > 0 \quad : \quad d\beta_k < 0$$

$$\left(\frac{\partial n}{\partial \beta_k}\right)^* < 0 \quad : \quad d\beta_k > 0$$

$$\therefore \quad d\beta_k = -|dp| \frac{\frac{1}{\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^*}{\left\{\sum_k \frac{1}{\alpha_k}\left(\frac{\partial n}{\partial \beta_k}\right)^{*2}\right\}^{1/2}} \tag{14}$$

For the next step, $\beta_k^* + d\beta_k$ are the starting points and when n_i is still the largest, the same procedure is repeated, During this process, if another n, for example n_2, becomes larger than n_1, n_2 should be chosen as the value to be optimized in place of n_1, and the same procedure must be repeated to the new root. This process can be illustrated in Fig. 1 schematically. We go down in the direction of the steepest-descent of one slope, but if we cross over an intersection or valley between two sloping surfaces, i.e. actually that of m-dimensional hypersurfaces, the point to be optimized must be changed to the point on another slope.

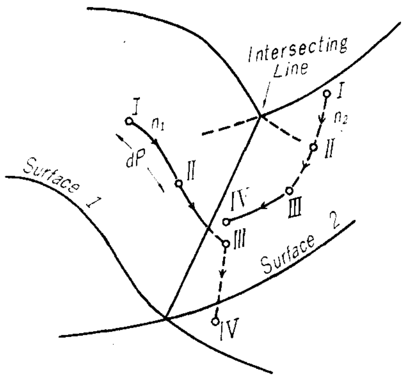

Fig. 1. Finding minimum of two n by steepest-descent method.

The precess should be repeated several times until the gradient dn/dp or

$$\frac{dn}{dp}=-\left\{\sum_k \cdot\frac{1}{a_k}\left(\frac{\partial n}{\partial \beta_k}\right)^{*2}\right\}^{1/2} \tag{15}$$

is nearly zero or the absolute values of all real parts of the roots become roughly the same. The maximum stability of this system is then obtained.

3. Example-Optimization of the Lateral Stability of a Compound Satellite with Gravity-Oriented Principle

As an example, the attitude stability of a compound satellite system, as shown in Fig. 2, is considered. The equations of motion of the system are linearized by the small perturbation principle and the assumption of small eccentricity of the orbit. For the particular system shown in

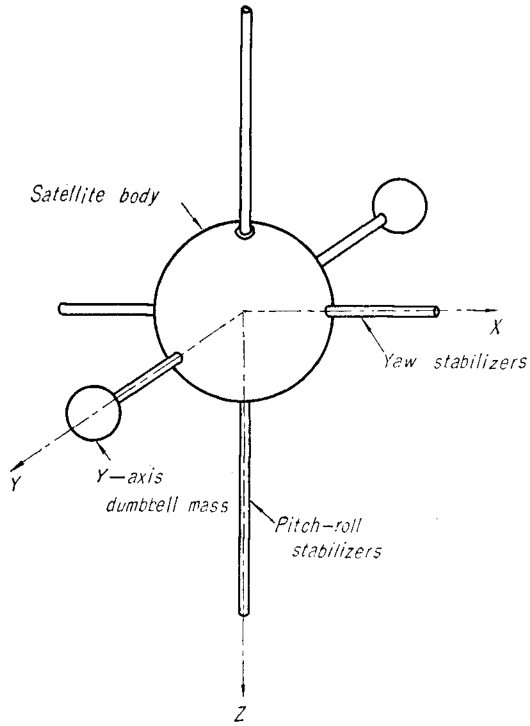

Fig. 2. A compound sotellite system.

Fig. 2, the characteristic equations are the quartic for the longitudinal motion and the sextic for the lateral motion, both associated with the antisymmetric modes[2),3)].

The lateral stability or the damping characteristics of lateral motion obtained in the initial series of calculations was unsatisfactory compared with that of the longitudinal motion. However, from the practical point of view, for example as an observatory base at high altitude of space, both the longitudinal motion and the lateral motion have the similar influences to the attitude of satellite body. It is, therefore, necessary to improve the stability of lateral motion with further parameter variations for achieving the damping of the same order as the longitudinal motion. The so-called "steepest-descent method", stated in the preceding section, was applied to optimize this problem.

Since the characteristic equation associated with the antisymmetric mode of the lateral motion is the sextic, it is expressed by

$$a_6\lambda^6+a_5\lambda^5+a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0 \tag{16}$$

where

$$\begin{aligned}
a_6 &= AC(A_2+A_3)-2A_1^2C\\
a_5 &= \Gamma AC\\
a_4 &= (AC'+FC+E^2)(A_2+A_3)+AC(A_2'+A_3)-2(2A_1A_1'C+A_1^2C')\\
a_3 &= \Gamma(AC'+FC+E^2)\\
a_2 &= FC'(A_2+A_3)+(AC'+FC+E^2)(A_2'+A_3)-2(A_1'^2C+2A_1A_1'C')\\
a_1 &= \Gamma FC'\\
a_0 &= FC'(A_2'+A_3)-2A_1'^2C'
\end{aligned}$$

$$\begin{aligned}
A &= A_b+2A_k+2m_k(a+b)^2 & F &= 4(A-E)\\
A_1 &= A_k+m_kb(a+b) & A_1' &= 4A_1\\
A_2 &= A_k+m_k\left(1-\frac{m_k}{m}\right)b^2 & A_2' &= 3A_1+A_2\\
A_3 &= \frac{m_k^2}{m}b^2 & C' &= C-E\\
C &= C_b+2C_\delta+2m_\delta(a'+b')^2 & &\\
E &= A_b+C_b-B_b & &
\end{aligned}$$

The numerical data which were used for calculation are as follows:

$$\begin{aligned}
\frac{m_k}{m_b} &= 0.005\frac{b}{a}, & \frac{m_\delta}{m_b} &= 0.005\frac{b'}{a}\\
A_b &= C_b = 0.4m_ba^2 & &\\
A_k &= \frac{1}{3}m_kb^2, & C_\delta &= \frac{1}{3}m_\delta b'^2
\end{aligned}$$

The principal variables are b/a, b'/a and Γ, and Fig. 3 shows plots of

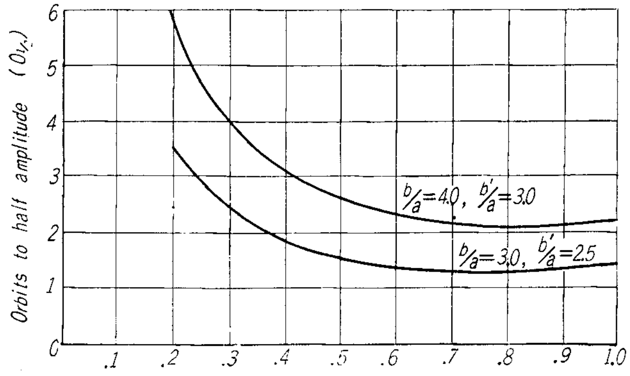

Fig. 3. Least damped mode (Lateral motion).

some sample results. The best performance, from the standpoint of the characteristic decay time or the number of orbits to half amplitude, was obtained for the combination $b/a=3.0$, $b'/a=2.5$, $\Gamma \fallingdotseq 0.7$ and the value is seen to be nearly 1.35 orbits. However, the best value of the longitudinal motion is only 0.3 orbits, and hence the above-stated value of damping of the lateral motion is unsatisfactory compared with that of the longitudinal motion, and further parameter variations should be made in a search for better performance.

The damping of the least damped mode depends strongly on the coupling between yawing and rolling notion of the satellite body, and therefore, a dumbbell mass attached along Y-axis is considered to be effective. The coefficients of the characteristic equation affected by the dumbbell mass are

$$\begin{aligned} A &= A_0+I \\ C &= C_0+I \\ C' &= C_0'-I \\ E &= E_0+2I \\ F &= F_0-4I \end{aligned} \tag{17}$$

where subscript 0 denotes the original values without dumbbell mass, and I is the additional moment of inertia by virtue of dumbbell mass in nondimensional form. Fig. 4 shows clearly, as expected, that the dumbbell mass is effective to improve the stability of lateral motion. The best performance or the minimum number of orbits to half amplitude is about 0.5 orbits at $b/a=3.0$, $b'/a=2.5$, $\Gamma=0.70$, $I=0.15$.

However, since those numerical values were chosen more-or-less arbitrarily, the better performance will be expected for another combination of variables

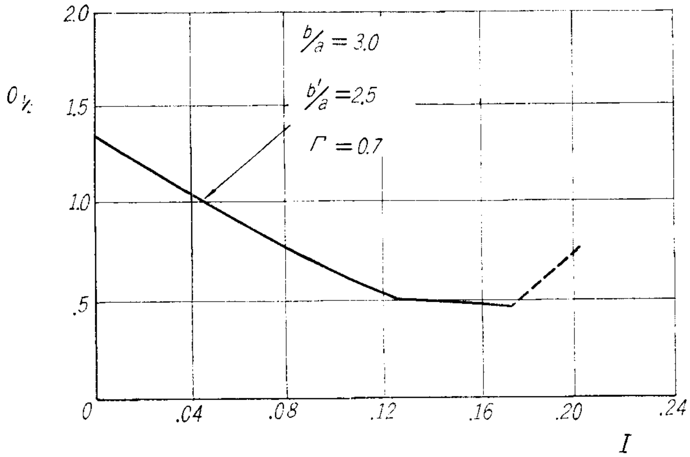

Fig. 4. Least damped mode (Dumbbell mass effect).

around those values. The steepest-descent method is, therefore, applied as follows:

Starting conditions:

$$\beta_1{}^* = b/a^* = 3.0\,, \qquad \beta_2{}^* = b'/a^* = 2.5$$
$$\beta_3{}^* = \Gamma^* = 0.70\,, \qquad \beta_4{}^* = I^* = 0.15$$

Small perturbations:

$$\Delta\beta_1 = \Delta\, b/a = 0.01\,, \qquad \Delta\beta_2 = \Delta\, b'/a = 0.01$$
$$\Delta\beta_3 = \Delta\Gamma = 0.001\,, \qquad \Delta\beta_* = \Delta I = 0.001$$

Weighting numbers:

$$\alpha_1 = \alpha_{b/a} = \frac{1}{100}\,, \qquad \alpha_2 = \alpha_{b'/a} = \frac{1}{100}\,,$$
$$\alpha_3 = \alpha_\Gamma = 1\,, \qquad \alpha_4 = \alpha_I = 1$$

The characteristic equation was solved with IBM 7090, and, at the starting condition, 6 roots of the characteristic equation are obtained as follows:

$$\lambda_1 = -1.8364\,, \qquad \lambda_3, \lambda_4 = -0.2204 \pm 1.5130i$$
$$\lambda_2 = -0.5974\,, \qquad \lambda_5, \lambda_6 = -0.2762 \pm 0.4190i$$

Hence, the largest real part of the roots is

$$n_1 = -0.2204$$

Substituting $\Delta\beta_1 = 0.01$, $\Delta\beta_2 = \Delta\beta_3 = \Delta\beta_4 = 0$ into the characteristic equation, the equation was solved again. From the small change of n_1 or Δn,

$$\left(\frac{\partial n}{\partial \beta_1}\right)^* = \frac{\Delta n}{\Delta \beta_1} = -0.0784$$

Since Δn is sometimes so sensitive to $\Delta\beta$, then the absolute value of $\Delta\beta$ must be chosen small enough to insure the assumption of linearization. The same procedure was repeated for the case of $\Delta\beta_2$, $\Delta\beta_3$ and $\Delta\beta_4$, and $\left(\frac{\partial n}{\partial \beta_2}\right)^*$, $\left(\frac{\partial n}{\partial \beta_3}\right)^*$ and $\left(\frac{\partial n}{\partial \beta_4}\right)^*$ were obtained as follows:

$$\left(\frac{\partial n}{\partial \beta_2}\right)^* = -0.0523$$

$$\left(\frac{\partial n}{\partial \beta_3}\right)^* = +0.442$$

$$\left(\frac{\partial n}{\partial \beta_4}\right)^* = -0.548$$

Using those results the following expression was computed,

$$D^2 = \left\{100\left(\frac{\partial n}{\partial \beta_1}\right)^{*2} + 100\left(\frac{\partial n}{\partial \beta_2}\right)^{*2} + \left(\frac{\partial n}{\partial \beta_3}\right)^{*2} + \left(\frac{\partial n}{\partial \beta_4}\right)^{*2}\right\}$$
$$= 1.3835$$

when $dp=0.01$,

$$\Delta\beta_1 = -\frac{0.01}{D}\cdot 100\cdot\left(\frac{\partial n}{\partial \beta_1}\right)^* = +0.0667$$

$$\Delta\beta_2 = -\frac{0.01}{D}\cdot 100\cdot\left(\frac{\partial n}{\partial \beta_2}\right)^* = +0.0444$$

$$\Delta\beta_3 = -\frac{0.01}{D}\cdot\left(\frac{\partial n}{\partial \beta_3}\right)^* = -0.0038$$

$$\Delta\beta_4 = -\frac{0.01}{D}\cdot\left(\frac{\partial n}{\partial \beta_4}\right)^* = +0.0047$$

For the second step, the starting conditions are therefore

$$\beta_1 = \beta_1^* + \Delta\beta_1 = 3.0667$$
$$\beta_2 = \beta_2^* + \Delta\beta_2 = 2.5444$$
$$\beta_3 = \beta_3^* + \Delta\beta_3 = 0.6962$$
$$\beta_4 = \beta_4^* + \Delta\beta_4 = 0.1547$$

The characteristic equation was solved for those β_k, and the roots are, at the second point,

$$\lambda_1 = -1.5020, \qquad \lambda_3, \lambda_4 = -0.2323 \pm 1.5275i$$
$$\lambda_2 = -0.7758, \qquad \lambda_5, \lambda_6 = -0.2718 \pm 0.4166i$$

Hence, the largest real part of the roots is

$$n_1 = -0.2323$$

The same procedure should be repeated again and again.

The results are shown in Table 1 and Fig. 5. Fig. 5 shows clearly that the least damped mode is improved remarkably by the use of this method, i.e. the damping or the number of orbits to half amplitude at the starting

Table 1.

Control variables	Orbits to $\frac{1}{2}$ amplitude $O_{1/2}$	Control variables	Orbits to $\frac{1}{2}$ amplitude $O_{1/2}$
$b/a=3.00$ $b'/a=2.50$ $\Gamma=0.70$ $I=0.15$	0.0601 0.1846 0.5006 0.3996	(8) $b/a=3.3158$ $b'/a=2.8997$ $\Gamma=0.6617$ $I=0.1941$	0.1595 0.3366 0.3205
(1) $b/a=3.0667$ $b'/a=2.5444$ $\Gamma=0.6962$ $I=0.1547$	0.0734 0.1422 0.4750 0.4059	(9) $b/a=3.3344$ $b'/a=2.9496$ $\Gamma=0.6558$ $I=0.2001$	0.1736 0.3205 0.3093
(2) $b/a=3.1267$ $b'/a=2.5916$ $\Gamma=0.6922$ $I=0.1597$	0.1034 0.4513 0.4089	(10) $b/a=3.3443$ $b'/a=2.9980$ $\Gamma=0.6469$ $I=0.2063$	0.1893 0.3055 0.2977
(3) $b/a=3.1800$ $b'/a=2.6408$ $\Gamma=0.6878$ $I=0.1650$	0.1103 0.4292 0.4070	(11) $b/a=3.3448$ $b'/a=3.0443$ $\Gamma=0.6432$ $I=0.2124$	0.2067 0.2916 0.2863
(4) $b/a=3.2266$ $b'/a=2.6914$ $\Gamma=0.6878$ $I=0.1650$	0.1179 0.4085 0.4009	(12) $b/a=3.3231$ $b'/a=3.0872$ $\Gamma=0.6368$ $I=0.2184$	0.2263 0.2787 0.2696
(5) $b/a=3.2135$ $b'/a=2.7470$ $\Gamma=0.6777$ $I=0.1767$	0.1267 0.3914 0.3433	(13) $b/a=3.3068$ $b'/a=3.1285$ $\Gamma=0.6301$ $I=0.2243$	0.2462 0.2667 0.2262 0.3085
(6) $b/a=3.2550$ $b'/a=2.7978$ $\Gamma=0.6727$ $I=0.1823$	0.1363 0.3720 0.3384	(14) $b/a=3.3970$ $b'/a=3.1156$ $\Gamma=0.6289$ $I=0.2204$	0.2323 0.2726 0.3092
(7) $b/a=3.2892$ $b'/a=2.8489$ $\Gamma=0.6673$ $I=0.1881$	0.1472 0.3537 0.3306		

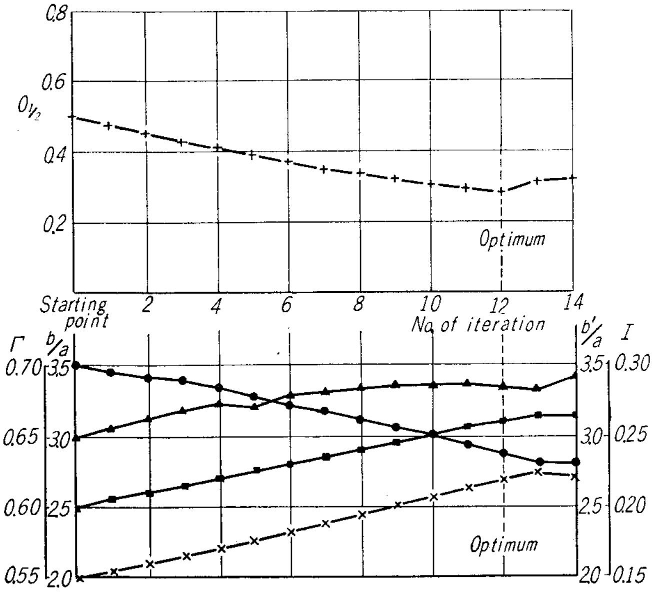

Fig. 5. An example of steepest-descent computation.

point is nearly 0.5 orbits in this example but it is about 0.28 orbits after 12 times of iteration of the computation. The optimum combination of variables corresponding to this optimum damping mode is as follows:

$$b/a = 3.3231, \quad b'/a = 3.0872$$
$$\Gamma = 0.6368, \quad I = 0.2184$$

4. Summary

To improve the performance of a linear dynamic system, a numerical method by the use of a high-speed digital computer is presented for optimizing the stability of the dynamic system. The process is an application of the so-called "steepest-descent method", and the practical procedure is as follows:

1) Solve the characteristic equation of a linear system at the starting point and find a root to optimize.

2) Consider small perturbations of control variables about the starting point, and determine the direction of maximum gradient in the m-hypersurface.

3) Choose a reasonable step dp which insures the linear perturbation, and find the second combination of variables.

4) Repeat the process several time until the optimum value or the maximum stability of the system is obtained.

A numerical example is presented for maximizing the damping of the lateral perturbed motion of a compound satellite. The configuration was found to provide the damping to half amplitude in about 0.28 orbits, which is satisfactory compared with that of the longitudinal motion.

Acknowledgement

The author wishes to express his sincere thanks to Professor B. Etkin, Institute for Aerospace Studies, University of Toronto, for his kind guidance and discussion.

Nomenclature

A_b, B_b, C_b	:	principal moments of inertia of a satellite body
A_k, C_δ	:	moments of inertir of stabilizers
a, a'	:	satellite body dimensions
b, b'	:	stabilizer dimensions
m	:	total mass of a satellite
m_k, m_δ	:	mass of stabilizers
$O_{1/2}$	:	orbits to half amplitude
α_k	:	weighting numbers
β_k	:	control variables
μ	:	Lagrange multiplier
Γ	:	damping coefficient of hinges (dimensionless)
λ_j	:	roots of characteristic equations
n_j	:	real part of roots
ω_j	:	imaginary part of roots

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