Variation in Head and Rate of Discharge in the Non-Steady Seepage Flow

By

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By field observation in an actual river-bank where the ground and the embankment are permeable, the seepage water can be regarded as going through them very fast to reach its steady state. Judging from the experimental and analytical considerations carried out to verify the fact, it is found out that it takes much more time to seep through an embankment on the impermeable foundation than that on the permeable one. The solutions proposed by many investigators for seepage through semi-infinite body are compared with each other and the equation of head distribution in the embankment on the thick permeable foundation is explicitly presented. It is also pointed out that the theoretical curves coincide closely with the experimental data, provided the parameters, the coefficient of permeability k and the effective porosity β, are adequately chosen, and then the approximate values of these parameters at the non-steady stage are calculated. The change in the rate of discharge is also investigated, corresponding to the head variation.

1. Introduction

The development of free surface in a river-bank in the Kizu Branch of the Yodo was observed when the flood came on October 28, 1961[1]. It was found that in a period of this flood the approximate steady state of high water level appeared during six hours and that simultaneously with the flood steady seepage flow occurred in the bank. This fact is very important for designing a sandy bank on the permeable foundation so that we cannot expect any factor of safety for the decrease in leakage and the stability of embankment due to the non-steady motion of seepage flow.

The first purpose of this paper is to examine this fact, comparing the seepage flow on the impermeable foundation with the seepage on the permeable one. The second is to establish the treatment of non-steady seepage problems.

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2. Analysis of Non-Steady Seepage

Fig. 1 indicates a rectangular sand model on the horizontal impermeable foundation with an initial water level H_0. We search for the variation in head in the sand model when a step variation in boundary water level is caused.

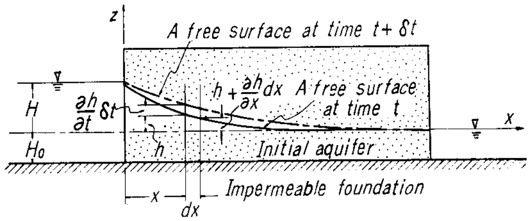


Fig. 1. Sand model.

Taking the assumptions that the seepage flow is approximately horizontal and the variation in head h at x is equal to the height from the initial water level to the free surface and that k and β are constant throughout the embankment, and designating the horizontal velocity by u, the coefficient of permeability by k, the effective porosity by β and the variation in head by $h=z+p/\rho g$, where p denotes the pressure, ρ the density of fluid and g the acceleration of gravity, we take the Darcy's law as the equation of motion without the inertia term

$$u = -k\cdot\frac{\partial h}{\partial x} \tag{1}$$

and the equation of continuity is written

$$\beta\frac{\partial h}{\partial t}+H_0\frac{\partial u}{\partial x}+h\frac{\partial u}{\partial x}+u\frac{\partial h}{\partial x} = 0 \tag{2}$$

Combining Eq. (1) and Eq. (2), we obtain

$$\frac{\partial h}{\partial t} = \frac{k}{\beta}\left[H_0\frac{\partial^2 h}{\partial x^2}+h\frac{\partial^2 h}{\partial x^2}+\left(\frac{\partial h}{\partial x}\right)^2\right] \tag{3}$$

This is the fundamental equation of the seepage flow through embankment on the permeable foundation with a certain initial water level H_0.

When it is $h\ll H_0$, we deduce from Eq. (3)

$$\frac{\partial h}{\partial t} = \frac{kH_0}{\beta}\cdot\frac{\partial^2 h}{\partial x^2} \tag{4}$$

When it is $H_0=0$ (the seepage on the impermeable foundation), we obtain

$$\frac{\partial h}{\partial t} = \frac{k}{2\beta}\cdot\frac{\partial^2 h^2}{\partial x^2} \tag{5}$$

Usually most investigators take the boundary conditions

$$h(x, 0) = 0, \quad h(0, t) = H \tag{6}$$

The condition is that the soil mass is semi-infinite body. Like a river-bank, however, the boundary condition contains the effect of tow boundaries; one is Eq. (6) and the other is

$$h(l, z, t) = z + p/\rho g \doteqdot z \tag{7}$$

In the quasi-one-dimensional seepage flow, however, the relation

$$h(l, t) = 0 \tag{8}$$

is used approximately, where l is the length of a finite body like a rectangular bank.

The solution of Eq. (4) which satisfies Eq. (6) is easily found out as follow:

$$\frac{h}{H} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x/l}{2\sqrt{T_0}}} e^{-\lambda^2} d\lambda = \operatorname{erfc}\left(\frac{x/l}{2\sqrt{T_0}}\right), \quad T_0 = \frac{kH_0 t}{\beta l^2} \tag{9}$$

The solution of Eq. (4) obeying the two boundary conditions is obtained as follows[2].

$$\frac{h}{H} = \sum_{n=0}^{\infty} \left[\operatorname{erfc}\left(\frac{2n + x/l}{2\sqrt{T_0}}\right) - \operatorname{erfc}\left(\frac{2(n+1) - x/l}{2\sqrt{T_0}}\right)\right], \quad T_0 = \frac{kH_0 t}{\beta l^2} \tag{10}$$

These solutions are drawn in Fig. 2 and they can be applied when $h \ll H_0$. However, Eq. (10) is only an estimate of the solution of Eq. (4) for small time. The estimate for large time is expressed by

$$\frac{h(x, t)}{H} = 1 - \frac{x}{l} - \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{\frac{1}{n} e^{-n^2 \pi^2 T_0} \cdot \sin \frac{n\pi x}{l}\right\}, \quad T_0 = \frac{kH_0 t}{\beta l^2} \tag{10'}$$

The necessary range of small "dimensionless time T_0" lies between 10^{-4} and unity and there is a slight difference between Eq. (10) and Eq. (10′) when the value of T_0 approaches to the extreme values.

Now let's consider the solution of Eq. (5). In many investigations the method of linearization using approximate substitution has been adopted owing to the non-linearity of Eq. (5). Vladimirescu and Lates[3], for example, transform Eq. (5) into

$$\frac{\partial h}{\partial t} = \frac{k}{\beta} \cdot \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) \doteqdot \frac{k\hat{h}}{\beta} \cdot \frac{\partial^2 h}{\partial x^2} \equiv a^2 \cdot \frac{\partial^2 h}{\partial x^2} \tag{11}$$

where $\hat{h}$ is a parameter for linearization which is determined experimentally. Then they obtain

$$\frac{h}{H} = \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right) \qquad (0 < t \leqq t_1) \tag{12a}$$

$$\left(\frac{h}{H}\right)^2 = \operatorname{erfc}\left(\frac{x}{2a\sqrt{t}}\right) \qquad (t > t_1) \tag{12b}$$

where erfc(z) is the complementary error function defined in Eq. (9) and H

denotes the sudden variation in boundary water level and t_1 the period of the flood.

Though Eq. (12a) can be deduced as the solution of Eq. (11), Eq. (12b) cannot be obtained as the same way, but from the equation:

$$\frac{\partial u}{\partial t}=\frac{kh}{\beta}\cdot\frac{\partial^2 u}{\partial x^2}, \qquad u=h^2 \tag{13}$$

which is called the second linearization method. In this way of solving Eq. (11) or Eq. (13), questions are in the coefficient a or kh/β which contains the variable h.

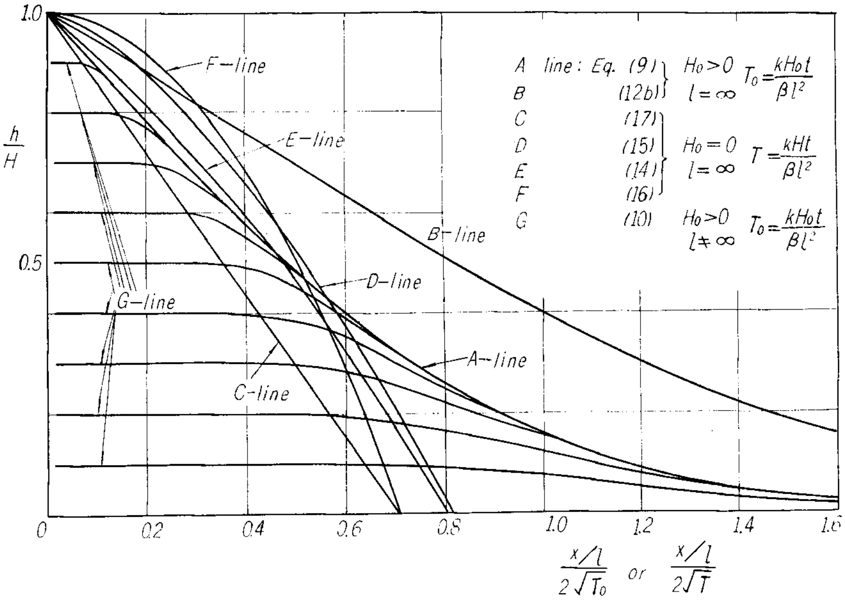


Fig. 2. Theoretical correlation between h/H and $(x/l)/2\sqrt{T_0}$ or $(x/l)/2\sqrt{T}$ obtained from various methods.

Polubarinova-Kochina[4] has solved this problems by means of differentiation of Eq. (5) and gives the solution:

$$u=-c(\xi-c)-\frac{1}{4}(\xi-c)^2-\frac{1}{72c}(\xi-c)^3+\frac{(\xi-c)^4}{576c^2}+\cdots \tag{14}$$

where $u=h/H$, $c=1.14277$, $\xi=x\sqrt{m}/\sqrt{2kHt}=\sqrt{2}\{(x/l)/2\sqrt{T}\}$ and m is porosity of the soil, corresponding to β in the present paper.
Solution (14) is also drawn in Fig. 2.

Uchida[5] has deduced his formula by means of graphical solution and experiments;

$$\frac{y_f}{H}=1-\left\{\frac{x_f/H}{\sqrt{8/3\alpha}\sqrt{kt/H}}\right\}^{3/2}=1-\left(\frac{3}{2}\right)^{3/4}\left\{\frac{x_f/l}{2\sqrt{T}}\right\},\quad T=\frac{kHt}{\alpha l^2} \tag{15}$$

where y_f denotes the height from the impermeable foundation to the free surface, x_f the abscissa from the upper boundary to the free surface and α the porosity.

Where all the stream lines are horizontal, the equation of free surface takes the form:

$$\frac{dx}{dt}=\frac{k}{m}\cdot\frac{H-y}{x}$$

in which y denotes the ordinate of free surface. The integration for the condition; $x=0$ for $t=0$, gives the solution for the free surface:

$$\frac{y_f}{H}=1-\frac{m}{2kH}\cdot\frac{x_f^2}{t}=1-2\left\{\frac{x_f/l}{2\sqrt{T}}\right\}^2,\quad T=\frac{kHt}{ml^2} \tag{16}$$

Considering the seepage water which is confined at the base and moves in a horizontal direction only, on the other hand, we write the next fundamental equations:

$$\frac{\partial h}{\partial x}=-\frac{H}{x_f}\quad\text{and}\quad u(t)=\beta\cdot\frac{dx_f}{dt}=-k\cdot\frac{\partial h}{\partial x}$$

where u denotes the horizontal component of stream velocity. By combining and integrating them, we obtain

$$\frac{h(x,t)}{H}=1-\sqrt{\frac{\beta}{2kH}}\cdot\frac{x}{\sqrt{t}}=1-\sqrt{2}\left\{\frac{x/l}{2\sqrt{T}}\right\},\quad T=\frac{kHt}{\beta l^2} \tag{17}$$

Eqs. (14), (15) and (16) express the equation of the free surface, whereas Eq. (17) is the equation of the head distribution at the base of media in the confined state and they are drawn as C-line, D-line, E-line, and F-line in Fig. 2, respectively. G-lines denote Eq. (10) which is the solution of seepage movement on the permeable foundation and satisfies two boundary conditions. Obviously the group of G-lines has an asymptote, A-line. It is also notable that G-lines become constant beyond the intersecting point with C-line (straight line).

Thus, we have considered the variation in head distribution in sand embankment. Now let's examine the rate of discharge q. Ordinarily it is expressed by

$$q(x,t)=-k(H_0+h)\cdot\frac{\partial h}{\partial x},\quad [\text{cm}^3/\text{sec}/\text{cm}] \tag{18}$$

By the results of calculation, the expression of head variation given by Eq. (10′) is more suitable when we consider the rate of discharge. When Eq.

(10′) is used, we have

$$q(x,\,t)=\frac{kH^2}{l}\left[\frac{H_0}{H}+\left(1-\frac{x}{l}\right)-\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n}e^{-n^2\pi^2T_0}\sin\frac{n\pi x}{l}\right\}\right]\cdot\left[1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2T_0}\cos\frac{n\pi x}{l}\right] \tag{19}$$

$$q(0,\,t)=\frac{kH(H_0+H)}{l}\left[1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2T_0}\right] \tag{20}$$

$$q(l,\,t)=\frac{kHH_0}{l}\left[1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2T_0}\cos n\pi\right] \tag{21}$$

$$q(0,\,\infty)=\frac{kH(H_0+H)}{l} \quad (22), \qquad q(l,\,\infty)=\frac{kHH_0}{l} \quad (23)$$

The difference between Eq. (22) and Eq. (23), kH^2/l, is peculiar to the seepage phenomena compared with the heat conduction through a column of costant section. The discrepancy is due to the fact that the section through which seepage water flows is constant in the case of heat conduction, but in seepage problems it varies with both distance x and time t. Therefore, it is considered that the gradient of head is not linear even though seepage flow becomes steady. Thus the theoretical solution of head variation given by Eq. (10) cannot be applied to actual seepage problems. Taking away the assumption $h\ll H_0$ on which the above analysis is based, the non-linear differential equation (5) has to be solved according to the condition without assumptions. Assuming that the rate of discharge at the steady state becomes an average of $q(0,\,\infty)$ and $q(l,\,\infty)$:

$$q_{\text{mean}}=k\left(H_0+\frac{H}{2}\right)\cdot\frac{H}{l}$$

In the unconfined ground water, however, the rate of discharge through the capillary zone cannot be neglected. The total rate of discharge in the steady state is

$$q_0=k\left(H_0+\frac{H}{2}+\frac{h_c}{2}\right)\cdot\frac{H}{l} \tag{24}$$

in which h_c is the maximum capillary rise at the static state and it is assumed that half of the maximum capillary zone is effective to permit the flow. Substitution of q_0 in Eq. (24) for $q(0,\,\infty)$ in Eq. (22) and $q(l,\,\infty)$ in Eq. (23) leads the following relations:

$$q(0,\,t)/q_0=1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2T_0} \tag{20′}$$

$$q(l,\,t)/q_0=1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2T_0}\cdot\cos n\pi \tag{21′}$$

3. Experiments

3.1. Experimental apparatus and procedure

A sand model was built up in the water channel made of steel which has a dimension of about 400 cm. in length, 50 cm. in height and 23 cm. in width. The front face of the channel has a glass window so that the seepage water flow can be observed. The equipments by which the water level as a boundary head is controlled are two drainage pipes which can be risen and dropped. The rates of discharge at the upstream and downstream boundaries and the variations in water head due to seepage flow in the sand model are measured by the pressure gauge and recorded on the automatic electronic recorder. The grain-size curve of sand used in the rectangular sand model is shown in Fig. 3. The dry density of sand is 1.58 g/cm^3 and the void ratio is 0.78. The water content of sand above the capillary zone is equal to about 5 per cent as shown in Fig. 4. The coefficient of permeability of sand model $k=0.332$ cm/sec and the maximum capillary rise $h_c=8$ cm. are measured in the steady state. The hydrograph as a step function of time is adopted as the boundary con-

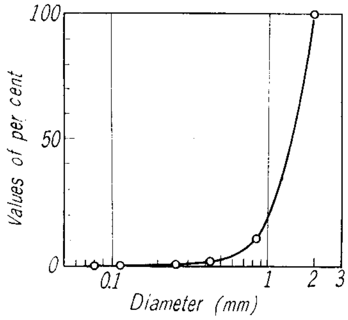


Fig. 3. Grain-size curve.

Table 1.

H_0 \ H	−30	−20	−10	10	20	30
0	/	/	/	A_0	B_0	C_0
10	/	/	$\bar{A}_{10}$	A_{10}	B_{10}	/
20	/	$\bar{B}_{20}$	$\bar{A}_{20}$	A_{20}	/	/
30	$\bar{C}_{30}$	$\bar{B}_{30}$	$\bar{A}_{30}$	/	/	/

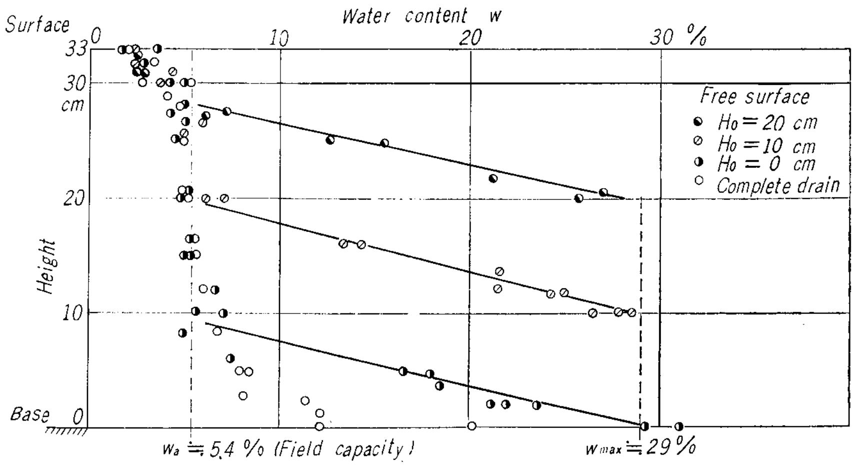


Fig. 4. Distribution of water content within sand model in experiment.

dition. A series of experiments is carried out as shown in Table-1. The notation A_{20} means the case of the initial aquifer thickness $H_0=20$ cm. and of the step variation in head $H=10$ cm..

3.2. Experimental results

By the consideration in the preceding section, the relationship between dimensionless head h/H and dimensionless value $(x/l)/2\sqrt{T_0}$ is shown in Fig. 2, which can be expressed by the correlation between h/H and x/l, taking T_0 or T as a parameter shown in Fig. 5. In this figure we can recognize that the

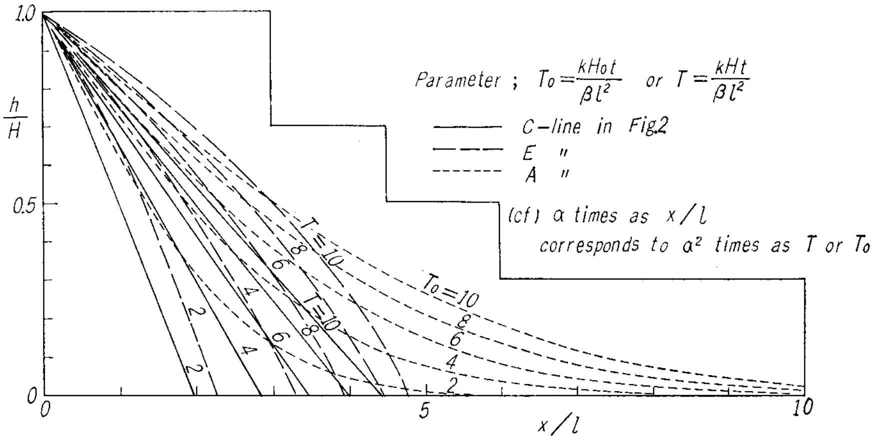


Fig. 5. Comparison of seepage movement on the permeable foundation with that on the impermeable one by the correlation between h/H and x/l.

propagation of head in a group of Eqs. (9) and (10) is faster than that of solutions (14), (15) and (17). The former group is of heat conduction type. Experimental results are shown in Figs. 6 and 7. Fig. 6 shows the variation in head distribution, *i.e.*, the development of free surface. In the case of the upward variation in head, when the ratio H/H_0 is small, the free surface develops as the heat conduction. As the ratio H/H_0 becomes large, *i.e.*, seepage water becomes the flow through bank on the impermeable foundation, the free surface tends to develop slowly and does not reveal the linear relationship between dimensionless head h/H and distance x even if the flow becomes steady and the seepage surface appears in the downstream side. Variation in height with time is unknown owing to the complexity of the boundary condition and so we do not consider it here.

The correlations between experimental results and theoretical curve are adjusted by the term, abscissa $x/2\sqrt{t}$ and ordinate h/H in Fig. 7. The abscissa is here taken as logarithmic scale not same as Fig. 2. This is for the con-

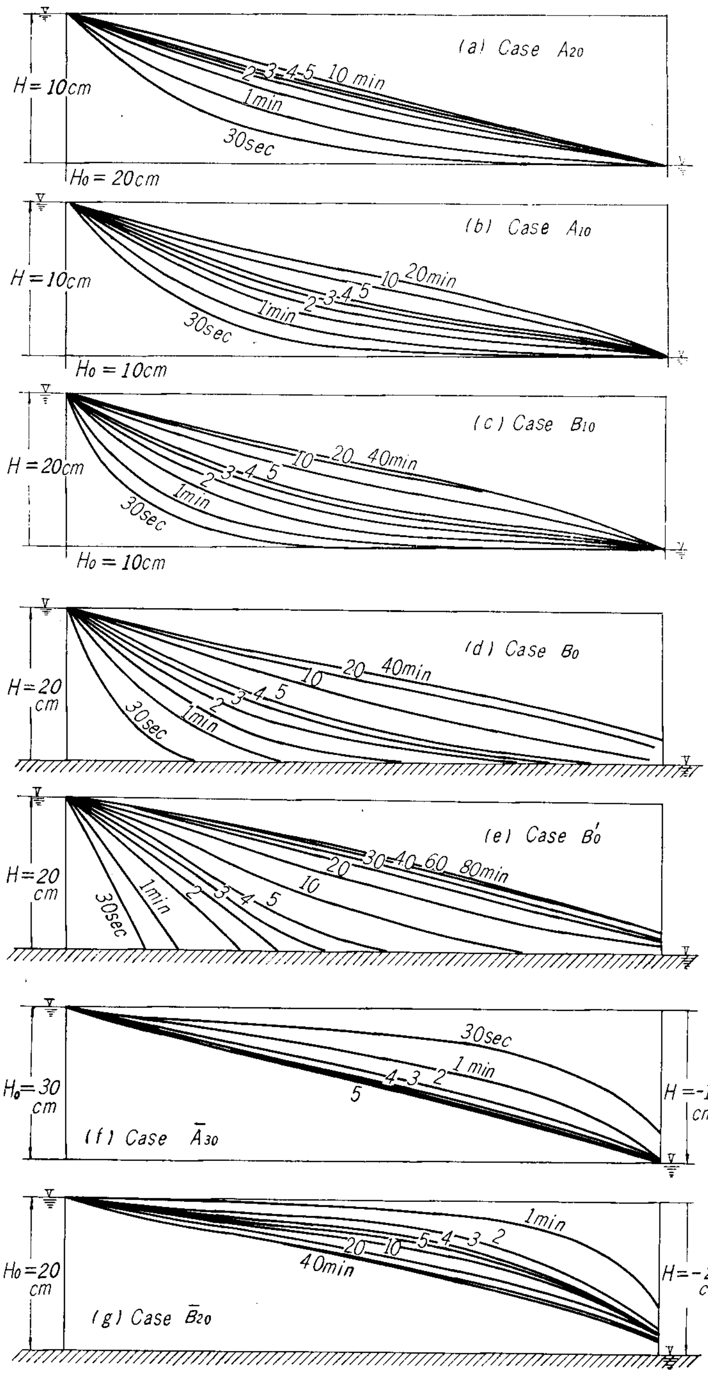


Fig. 6. Correlation between h/H and x in experiment.

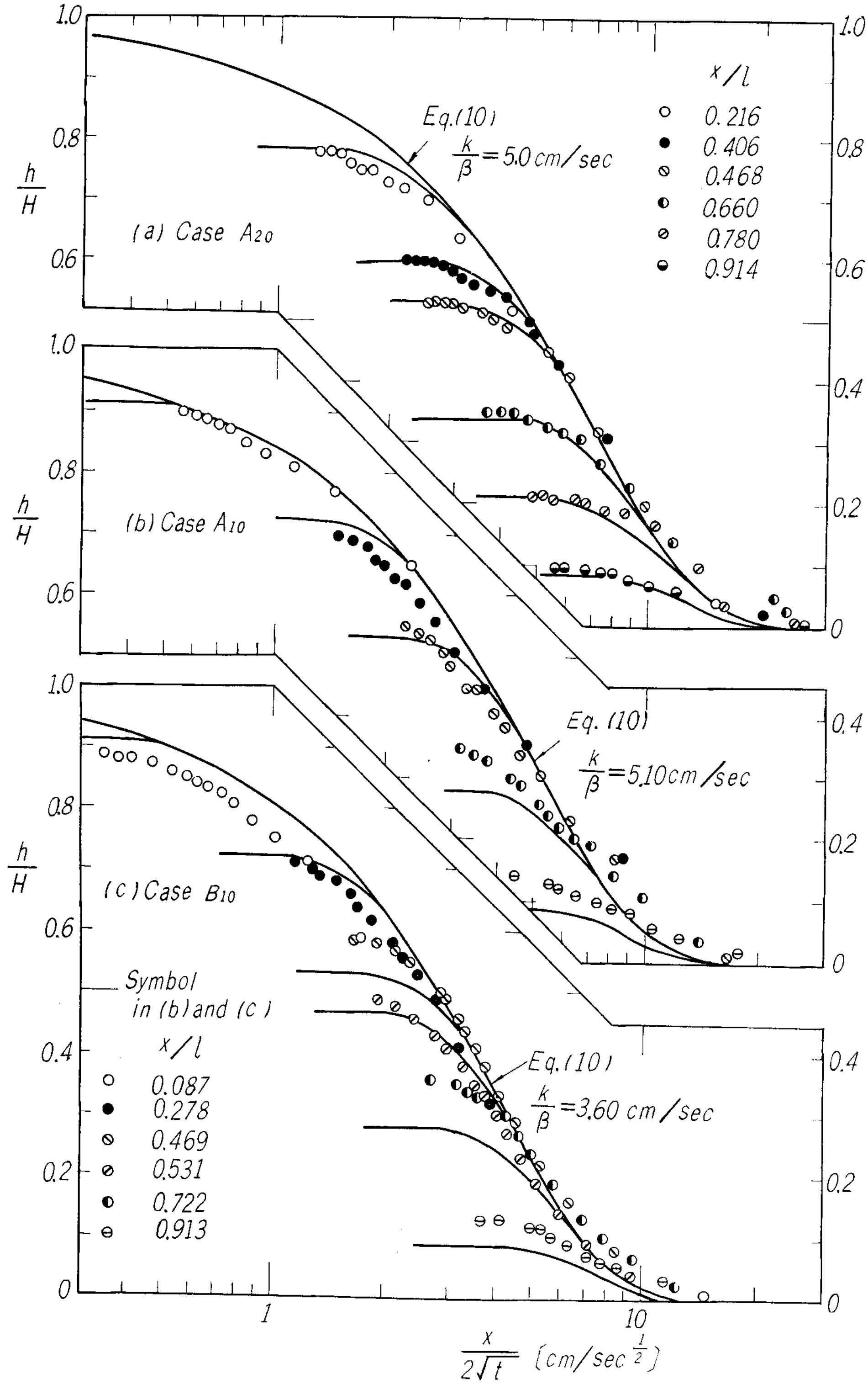
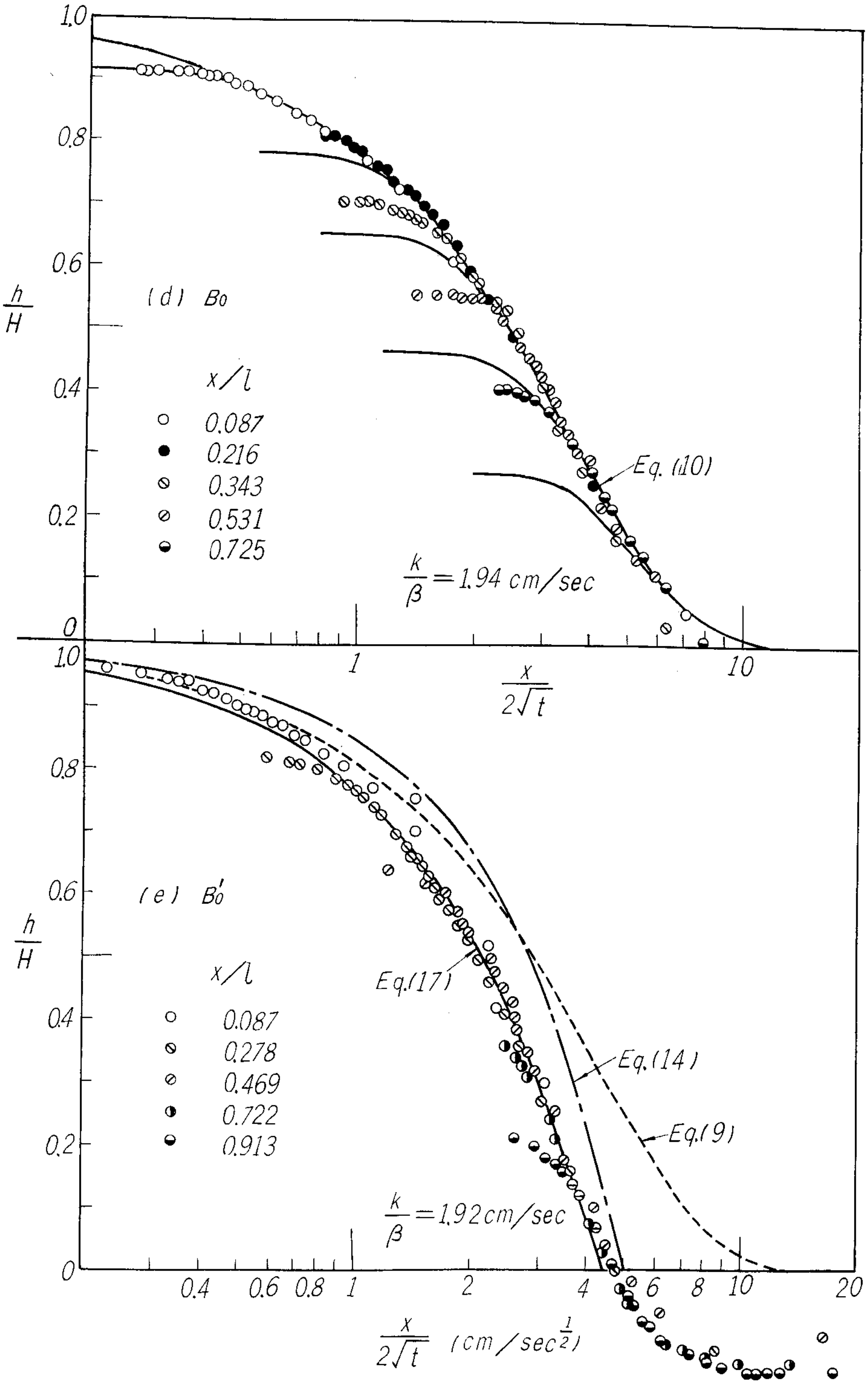
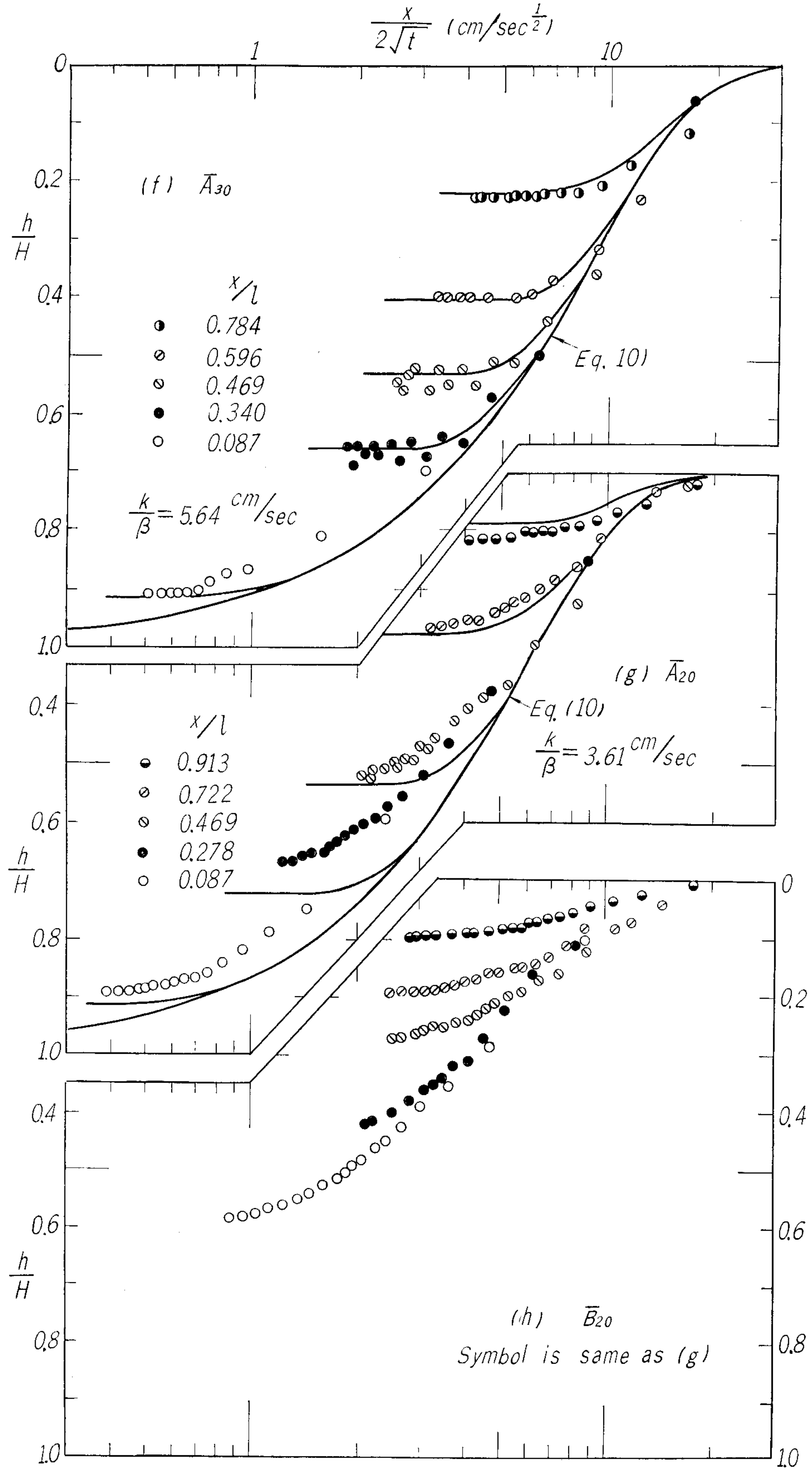


Fig. 7. Correlation between h/H and $x/(2\sqrt{t})$.


(d) Bo
x/l
0.087
0.216
0.343
0.531
0.725
Eq. (10)
k/β = 1.94 cm/sec
h/H
x/2√t
(e) Bo'
x/l
0.087
0.278
0.469
0.722
0.913
Eq.(17)
Eq.(14)
Eq.(9)
k/β = 1.92cm/sec
x/2√t (cm/sec^½)


x / 2√t (cm/sec^1/2)
1
10
h/H
(f) Ā30
x/l
0.784
0.596
0.469
0.340
0.087
k/β = 5.64 cm/sec
Eq. 10)
(g) Ā20
x/l
0.913
0.722
0.469
0.278
0.087
Eq. (10)
k/β = 3.61 cm/sec
(h) B̄20
Symbol is same as (g)
0
0.2
0.4
0.6
0.8
1.0

venience of calculating the parameter k/β. Observing Fig. 7, it is found that the experimental value and theoretical one are well in accordance with each other, and that the smaller the ratio H/H_0 becomes, the more closely the development of free surface resembles that of the isothermal line in heat conduction system, *i.e.*, that the smaller the influence of the impermeable foundation is, the greater the rising speed of head due to the non-steadiness turns.

Case A_{20} ($H/H_0=0.5$), Case A_{10} ($H/H_0=1.0$) and Case B_{10} ($H/H_0=2.0$) which correspond to the condition that the flow is on the permeable foundation are well in accordance with Eq. (10) or Eq. (10′). The variation in head in Case A_0, B_0 and C_0 which are on the impermeable foundation also coincides with Eq. (10) or Eq. (10′), provided the parameter k/β is taken adequately. In such a case the variation in head at Case B_0 is subjected to Eq. (10) to a degree, although $H_0=0$, H_0 in the term T_0 would rather be substituted for H, because the flow is of the type on the impermeable foundation. The head variation in Case A_0', B_0' ane C_0', which are in the case of the same boundary condition as those A_0, B_0 and C_0 and whose distribution of initial water content is different, is in agreement with Eq. (17) which is the linear relation between h/H and $x/2\sqrt{t}$. (The negative pressure is measured in Fig. 7(e) which is the case that the water content at initial stage is nearly constant, not the same as the Case B_0 (see. Fig. 4)). This means the non-steady development with the linear free surface. As the experimental value h/H is observed at the base in water channel, however, it must be examined whether the head at the same section but at the different depth is the same. The head at the point 15 cm. in height from the impermeable base is observed and the ratios of head at the point to head at the impermeable base are calculated, which are shown in Fig. 8.

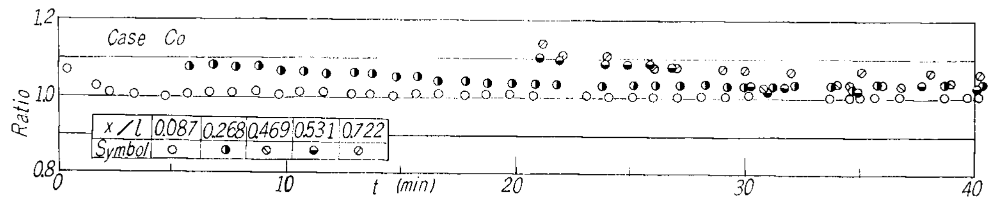
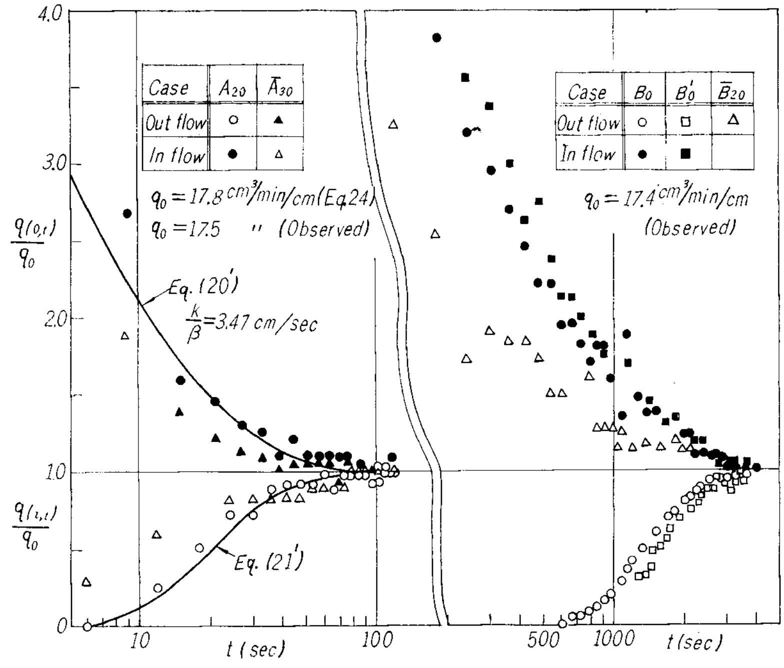


Fig. 8. Ratio of head measured at the point in the height of 15 cm to that on the impermeable base.

From this the ratios are almost equal to unity; the head throughout the vertical section at the same distance is nearly equal, whose amount of fluctuations is within 10% and the seepage flow can be regarded as quasi-one-dimensional.

Experiments in the case of downward variation in head caused are shown in Fig. 7 (f), (g). The ratio H/H_0 in Case $\bar{A}_{30}$ is -0.333, the smallest of our experiments and the observed point in this case is better. As the ratio H/H_0 becomes large, the variation in head deviates from Eq. (10) and its rate turns into a slower one. Moreover, experimental curves at different positions do not lie on one line as is expected in the case of upward variation in head. Therefore, it seems that the seepage movement in this case is a different type from the one when the head rises. Although explicit interpretation about the discrepancies is not given in present, it seems mainly due to the capillary effect. Only when the absolute ratio $|H/H_0|$ is small $(H<0)$, Eq. (10) can be applied, whereas when $H>0$, Eq. (10) can be used even if the ratio H/H_0 becomes somewhat large.

Secondly, let's consider the aspect of variation in rate of discharge, some cases of which are shown in Fig. 9. From this figure we can know the rate



(a) (b)

Fig. 9. Variation in the rate of discharge.

of discharge in the case of large initial aquifer thickness which is well subjected to Eqs. (20′) and (21′), provided q_0 is taken as the observed value shown in the figure. The rate of discharge calculated by Eq. (24) is also written in Fig. 9 (a) which shows good accordance. But the parameter k/β calculated from the rate of discharge is slightly different from the one obtained from variation in head, which may be mainly due to the error in measurement. Some amount of time lag, for example, enters inevitably into the system measuring the rate of discharge and the parameter calculated from it is somewhat smaller. When the rate of discharge in the case of falling head, $\bar{A}_{30}$, is compared with that in the rising head, A_{20}, the former changes somewhat smaller than the latter. This tendency corresponds to the head variation.

It is notable that the coefficient of permeability or the effective porosity depends on the initial condition. The value of parameter $k/\beta=5.0$ cm/sec calculated in Fig. 7 (a) in the Case A_{20}, for example, is greater than the one $k/\beta=1.92$ cm/sec in Fig. 7 (e) in the Case B_0', because the initial water content in the Case A_{20} is greater than that in the Case B_0' as shown in Fig. 4. As the values of parameter k/β in the other cases also hold this trend, this method for calculating the parameter in the non-steady state may be recommended considerably.

4. Conclusions

The results from this investigation may be summarized as follows ;

(1) The seepage flow like that in the actual river-bank becomes steady state very much faster than is expected. This is attributed to the existence of thick permeable foundation and the moisture in soil.

(2) Assumptions in the quasi-one-dimensional flow in which the seepage flow is approximately horizontal and the variation in head at the same distance is equal to the height from the initial water level to the free surface are verified to be approximately correct within 10 per cent in accuracy, which is judged from Fig. 8.

(3) As the general tendency, variation in the head and the rate of discharge is well expressed by the mechanism of heat conduction type ; for example, Eq. (9) or Eq. (10) and Eq. (19). The head variation in the sand model whose pore water is drained only by the gravity and whose water content is nearly constant is well subjected to Eq. (17) not so much to Eq. (14), Eq. (15) and Eq. (16) ; that in the dry sand model whose water content is completely constant may obey one of the latter equations.

(4) The value of parameter, the coefficient of permeability or the effective porosity is subjected to the moisture condition.

(5) Variation in the head or the rate of discharge at the rising head and at the falling head are quite different with each other, which is due to the difference of the capillary effect and of the pressure propagation ; at the rising

head they vary rapidly than at the falling head. This means that there are very few possibilities to apply the method of superposition in calculating the variation in head or rate of discharge on the non-steady seepage flow.

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