On the Absolute Stability of Automatic Control System with Many Nonlinear Characteristics

By

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Many authors, mainly in the Soviet, have discussed the absolute stability of the automatic control system with one nonlinear element. In practical problems, however, we sometimes encounter the control system with many nonlinear elements. In this paper the absolute stability of the automatic control system with many nonlinear elements is discussed. The discussions are based on the methods :

1. Lyapunov's direct method.
2. Popov's method.

The relations between the results obtained by means of these two methods are also described.

1. Introduction

In practice of automatic control we are usually obliged to use the controllers possessing nonlinear characteristics. In each particular case, sometimes it is impossible to fix rigorously the functions of the nonlinear characteristics under the real operating conditions of the control system. However, it is required that the control system should be stable. Further, in practical problems it is required that the control error decays after not only small but also any arbitrary, finite, initial displacements have been imposed. In other words, the equilibrium should be asymptotically stable in the whole.

From such technical viewpoints the concept of "absolute stability" was proposed about twenty years ago. A control system is called to be absolutely stable, if its equilibrium is asymptotically stable in the whole for any characteristic $y=\varphi(\sigma)$ of a nonlinear element, which belongs to a class of functions say, a class of functions $\varphi(\sigma)$ such that $\varphi(\sigma)\sigma>0$, $\sigma\neq 0$ and $\varphi(o)=0$.

At first the problem of the absolute stability of a control system with one nonlinear element was formulated by A. I. Lur'e and V. N. Postnikov[1,2].

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They investigated the problem, using the Lyapunov's direct method. Applying the so-called Lur'e transformation and constructing the Lyapunov functions of the type "a quadratic form plus the integral of the nonlinear function $\varphi(\sigma)$". They obtained a system of quadratic equations and showed that if the system of quadratic equations has real solutions, then the control system is absolutely stable. Following this method V. A. Yakubovich[5,8], E. N. Rozenvasser[9], A. M. Letov[4] and other Soviet authors[13] have discussed the problem in greater detail.

I. G. Malkin, using the same type of Lyapunov functions as the above, but not reducing the problem to the discussion of the system of quadratic equations, gave an inequality as a sufficient condition for the absolute stability by means of the Sylvester's criteria[3].

V. M. Popov introduced a new method of the investigation of the absolute stability, which is different from the Lyapunov's direct method[7,11]. He gave a sufficient condition for the absolute stability in terms of the frequency characteristic of the linear parts of the system and showed that all the results obtained by means of the above mentioned methods are included in his criteria, that is, if for the system there exists a Lyapunov function of the above type, then the Popov's sufficient condition is satisfied. Moreover V. A. Yakubovich proved its inverse proposition[12].

The investigations above mentioned were carried out for a system containing only one nonlinear characteristic. In the practical problems, however, we sometimes encounter the control system containing many nonlinear characteristics. As to such control systems, extension of the Lur'e method was discussed by A. M. Letov[4] and I. A. Sultanov[14]. It is reported that V. M. Popov extended his method to the system with many nonlinear characteristics[10], but unfortunately the paper is written in Rumanian and is not in the hands of the authors.

In this paper the problem of the absolute stability of the system containing m nonlinear characteristics will be discussed in detail.

2. Statement of the Problem

We shall consider a dynamical system which is described by the system of the form

$$\frac{dx}{dt} = Ax + By, \tag{2-1$_1$}$$

$$y = \varphi(\sigma) = \begin{pmatrix} \varphi_1(\sigma_1) \\ \vdots \\ \varphi_m(\sigma_m) \end{pmatrix}, \qquad \sigma = C' \cdot x$$

where x : n-dimensional state vector,
y : m-dimensional vector,
A : $n \times n$ constant matrix,
B, C: $n \times m$ constant matrices,

and the prime denotes the transpose of matrix. $\varphi_j(\sigma_j)$ $(j=1, \cdots, m)$ are one-valued continuous functions which are defined for all real values σ_j $(j=1, \cdots, m)$ and satisfy the following conditions

$$\varphi_j(o) = 0$$
$$0 \leq \frac{\varphi_j(\sigma_j)}{\sigma_j} \leq k_j, \qquad (j = 1, \cdots, m) \tag{2-1$_2$}$$

where $k_1, \cdots, k_m$ are finite postive numbers or some of them infinite*. We assume that the system (2-1) satisfies the conditions of existence and uniqueness of solutions for all $t \geq 0$ and for any initial conditions $x(0)$.

The system $(2\text{-}1)_1$ can be shown in the form of a block diagram as in Fig. 1, where the characteristic of the linear block L. is represented by

$$\frac{dx}{dt} = Ax + By, \qquad -\sigma = -C' \cdot x$$

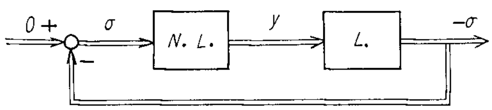

Fig. 1.

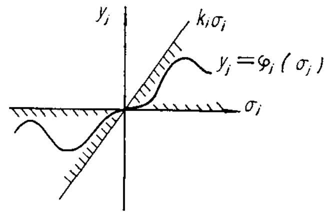

Fig. 2.

and that of the nonlinear block N. L. is represented by

$$y = \varphi(\sigma).$$

The conditions $(2\text{-}1)_2$ are represented graphically in Fig. 2. That is to say, the curve $y_j = \varphi_j(\sigma_j)$ in the (σ_j, y_j) plane is laid in the angle formed by the σ_j-axis and the straight line $y_j = k_j\sigma_j$.

Now let us define the absolute stability of the system (2-1). Let the constant matrix A to be asymptotically stable. Namely, all the roots of the characteristic equation of A:

$$det(\lambda E - A) = 0$$

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* For $k_j = \infty$ the conditions $(2\text{-}1)_2$ are reduced to the inequality $0 \leq \frac{\varphi_j(\sigma_j)}{\sigma_j}$.

have negative real parts, where E is unit matrix. The system (2-1) is called to be absolutely stable in $[O, K]$, provided that the equilibrium $x=0$ of the system (2-1) is asymptotically stable in the whole for any functions $\varphi_j(\sigma_j)$, $(j=1,\cdots,m)$ satisfying the conditions $(2\text{-}1)_2$, where $K=\begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_m \end{pmatrix}$.

In the following paragraphs we shall investigate what conditions are to be satisfied for the system parameters A, B and C to guarantee the absolute stability in $[O, K]$ of the system (2-1).

Relating to the problem of the absolute stability, the following question arises. Is the system (2-1) asymptotically stable in the whole for any nonlinear functions satisfying the conditions $(2\text{-}1)_2$, if the linear system obtained by putting $y_j=h_j\sigma_j$ is asymptotically stable for all h_j such that $0\leq h_j\leq k_j$ $(j=1,\cdots,m)$? However the example of a system of the third order was shown, for which the above question has negative answer[13]. Thus we must investigate the problem of the absolute stability as that of nonlinear theory.

3. Investigation of the Absolute Stability by Means of the Lyapunov's Direct Method

Let us consider an ordinary differential equation of the form

$$\frac{dx}{dt}=f(x), \quad f(o)=0 \tag{3-1}$$

where $x=\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ is a vector in the n-dimensional vector space and $f(x)=\begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$ is a vector function which is defined for all x. We assume that the equation (3-1) has unique solution for all $t\geq 0$ and for any initial condition $x(o)$. The equilibrium $x=0$ of the equation (3-1) is asymptotically stable in the whole, if there exists a scalar function $V(x)$ which is continuous and differentiable with respect to $x_1,\cdots,x_m$ and satisfies the following three conditions.

1°. The function $V(x)$ is positive definite over the whole space x, that is,

$$V(x)>0 \qquad \text{for } x\neq 0, \quad V(o)=0. \tag{3-2}$$

2°. The derivative $\dfrac{dV}{dx}$ along the trajectories of the equation (3-1) is negative definite over the whole space x, that is,

$$\begin{aligned} \frac{dV}{dt}=\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}\cdot f_j(x) &< 0 \qquad \text{for} \quad x\neq 0 \\ &=0 \qquad \text{for} \quad x=0. \end{aligned} \tag{3-3}$$

3°. The function $V(x)$ becomes infinitely large with $||x||$, that is,

$$\lim_{||x||\to\infty} V(x) = \infty . \tag{3-4}$$

where $||x||$ is the norm of vector x, defined by $||x|| = \sup_j |x_j|$.

This continuous and differentiable scalar function $V(x)$ satisfying the above three conditions $1^\circ, 2^\circ$ and 3° is called a Lyapunov function for the equation (3-1), which guarantees the asymptotic stability in the whole.

We shall investigate the absolute stability of the system (2-1), using the Lyapunov function. At first let all $k_1, \cdots, k_m$ to be finite positive numbers. We can formulate the following theorem.

Theorem 1: If we can take the diagonal matrix $\beta = \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_m \end{pmatrix}$ such that the $m \times m$ symmetric matrix H is positive definite and for some $n \times n$ positiv definite P the system of quadratic equations

$$\alpha - \frac{1}{2}\beta C'A - B'M - B'P - \frac{1}{2}C' = 0 \tag{3-5}$$

has real solutions $\alpha = (\alpha_{ij})$, $(i = 1, \cdots, m,\ j = 1, \cdots, n)$, then the system (2-1) is absolutely stable in $[O, K]$, where

$$H = K^{-1} - \frac{1}{2}(\beta C'B + B'C\beta), \tag{3-6}$$

$$K^{-1} = \begin{pmatrix} \frac{1}{k_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{k_m} \end{pmatrix},$$

$$M = \mathcal{L}[\alpha' H^{-1} \alpha] \quad \text{or} \quad A'M + MA = -\alpha' H^{-1} \alpha . \tag{3-7}$$

For the absolute stability of the system (2-1) in $[O, K]$ it is necessary that the linear system obtained by putting $y_j = h_j \sigma_j$ in (2-1) is asymptotically stable for any h_j such that $0 \leq h_j \leq k_j$.

Proof: We look for a Lyapunov function of the form

$$V(x) = x'Lx + \frac{1}{2}\beta_1 \int_0^{\sigma_1} \varphi_1(\sigma_1) d\sigma_1 + \cdots + \frac{1}{2}\beta_m \int_0^{\sigma_m} \varphi_m(\sigma_m) d\sigma_m . \tag{3-8}$$

Where the real symmetric matrix L is obtained by operating the Lyapunov's operator $\mathcal{L}$ on a real symmetric positive definite matrix G, that is to say,

$$\mathcal{L}[G] = L \quad \text{or} \quad A'L + LA = -G . \tag{3-9}$$

For convenience we shall use the following notations

$$\varphi(\sigma) = \begin{pmatrix} \varphi_1(\sigma_1) \\ \vdots \\ \varphi_m(\sigma_m) \end{pmatrix}, \qquad d\sigma = \begin{pmatrix} d\sigma_1 \\ \vdots \\ d\sigma_m \end{pmatrix}.$$

Then the function $V(x)$ of (3-8) is rewritten in the following form

$$V(x) = x'Lx+\int_0^{\sigma}\varphi(\sigma)'\beta d\sigma\,. \tag{3-8}'$$

At first let $\beta_j\geq 0$, $(j=1,\cdots,m)$. By the Lyapunov's theorem (Appendix I) $x'Lx$ is positive definite. And $\int_0^{\sigma_j}\varphi_j(\sigma_j)d\sigma_j\geq 0$, because the nonlinear characteristics $\varphi_j(\sigma_j)$, $(j=1,\cdots,m)$ satisfy the conditions $(2\text{-}1)_2$. Then the function $V(x)$ of (3-8) is positive definite over the whole space x. In our case $(\beta_j\geq 0,\ (j=1,\cdots,m))$ $V(x)\geq x'Lx>0$ and $\lim_{\|x\|\to\infty} x'Lx=\infty$, so $\lim_{\|x\|\to\infty} V(x)=\infty$. Thus, for $\beta_j\geq 0$ $(j=1,\cdots,m)$ conditions 1° and 3° are satisfied. If some of β_j $(j=1,\cdots,m)$ are negative, positive definiteness of the function $V(x)$ of (3-8) does not follow immediately. But, as we shall show later, even if some of β_j $(j=1,\cdots,m)$ are negative, the conditions 1° and 3° follow from the condition 2° under the condition that the linear system obtained by putting $y_j=h_j\sigma_j$ $(j=1,\cdots,m)$ in (2-1) is asymptotically stable for any h_j such that $0\leq h_j\leq k_j$ $(j=1,\cdots,m)$. Let us calculate the derivative $\frac{dV}{dx}$ along the trajectories of the system (2-1).

$$\begin{aligned}\frac{dV}{dt} &= \sum_{j=1}^{n}\frac{\partial V}{\partial x_j}\frac{dx_j}{dt}\\ &= \sum_{j=1}^{m}\{2\sum_{i=1}^{m}l_{ji}x_i+\sum_{i=1}^{m}\beta_i c_{ji}\varphi_i\}\{\sum_{i=1}^{n}a_{ji}x_i+\sum_{i=1}^{m}b_{ji}\varphi_i\}\,.\end{aligned} \tag{3-10}$$

Using (3-9)

$$\frac{dV}{dt} = -x'Gx+\varphi(\sigma)(2B'L+\beta C'A)x+\varphi(\sigma)\beta C'B'\varphi(\sigma)\,. \tag{3-10}'$$

Our problem is how to guarantee negative definiteness of $\frac{dV}{dt}$ of (3-10) for any nonlinear functions $\varphi_j(\sigma_j)$, $(j=1,\cdots,m)$ satisfying the conditions $(2\text{-}1)_2$. The right-hand side of (3-10) contains the nonlinear functions $\varphi_j(\sigma_j)$, $(j=1,\cdots,m)$ and is not a quadratic form of vector x. Then there is no simple criterion which guarantees negative definiteness of the right-hand side of (3-10).

If we replace $\varphi(\sigma)$ by variable vector y, $\frac{dV}{dt}$ of (3-10) can be considered as a quadratic form of the $(n+m)$-dimensional vector $\binom{x}{y}$. This quadratic form may be negative difinite. But this is not the case. In fact, from the representation of (3-10), $\frac{dV}{dt}$ can be zero for the vector $\binom{x}{y}$ satisfying

$$\sum_{i=1}^{n}a_{ji}x_i+\sum_{i=1}^{m}b_{ji}y_i = 0 \qquad (j = 1,\cdots,n)\,. \tag{3-11}$$

This is a system of n linear homogeneous equation of $n+m$ unknowns $x_1,\cdots,x_n$, $y_1,\cdots,y_m$, so cleary this system has non-trivial solutions. This means that the quadratic form $\frac{dV}{dt}$ of (3-10) of the $(m+n)$-dimensional vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can not be sign definite.

Then we use the so-called S-process[13], as in the case of the system with one nonlinear characteristic. We add and subtract the expression

$$\{\sigma - K^{-1}\varphi(\sigma)\}'\varphi(\sigma) = \sum_{j=1}^{m} \left(\sigma_j - \frac{\varphi_j(\sigma_j)}{k_j}\right)\varphi_j(\sigma_j) \tag{3-12}$$

from the right-hand side of (3-10). Under the conditions $(2\text{-}1)_2$ this expression is non-negative

$$\{\sigma - K^{-1}\varphi(\sigma)\}'\varphi(\sigma) \geq 0. \tag{3-13}$$

Introducing the following expression

$$\begin{aligned} S[x, \varphi(\sigma)] &= x'Gx - \varphi(\sigma)'(2B'L + \beta C'A)x \\ &\quad - \{\sigma - K^{-1}\varphi(\sigma)\}'\varphi(\sigma) - \varphi(\sigma)'(\beta C'B)\varphi(\sigma), \end{aligned} \tag{3-14}$$

$\frac{dV}{dt}$ of (3-10) can be rewritten in the following form

$$\frac{dV}{dt} = -S[x, \varphi(\sigma)] - \{\sigma - K^{-1}\varphi(\sigma)\}'\varphi(\sigma). \tag{3-15}$$

From (3-13) and (3-15) if $S[x, \varphi(\sigma)]$ is positve definite, $\frac{dV}{dt}$ becomes negative definite.

Now let us replace $\varphi(\sigma)$ in S by variable vector y and find the condition for positive definiteness of $S(x, y)$, because it is very difficult to discuss positive definiteness of $S[x, \varphi(\sigma)]$. $S(x, y)$ is a quadratic form of the $(n+m)$-dimensional vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as follows

$$\begin{aligned} S(x, y) &= x'Gx - 2y'\left(B'L + \frac{1}{2}\beta C'A + \frac{1}{2}C'\right)x + y'(K^{-1} - \beta C'B)y \\ &= (x', y')\begin{pmatrix} G & -\alpha' \\ -\alpha & H \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned} \tag{3-16}$$

where

$$\begin{aligned} \alpha &= B'L + \frac{1}{2}\beta C'A + \frac{1}{2}C', \\ H &= K^{-1} - \frac{1}{2}(\beta C'B + B'C\beta). \end{aligned} \tag{3-17}$$

Therefore, if $(n+m)\times(n+m)$ symmetric matrix

$$\tilde{G} = \begin{pmatrix} G & -\alpha' \\ -\alpha & H \end{pmatrix}$$

is positive definite, $S(x, y)$ becomes a positive definite quadratic form. On the other hand, if the matrix H is non-singular, $S(x, y)$ can be rewritten in the following form

$$S(x, y) = (y-H^{-1}ax)'H(y-H^{-1}ax)+x'(G-a'H^{-1}a)x.$$

Therefore, in order that $S(x, y)$ is positive definite, it is necessary and sufficient that $m\times m$ matrix H be positive definite and $n\times n$ matrix

$$R = G-a'H^{-1}a \tag{3-18}$$

is positive definite.

In view of (3-9) and (3-17), the elements g_{ij} $(i, j=1, \cdots, n)$ of the symmetric matrix G and the elements a_{ij} $(i=1, \cdots, m,\ j=1, \cdots, n)$ of the matrix a are dependent linearly on the elements l_{ij} $(i, j=1, \cdots, n)$ of the symmetric matrix L. Consequently if positive definite matrix R is given arbitrarily, the relation (3-18) is a system of $\frac{1}{2}n(n+1)$ quadratic equations of $\frac{1}{2}n(n+1)$ unknowns l_{ij}.

Thus, if the symmetric matrix H is positive definite and for some positive definite matrix R the system of quadratic equations (3-18) has real solutions l_{ij} $(i, j=1, \cdots, n)$, the obtained symmetric matrix L in this manner is positive definite. In fact, the matrix

$$G = R+a'H^{-1}a \tag{3-18$'$}$$

is positive definite and the matrix L is obtained by operating the Lyapunov's operator $\mathcal{L}$ on the symmetric matrix G.

Thus if we take $\beta=\begin{pmatrix}\beta_1 & & 0\\ & \ddots & \\ 0 & & \beta_m\end{pmatrix}$ such that the symmetric matrix H is positive definite and for some positive definite matrix R there exists a real symmetric matrix L satisfying the relation (3-18), $-S(x,y)$, hence, $\frac{dV}{dt}$ becomes negative definite.

Now let us prove that (3-18), the system of $\frac{1}{2}n(n+1)$ quadratic equations, can be reduced to a system of mn quadratic equations of the mn unknowns a_{ij} $(i=1, \cdots, m,\ j=1, \cdots, n)$. Operating the Lyapunov's operator $\mathcal{L}$ on the right- and left-hanp sides of (3-18) and putting

$$\mathcal{L}[R] = P, \quad \mathcal{L}[a'H^{-1}a] = M, \tag{3-19}$$

we obtain

$$P = L-M. \tag{3-20}$$

When R is any positive definite matrix, P is also a positive definite matrix, and vice versa, because P and R are connected by the Lyapunov's operator with each other. Then, we can consider P given arbitrary instead of R.

The elements m_{ij} $(i, j=1, \cdots, n)$ of the symmetric matrix M depend quadratically on α_{ij} $(i=1, \cdots, m,\ j=1, \cdots, n)$. Using (3-20) and eliminating the matrix L from (3-17) we obtain

$$\alpha-\frac{1}{2}\beta C'A-B'M-B'P-\frac{1}{2}C'=0. \tag{3-5}$$

This relation can be considered as a system of mn quadratic equations of mn unknowns α_{ij} $(i=1, \cdots, m, j=1, \cdots, n)$. If we can take the positive definite matrix P such that the system of quadratic equations (3-5) has real solutions α_{ij} $(i=1, \cdots, m, j=1, \cdots, n)$, then the positive definite matrix G is determined from (3-18)′, hence $L=\mathcal{L}[G]$. Thus there exists a Lyapunov function of the form (3-8) (in our case $\beta_j \geq 0$ $(j=1, \cdots, m)$) and the absolute stability of the system (2-1) is established.

In the above arguments, we have restricted our discussions to the case where $\beta_j \geq 0$ $(j=1, \cdots, m)$. But for any real numbers β_j as we show in the following, if the linearized system of (2-1) is asymptotically stable for any h_j such that $0 \leq h_j \leq k_j$ $(j=1, \cdots, m)$ and the condition 2° is satisfied for any $\varphi_j(\sigma_j)$ $(j=1, \cdots, m)$ satisfying $(2\text{-}1)_2$, then the conditions 1° and 3° are also satisfied. The following two lemmata are extentions of those of V. A. Pliss and E. N. Rozenvasser for the system containing one nonlinear charracteristic[13].

Lemma 1: If the function $V(x)$ of (3-8), in which $\varphi_j(\sigma_j)$ is replaced by $h_j\sigma_j$, satisfies the conditions 1°, 2° and 3° for any h_j such that $0 \leq h_j \leq k_j$, $(j=1, \cdots, m)$, then the function $V(x)$ for any $\varphi_j(\sigma_j)$ satisfying the conditions $(2\text{-}1)_2$ satisfies the conditions 1°, 2° and 3°.

Proof: If we take the real numbers β_j $(j=1, \cdots, m)$ such that $\beta_1, \cdots, \beta_r \geq 0$ $\beta_{r+1}, \cdots, \beta_m < 0$, the following inequality is obtained

$$\begin{aligned} V_1(x) &\equiv x'Lx+\frac{1}{2}\beta_{r+1}k_{r+1}\sigma_{r+1}^2+\cdots+\frac{1}{2}\beta_m k_m \sigma_m^2 \\ &\leq V(x) \leq x'Lx+\frac{1}{2}\beta_1 k_1 \sigma_1^2+\cdots+\frac{1}{2}\beta_r k_r \sigma_r^2 \equiv V_2(x) \end{aligned} \tag{3-21}$$

Form the assumption, the functions $V_1(x)$ and $V_1(x)$ satisfy the conditions 1° and 3°. Thus, from (3-21) the function $V(x)$ also satisfies the conditions 1° and 3°. Now any functions $\varphi_j(\sigma_j)$ $(j=1, \cdots, m)$ are represented as follows

$$\varphi_j(\sigma_j)=\kappa_j(\sigma_j)\sigma_j, \qquad 0 \leq \kappa_j(\sigma_j) \leq k_j$$

Therefore, it is sufficient to verify the sign definiteness of $V(x)$ only for linearized system such that $y_j=h_j\sigma_j$, $0 \leq h_j \leq k_j$ $(j=1, \cdots, m)$. Thus, if the function

$$V_0(x) = x'Lx + \frac{1}{2}\beta_1 h_1 \sigma_1^2 + \cdots + \frac{1}{2}\beta_m h_m \sigma_m^2 \qquad (3\text{-}22)$$

satisfies the conditions 1°, 2° and 3° for any h_j, $0 \leq h_j \leq k_j$ $(j=1,\cdots,m)$, then so does the function $V(x)$ with the same L and β as $V_0(x)$ for any $\varphi_j(\sigma_j)$ $(j=1,\cdots,m)$ satisfying the conditions $(2\text{-}1)_2$.

Lemma 2: In order that the function $V(x)$ of (3-8) guarantees the absolute stability of the system (2-1), it is necessary and sufficient that the following two conditions are satisfied.

(a) The linear system obtained by putting $y_j = \delta_j k_j \sigma_j$, $\delta_j = 0$ or $1 (j=1,\cdots,m)$ are asymptotically stable.

(b) The perivative $\frac{dV}{dt}$ along the trajectories of the system (2-1) in which $\varphi_j(\sigma_j) = h_j \sigma_j$, $0 \leq h_j \leq k_j$ $(j=1,\cdots,m)$ is negative definite.

Proof: It is evident that the conditions (a) and (b) are necessary. Let us prove the sufficiency of the conditions (a) and (b). As we showed previously, if the condition (b) is satisfied, the derivative $\frac{dV}{dt}$ is negative definite for any nonlinear characteristics $\varphi_j(\sigma_j)$ $(j=1,\cdots,m)$ satisfying the conditions $(2\text{-}1)_2$. We assume that the constants β_j $(j=1,\cdots,m)$ are taken such that $\beta_1,\cdots,\beta_r \geq 0$, $\beta_{r+1},\cdots,\beta_m < 0$. From the condition (a) the two linear systems obtained by replacing $y_1=0,\cdots,y_r=0$, $y_{r+1}=k_{r+1}\sigma_{r+1},\cdots,y_m=k_m\sigma_m$ and $y_1=k_1\sigma_1,\cdots$, $y_r=k_r\sigma_r$, $y_{r+1}=0,\cdots,y_m=0$ are stable. From the condition (b) the derivatives $\frac{dV_1}{dt}$ and $\frac{dV_2}{dt}$ of the following functions

$$V_1(x) = x'Lx + \frac{1}{2}\beta_{r+1}k_{r+1}\sigma_{r+1}^2 + \cdots + \frac{1}{2}\beta_m k_m \sigma_m^2$$

$$V_2(x) = x'Lx + \frac{1}{2}\beta_1 k_1 \sigma_1^2 + \cdots + \frac{1}{2}\beta_r k_r \sigma_r^2$$

are negative definite. Since these functions are quadratic forms of vector x, by the Lyapunov's theorem the functions $V_1(x)$ and $V_2(x)$ are positive definite and $\lim_{||x|| \to \infty} V_j(x) = \infty$ $(j=1, 2)$. Moreover, since for the function $V(x)$ in which nonlinear characteristics $\varphi_j(\sigma_j)$ $(j=1,\cdots,m)$ satisfy the conditions $(2\text{-}1)_2$, the inequality

$$V_1(x) \leq V(x) \leq V_2(x)$$

holds, conditions 1° and 3° are satisfied for also $V(x)$.

In view of the above two lemmata it can be concluded that if the linear system obtained by putting $y_j = \delta_j k_j \sigma_j$, $\delta_j = 0$ or 1 $(j=1,\cdots,m)$ is asymptotically stable and the function $V(x)$ of (3-8) satisfyies the condition 2° under the restriction of the S-process, that is, the system of quadratic equations (3-5) has real solutions the system (2–1) is absolutely stable in $[O, K]$. This completes the proof of Theorem 1.

Now the above two lemmata give criteria of the absolute stability of the system (2–1), which contain not only the case of the S-process but also all the cases of the Lyapunov function of the type (3–8) with any real numbers β_j $(j=1,\cdots,m)$.

Let us consider the case in which $k_1,\cdots,k_l$ are infinite and $k_{l+1},\cdots,k_m$ are finite positive numbers in the conditions $(2\text{–}1)_2$. In this case we can discuss analogously to the previous arguments, if we replace the matrix

$$K^{-1}=\begin{pmatrix}\frac{1}{k_1} & & 0\\ & \ddots & \\ 0 & & \frac{1}{k_m}\end{pmatrix}$$

by the matrix

$$\begin{pmatrix}0 & & & 0\\ & \ddots & & \\ & & 0 & \\ 0 & & & K_2^{-1}\end{pmatrix},$$

where $K_2=\begin{pmatrix}k_{l+1} & & 0\\ & \ddots & \\ 0 & & k_m\end{pmatrix}$. Namely we look for a Lyapunov function of the form

$$V(x)=x'Lx+\beta_1\int_0^{\sigma_1}\varphi_1(\sigma_1)d\sigma_1+\cdots+\beta_m\int_0^{\sigma_m}\varphi_m(\sigma_m)d\sigma_m. \tag{3-23}$$

The derivative $\frac{dV}{dt}$ along the trajectories of the system (2–1) becomes

$$\frac{dV}{dt}=-S[x,\varphi(\sigma)]-\left\{\sum_{j=1}^{l}\sigma\varphi_j(\sigma_j)+\sum_{j=l+1}^{m}\left(\sigma_j-\frac{\varphi_j(\sigma_j)}{k_j}\right)\varphi_j(\sigma_j)\right\}$$

$$S[x,\varphi(\sigma)]=x'Gx-2\varphi(\sigma)'ax+\varphi(\sigma)'H\varphi(\sigma) \tag{3-24}$$

where

$$\begin{aligned}a&=B'L+\frac{1}{2}\beta C'A+\frac{1}{2}C'\\ H&=\begin{pmatrix}0 & 0\\ 0 & K_2^{-1}\end{pmatrix}-\frac{1}{2}(\beta C'B+B'C\beta).\end{aligned} \tag{3-25}$$

And the system of quadratic equations of the unknowns a_{ij} $(i=1,\cdots,m,$ $j=1,\cdots,n)$ becomes

$$a-\frac{1}{2}\beta C'A-B'M-B'P-\frac{1}{2}C'=0 \tag{3-26}$$

where P and M are obtained by the relations

$$P=\mathcal{L}[R],\quad M=\mathcal{L}[a'H^{-1}a].$$

In this case, differently from the case where all k_j $(j=1,\cdots,m)$ are finite positive numbers, the real numbers β_j $(j=1,\cdots,l)$ must be non-negative. In fact, if we take some of $\beta_1, \cdots, \beta_l$ negative, the values of the function

$$V_0(x) = x'Lx + \frac{1}{2}\beta_1 h_1 \sigma_1^2 + \cdots + \frac{1}{2}\beta_l h_l \sigma_l^2 + \cdots + \frac{1}{2}\beta_m h_m \sigma_m^2$$

can be negative for sufficiently large h which have the same suffices as the negative β.

Thus, we obtain:

Theorem 2: If we can take the diagonal matrix $\beta = \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_m \end{pmatrix}$ such that the $m \times m$ symmetric matrix H of (3–25) is positive definite and for some $n \times n$ positive definite matrix P the system of quadratic equations (3–26) has real solutions $\alpha = (\alpha_{ij})$ $(i = 1, \cdots, m,\ j = 1, \cdots, n)$, the system (2–1) is absolutely stable in $[O, K]$. Where $K = \begin{pmatrix} \infty & & & & 0 \\ & \ddots & & & \\ & & \infty & & \\ & & & k_{l+1} & \\ & & & & \ddots \\ 0 & & & & & k_m \end{pmatrix}$. In this case the real numbers β_j $(j = 1, \cdots, l)$ must be non-negative. Of course it is necessary that the linear system obtained by putting $y_j = h_j \sigma_j$ $(j = 1, \cdots, m)$ in (2–1) is asymptotically stable for any h_j such that $0 \le h_1, \cdots, h_l$, $0 \le h_j \le k_j$, $(j = l+1, \cdots, m)$.

So far we have discussed the case where $m \times m$ symmetric matrix H is positive definite. When the matrix H is not positive definite, but non-negative, the quadratic form $S(x, y)$ can not be positive definite. However, if we require that the quadratic form $S(x, y)$ is non-negative and not degenerate with respect to x, that is, $S(x, y) > 0$ for $x \neq 0$, $-S(x, \varphi(\sigma))$ and therefore $\frac{dV}{dt}$ of (3–10) becomes negative definite over the whole space x. And as we showed previously, also in this case, negative definiteness of $\frac{dV}{dt}$ of (3–10) implies the absolute stability of the system (2–1) under the condition that the linear system obtained by putting $y_j = h_j \sigma_j$ is asymptotically stable for any h_j such that $0 \le h_j \le k_j$ $(j = 1, \cdots, m)$.

Let the $m \times m$ symmetric matrix H to be non-negative and let rank $H = l$, $(0 \le l \le m)$. Then there exists an orthogonal matrix T such that

$$T'HT = \begin{pmatrix} H_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_l \end{pmatrix}, \quad h_1, \cdots, h_e > 0\,.$$

If we transform the vector y to the vector z by

$$y = Tz\,,$$

S is represented in the following

$$S(x, z) = x'Gx - z'T'\alpha x - x'\alpha Tz + z' \begin{pmatrix} H_0 & 0 \\ 0 & 0 \end{pmatrix} z. \tag{3–27}$$

We represent the m-dimensional vector z as follows

$$z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$$

where z_0 is l-dimentional, z_1 is $(m-l)$-dimensional. Put the matrix $T'\alpha$ as follows

$$T'\alpha = \begin{pmatrix} T_0' \\ T_1' \end{pmatrix}\alpha = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} \equiv \omega\,, \tag{3-28}$$

where T_0': $l\times m$-matrix, T_1': $(m-l)\times m$-matrix,
ω_0 : $l\times n$-matrix, ω_1 : $(m-l)\times n$-matrix.

The function $S(x, z)$ can be rewritten in the following

$$S(x, z) = (H_0^{1/2}z_0 - H_0^{-1/2}\omega_0 x)'(H_0^{1/2}z_0 - H_0^{-1/2}\omega_0 x) + x'(G-\omega_0'H^{-1}\omega_0)x - 2z_1'\omega_1 x\,. \tag{3-27)'$$

If we put

$$\omega_1 = 0 \tag{3-29}$$

and the matrix

$$R \equiv G - \omega_0'H^{-1}\omega_0 \tag{3-30}$$

is positive definite, $S(x, z)$ becomes a non-negative quadratic form of the $(n+m)$-dimensional vector $\begin{pmatrix} x \\ z \end{pmatrix}$ and not degenerate with respect to x. Operating the Lyapunov's operator on the both sides of the relation (3-30), we obtain

$$P = L - M \tag{3-31}$$

where

$$P = \mathcal{L}[R]\,, \quad L = \mathcal{L}[G]\,,$$
$$M = \mathcal{L}[\omega_0'H_0^{-1}\omega_0] = \mathcal{L}[\alpha'T_0H_0^{-1}T_0\alpha]\,. \tag{3-32}$$

From the relation

$$\alpha = B'L + \frac{1}{2}\beta C'A + \frac{1}{2}C'$$

and from (3-31) eliminating the matrix L, we obtain

$$\alpha = B'P + B'M + \frac{1}{2}\beta C'A + \frac{1}{2}C'\,.$$

Multiplying the matrix T' on the both sides of this relation from the left,

$$\omega = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} T_0' \\ T_1' \end{pmatrix}\left(B'P + B'M + \frac{1}{2}\beta C'A + \frac{1}{2}C'\right)$$
$$= \begin{pmatrix} T_0'\left(B'P + B'M + \frac{1}{2}\beta C'A + \frac{1}{2}C'\right) \\ T_1'\left(B'P + B'M + \frac{1}{2}\beta C'A + \frac{1}{2}C'\right) \end{pmatrix}.$$

In view of (3-28) and (3-29) we obtain

$$\begin{aligned} T_0'a &= T_0'\Big(B'P+B'M+\frac{1}{2}\beta C'A+\frac{1}{2}C'\Big), \\ 0 &= T_1'\Big(B'P+B'M+\frac{1}{2}\beta C'A+\frac{1}{2}C'\Big). \end{aligned} \tag{3-33}$$

Since from the definition of the matrix M its elements m_{ij} $(i, j=1,\cdots,n)$ are dependent quadratically on a_{ij} $(i=1,\cdots,m, j=1,\cdots,n)$, the relation (3-33) can be considered as a system of mn quadratic equations of unknowns a_{ij} $(i=1,\cdots,m,\ j=1,\cdots,n)$. If this system of quadratic equations has real solutions, the quadratic form $S(x, y)$ is nonnegative and not degenerate with respect to x.

Thus, if the real numbers β_j $(j=1,\cdots,m)$ are chosen such that the symmetric matrix H is non-negative and for some positive definite matrix R the system of quadratic equations (3-33) has real solutions, the system (2-1) is absolutely stable in $[O, K]$.

4. The Sufficient Condition for the Absolute Stability in Terms of the Frequency Characteristic of the Linear Parts of the System

In the paragraph 3 we discussed the absolute stability of the system (2-1) by means of the Lyapunov's direct method. By a different method from it V. M. Popov gave a sufficient condition for the absolute stability of the system with one nonlinear characteristic. The property of the Popov's method consists in the use of the frequency characteristic of the linear parts of the system.

In this paragraph, extending the Popov's method, we shall give a sufficient condition for the absolute stability of the system (2-1) contaning m nonlinear characteristics.

(i) The case where all $k_1,\cdots,k_m$ are finite.

The following theorem holds.

Theorem 3: Let the constant matrix A to be asymptotically stable. Let us represent the frequency characteristic of the linear parts of the system (2-1) $W(i\omega)$ and define the $m\times m$ matrix

$$N(i\omega) = K^{-1}+(E+i\omega Q)W(i\omega), \tag{4-1}$$

where

$$K=\begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_m \end{pmatrix}.$$

If there exists a real diagonal matrix $Q=\begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_m \end{pmatrix}$ such that the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is positive definite for all $\omega\geq 0$ and for $\omega\to\infty$, the system (2–1) is absolutely stable in $[O, K]$, where the asterisk $*$ denotes the transpose and complex conjugate of matrices. For the absolute stability of the system (2–1) in $[O, K]$ it is necessary that the linear system obtained by putting $y_j = h_j\sigma_j$ in (2–1) is asymptotically stable for any h_j such that $0 \le h_j \le k_j$ $(j=1,\cdots,m)$.

Note 1. The frequency characteristic $W(i\omega)$ of the linear parts of the system, whose input y and out put $-\sigma$, is

$$W(i\omega) = C'(A - i\omega E)^{-1}B\,, \tag{4–2}$$

where E is unit matrix.

Note 2. When the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is positive definite for all $\omega \ge 0$ and for $\omega \to \infty$, there is a positive constant δ independent of ω such that for any m-dimensional vector z the inequality

$$z^* \frac{1}{2}\{N(i\omega)+N(i\omega)^*\}z \ge \delta z^* z \tag{4–3}$$

takes place for all $\omega \ge 0$ and for $\omega \to \infty$. We shall write $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\} \ge \delta E$ in place of (4–3) for simplicity where E is $m \times m$ unite matrix. It is noteworthy to indicate the following relation:

$$\lim_{||x||\to\infty} \frac{1}{2}\{N(i\omega)+N(i\omega)^*\} = K^{-1} - \frac{1}{2}(QC'B + B'CQ) \tag{4–4}$$

Proof of Theorem 3: At first let all $q_1,\cdots,q_m$ to be positive. A solution of the system (2–1) with the initial condition $x(o) = x_0$ is

$$x(t) = X(t)x_0 + \int_0^t X(t-\tau)B\varphi[\sigma(\tau)]\,d\tau \tag{4–5}$$

where the $n \times n$ matrix $X(t)$ is the solution of the matrix equation

$$\frac{dX}{dt} = AX \tag{4–6}$$

with the initial condition $X(o) = E$ and the vector function $\varphi[\sigma(t)]$ is one such that

$$\varphi[\sigma(t)] = \begin{pmatrix} \varphi_1[\sigma_1(t)] \\ \vdots \\ \varphi_m[\sigma_m(t)] \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} = C'x(t)$$

From our assumption the null solution of the equation (4–6) is asymptotically stable and therefore two positive constants M_0 and M_1 can be found such that for all $t \ge 0$ the inequality

$$||X(t)|| < M_1 e^{-M_0 t} \tag{4–7}$$

holds, where the norm $||X||$ of the matrix X is defined such that $||X|| = \sup_j |X_j|$.

Using the vector function $y(t)=\varphi[\sigma(t)]$, we define the auxiliary vector function

$$y_T(t) = \begin{cases} y(t) & \text{for} \quad 0 \leq t \leq T \\ 0 & \text{for} \quad t \geq T \end{cases} \tag{4-8}$$

where T is any positive constant. Let us consider the linear non-homogeneous system

$$\frac{dx_T}{dt} = Ax_T + By_T(t) \tag{4-9}$$

and put

$$\sigma_T(t) = C'x_T(t). \tag{4-10}$$

If the same initial condition is given for the system (2-1) and for the system (4-9), then the equality

$$\sigma(t) = \sigma_T(t) \qquad \text{for} \quad 0 \leq t \leq T \tag{4-11}$$

yields. From the well known properties of the linear non-homogeneous system $\sigma_T(t)$ can be represented as follows

$$\sigma_T(t) = \tilde{\sigma}_T(t) + p(t) \tag{4-12}$$

where $p(t)$ is the term corresponding to the solution of the equation $\frac{dx_T}{dt} = Ax_T$ with the initial value $x_T(o)$ and $\tilde{\sigma}_T(t)$ the term corresponding to the solution of (4-9) with zero initial value. In agreement with (4-7) there exist positive constants M_2 and M_3 such that for all $t \geq 0$ the following inequalities hold

$$\|p(t)\| < M_2 e^{-M_0 t}, \qquad \left\| \frac{dp}{dt} \right\| < M_3 e^{-M_0 t}. \tag{4-13}$$

Now let us define the vector function $f(t)$ and $l(t)$ by the formulas

$$\begin{aligned} -f(t) &= \sigma_T(t) + Q\frac{d\sigma_T}{dt} - K^{-1} y_T(t) \\ &= \tilde{\sigma}_T(t) + Q\frac{d\tilde{\sigma}_T}{dt} - K^{-1} y_T(t) + p(t) + Q\frac{dp}{dt}, \end{aligned} \tag{4-14}$$

$$l(t) = -p(t) - Q\frac{dp}{dt}. \tag{4-15}$$

Let the Fourier transforms of the vector functions $f(t)$, $l(t)$, $y_T(t)$ and $\tilde{\sigma}_T(t)$ to be $F(i\omega)$, $L(i\omega)$, $Y_T(i\omega)$ and $\tilde{\Sigma}_T(i\omega)$, respectively. As can be seen from (4-8), (4-9), (4-10) and (4-13) these Fourier transforms exist. By the definition of the frequency characteristic $W(i\omega)$ the equality

$$-\tilde{\Sigma}_T(i\omega) = W(i\omega) Y_T(i\omega) \tag{4-16}$$

takes place. Taking the Fourier transforms of the both sides of the equality

$$f(t) = -\tilde{\sigma}_T(t) - Q\frac{d\tilde{\sigma}_T}{dt} + K^{-1}y_T(t) + l(t)$$

and using (4–16), we obtain the equality

$$F(i\omega) = \{(E+i\omega Q)W(i\omega)+K^{-1}\}Y_T(i\omega)+L(i\omega) = N(i\omega)Y_T(i\omega)+L(i\omega). \quad (4\text{–}17)$$

When the condition of Theorem 3:

$$\frac{1}{2}\{N(i\omega)+N(i\omega)^*\} \geq \delta E$$

is satisfied, in view of Appendix II we obtain the inequality

$$-\int_0^\infty f(t)'y_T(t)dt \leq D \quad (4\text{–}18)$$

where

$$D = \frac{1}{8\pi\delta}\int_{-\infty}^\infty L(i\omega)^*L(i\omega)d\omega. \quad (4\text{–}19)$$

As can be seen from (4–15) the vector function $l(t)$ is given in terms of the solution of the linear homogeneous system, and then $l(t)$ is dependent merely on the initial condition $x_T(o)=x(o)$ and independent of the constant T. Thus, the constant D depends only on the initial conditions $x(o)$ and $D\to 0$ as $\|x(o)\|\to 0$. By substituting (4–14) into the left-hand sids of (4–18) we obtain

$$\int_0^T \{\sigma(t)-K^{-1}\varphi[\sigma(t)]\}'\varphi[\sigma(t)]dt + \int_0^{\sigma(T)}\varphi(\sigma)'Qd\sigma \leq \bar{D}(x_0) \quad (4\text{–}20)$$

where

$$\bar{D}(x_0) = D(x_0)+\sum_{j=1}^m q_j\int_0^{\sigma_j(0)}\varphi_j(\sigma_j)d\sigma_j. \quad (4\text{–}21)$$

From this inequality the following inequalities are obtained

$$\int_0^T \{\sigma(t)-K^{-1}\varphi[\sigma(t)]\}'\varphi[\sigma(t)]dt \leq \bar{D}, \quad (4\text{–}22)$$

$$\int_0^{\sigma(T)}\varphi(\sigma)'Qd\sigma = \sum_{j=1}^m q_j\int_0^{\sigma_j(T)}\varphi_j(\sigma_j)d\sigma_j \leq \bar{D}. \quad (4\text{–}23)$$

At first, let us assume $\varphi_j(\sigma_j)$ satisfies the condition

$$0<\varepsilon\leq\frac{\varphi_j(\sigma_j)}{\sigma_j}\leq k_j, \quad \varphi_j(o)=0, \quad (j=1,\cdots,m) \quad (4\text{–}24)$$

instead of $(2\text{–}1)_2$, where ε is a small positive constant. Then from (4–23) we can easily obtain the inequality

$$\frac{1}{2}\varepsilon\sum_{j=1}^m q_j\sigma_j(T)^2 \leq \bar{D}. \quad (4\text{–}25)$$

As we assumed all $q_1,\cdots,q_m$ to be positive and $\bar{D}$ is independent of the constant T. We can conclude that

$$||\sigma(t)|| < M_4 \qquad \text{for all} \quad t \geq 0, \tag{4-26}$$

where

$$M_4 = \sqrt{\frac{2\bar{D}}{\varepsilon q}}, \qquad q = \min(q_1, \cdots, q_m).$$

For the solution $x(t)$ of (4-5) the inequality

$$||x(t)|| \leq ||X(t)x_0|| + ||\int_0^t X(t-\tau)B\varphi[\sigma(\tau)]d\tau|| \tag{4-27}$$

holds. From (4-7)

$$||X(t)x_0|| \leq M_5||x_0|| \qquad \text{for all} \quad t \geq 0, \ M_5 = nM_1. \tag{4-28}$$

From (4-24) and (4-26)

$$\begin{aligned}\left\| \int_0^t X(t-\tau)B\varphi[\sigma(\tau)]d\tau \right\| &\leq nm \int_0^t ||X(t-\tau)|| \, ||B|| ||\varphi(\sigma(\tau)]|| d\tau \\ &\leq nm ||B|| \int_0^t e^{-M_0(t-\tau)} \cdot M_1 \cdot ||K|| ||\sigma(\tau)|| d\tau \\ &\leq nm M_1 M_4 ||B|| \int_0^t e^{-M_0(t-\tau)} d\tau \leq \frac{nm M_1 M_4 ||B||}{M_0}. \end{aligned} \tag{4-29}$$

From (4-26), (4-27) and (4-28) we obtained the following estimation

$$||x(t)|| \leq M_6, \tag{4-30}$$

where

$$M_6 = M_5||x_0|| + \frac{nm M_1 M_4 ||B||}{M_0}.$$

Since $\bar{D} \to 0$ as $||x_0|| \to 0$, M_4, $M_6 \to 0$ as $||x_0|| \to 0$.

Consequently the equilibrium $x=0$ of the system (2-1) is stable in the sense of Lyapnov. Next, we prove the equilibrium $x=0$ to be asymptotically stable in the whole. From (4-22) the inequality

$$\int_0^T \left\{ \sigma_j - \frac{\varphi_j(\sigma_j)}{k_j} \varphi(\sigma_j) dt \leq \bar{D}\right.$$

holds for each j $(1 \leq j \leq m)$. Under the conditions (4-24) the functions $G_j(\sigma_j) = \left\{\sigma_j - \frac{\varphi_j(\sigma_j)}{k_j}\right\}\varphi_j(\sigma_j)$ satisfy the conditions of Appendix III and from (4-26) $\sigma_j(t)$ and $\frac{d\sigma_j(t)}{dt}$ are bounded, and then according to Appendix III we obtain

$$\lim_{t\to\infty} \sigma_j(t) = 0, \qquad (j = 1, \cdots, m) \tag{4-31}$$

for any initial condition. And therefore

$$\lim_{t\to\infty} \varphi_j[\sigma_j(t)] = 0 \qquad (j=1, \cdots, m). \tag{4-32}$$

For the first term of the right-hand side of (4-27) the equality

$$\lim_{t\to\infty} ||X(t)x_0|| = 0$$

yields from (4–7) and for the second term

$$\left\| \int_0^t X(t-\tau)B\varphi[\sigma(\tau)]d\tau \right\| \leq nm\,||B||\,M_1 \int_0^t e^{-M_0(t-\tau)}||\varphi[\sigma(\tau)]||d\tau$$

$$\leq nm\,||B||\,M_1 \int_0^t e^{M_0\tau}||\varphi[\sigma(\tau)]||d\tau/e^{M_0 t}\,.$$

Applying de l'Hopital rule we obtain

$$\lim_{t\to\infty} \frac{\int_0^t e^{M_0\tau}||\varphi[\sigma(\tau)]||d\tau}{e^{M_0 t}} = \lim_{t\to\infty}\frac{1}{M_0}||\varphi[\sigma(t)]|| = 0\,.$$

Thus, we obtain

$$\lim_{x\to\infty} ||x(t)|| = 0\,. \tag{4-33}$$

Consnquently it has been proved that the equilibrium $x=0$ of the system (2–1) containing m nonlinear characteristics satisfying the conditions (4–24) is asymptotically stable in the whole.

Now let us prove the asymptotical stability in the whole of the equilibrium $x=0$ of the system (2–1) with m nonlinear characteristics satisfying the conditions $(2\text{–}1)_2$. For that purpose we define the vector function $\varphi_\varepsilon(\sigma)$ as follows

$$\varphi_\varepsilon(\sigma) = \varphi(\sigma)+\varepsilon E\sigma \tag{4-34}$$

or

$$\varphi_{\varepsilon j}(\sigma_j) = \varphi_j(\sigma_j)+\varepsilon\sigma_j\,, \qquad (j = 1,\cdots,m)\,,$$

where ε is the same constant as in (4–24). Then the system (2–1) can be rewritten in the following form

$$\frac{dx}{dt} = A_\varepsilon x+By_\varepsilon\,,\quad y_\varepsilon = \varphi_\varepsilon(\sigma)\,,\quad \sigma = C'\cdot x\,, \tag{4-35}$$

where

$$A_\varepsilon = A-\varepsilon BC'\,. \tag{4-36}$$

When $\varphi(\sigma)$ satisfies the conditions $(2\text{–}1)_2$, $\varphi_\varepsilon(\sigma)$ satisfies the conditions

$$\varepsilon \leq \frac{\varphi_{\varepsilon j}(\sigma_j)}{\sigma_j} \leq k_j+\varepsilon \qquad (j = 1,\cdots,m)\,.$$

The new frequency characteristics $W_\varepsilon(i\omega)$ of the linear parts of the system (4–35), whose input is y_ε, output $-\sigma$, are expressed by means of the old one $W(i\omega)$ as follows

$$W_\varepsilon(i\omega) = \{E-\varepsilon W(i\omega)\}^{-1}W(i\omega)\,.$$

Let

$$N_\varepsilon(i\omega) = K_\varepsilon^{-1}+(E+i\omega Q)W_\varepsilon(i\omega)\,.$$

where

$$K = \begin{pmatrix} k_1+2\varepsilon & & 0 \\ & \ddots & \\ 0 & & k_m+2\varepsilon \end{pmatrix}.$$

Since all the elements of the matrices $W(i\omega)$ and $i\omega W(i\omega)$ are bounded functions of ω, there exists positive constants M_7 and M_8 such that

$$\|W(t\omega)\| \leq M_7, \quad \|i\omega W(i\omega)\| \leq M_8 \quad \text{for all} \quad \omega \geq 0. \tag{4-37}$$

As the difference between $W(i\omega)$ and $W_\varepsilon(i\omega)$ becomes

$$W_\varepsilon(i\omega) - W(i\omega) = \{(E - \varepsilon W(i\omega))^{-1} - E\} W(i\omega) = \varepsilon W(i\omega)^2 + 0(\varepsilon^2) \tag{4-38}$$

norm of the difference between $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ and $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is

$$\begin{aligned}
\left\|\frac{1}{2}(N_\varepsilon + N_\varepsilon^*) - \frac{1}{2}(N+N^*)\right\| &= \frac{1}{2}\|2(K_\varepsilon^{-1} - K^{-1}) \\
&\quad + (E+i\omega Q)(W_\varepsilon - W) + (W_\varepsilon - W)^*(E - i\omega Q)\| \\
&\simeq \frac{1}{2}\|4\varepsilon(K^{-1})^2 + \varepsilon W^2 + \varepsilon W^{*2} + \varepsilon Q(i\omega W)W + \varepsilon W^*(i\omega W)^* Q\| \\
&\leq \varepsilon(m\|W\|^2 + m^2\|Q\|\|W\|\|i\omega W\| + \|(K^{-1})^2\|) \\
&\leq \varepsilon(mM_7^2 + m^2 M_7 M_8\|Q\| + m\|K^{-1}\|^2).
\end{aligned}$$

In other words the difference between $\frac{1}{2}\{N_\varepsilon(i\omega)+N_\varepsilon(i\omega)^*\}$ and $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is of order ε. Thus, the condition

$$\frac{1}{2}\{N(i\omega)+N(i\omega)^*\} \geq \delta E$$

can be replaced by the inequality

$$\frac{1}{2}\{N_\varepsilon(i\omega)+N_\varepsilon(i\omega)^*\} \geq \delta_0 E \tag{4-39}$$

where δ_0 is a postive constant slightly different from δ. This inequality guarantees the asymptotical stability in the whole of the system (4-35) with the nonlinear characteristics $\varphi_{\varepsilon j}(\sigma_j)$ satisfying the condition $\varepsilon \leq \frac{\varphi_{\varepsilon j}(\sigma_j)}{\sigma_j} \leq k_j + 2\varepsilon$ hence so does for $\varepsilon \leq \frac{\varphi_{\varepsilon j}(\sigma_j)}{\sigma_j} \leq k_j + \varepsilon$. Consequently the equilibrium $x=0$ of the system (2-1) is asymptotically stable in the whole for any nonlinear characteristics $\varphi_j(\sigma_j)$ satisfying the conditions $(2\text{-}1)_2$. In other words the system (2-1) is absolutely stable in $[0, K]$.

So far we have assumed all the diagonal elements $q_1, \cdots, q_m$ of the diagonal matrix Q to be positive. Now let us show that the proof of Theorem 3 in the case, where $q_1, \cdots, q_r$ are negative and $q_{r+1}, \cdots, q_m$ positive, is reduced to the above proof. We represent the diagonal matrix Q as follows

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

where

$$Q_1 = \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_r \end{pmatrix}, \; q_1, \cdots, q_r < 0, \quad Q_2 = \begin{pmatrix} q_{r+1} & & 0 \\ & \ddots & \\ 0 & & q_m \end{pmatrix}, \; q_{r+1}, \cdots, q_m > 0.$$

The transformation of variable

$$y = \begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}\sigma - \begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix}\mathfrak{y} \tag{4-40}$$

reduces the system (2-1) to the system

$$\frac{dx}{dt} = \left\{A + B\begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}C'\right\}x - B\begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix}\mathfrak{y}, \tag{4-41}$$

$$\mathfrak{y} = \begin{pmatrix} \tilde{\varphi}_1(\sigma_1) \\ \vdots \\ \tilde{\varphi}_m(\sigma_m) \end{pmatrix}, \quad \sigma = C' \cdot x,$$

where

$$K_1 = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_m \end{pmatrix}$$

and E_1 is r-dimensional unite matrix and E_2 is $(m-r)$-dimensional unit matrix. Under the transformation (4-40) the conditions $(2\text{-}1)_2$ are preserved

$$0 \leq \frac{\tilde{\varphi}_j(\sigma_j)}{\sigma_j} \leq k_j, \quad \tilde{\varphi}_j(o) = 0, \qquad (j = 1, \cdots, m). \tag{4-41}$$

Because of necessity of the asymptotical stability of the linear system obtained by replacing $\varphi_1(\sigma_1) = k_1\sigma_1, \cdots, \varphi_r(\sigma_r) = k_r\sigma_r,\ \varphi_{r+1}(\sigma_{r+1}) = 0, \cdots, \varphi_m(\sigma_m) = 0$ in the system (2-1), the constant matrix $A + B\begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}C'$ is asymptotically stable. The frequency characteristic $\tilde{W}(i\omega)$ of the linear parts of the system (4-41), whose input is $\mathfrak{y}$ and output $-\sigma$, is given as follows

$$\begin{aligned} \tilde{W}(i\omega) &= \left(E + W(i\omega)\begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1} W(i\omega)\begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix} \\ &= \begin{pmatrix} -(E_1 + W_{11}K_1)^{-1}W_{11}, & (E_1 + W_{11}K_1)^{-1}W_{12} \\ W_{21}\{K_1(E_1 + W_{11}K_1)^{-1}W_{11} - E_1\}, & -W_{22} - W_{21}K_1(E_1 + W_{11}K_1)^{-1}W_{12} \end{pmatrix} \end{aligned} \tag{4-42}$$

where W_{jk} is partitioned matrix of $W(i\omega)$ as follows

$$W(i\omega) = \begin{pmatrix} W_{11}(i\omega), & W_{12}(i\omega) \\ W_{21}(i\omega), & W_{22}(i\omega) \end{pmatrix}.$$

For the system (4-41) we define the matrix

$$\tilde{N}(i\omega) = K^{-1} + (E + i\omega\tilde{Q})\tilde{W}(i\omega) \tag{4-43}$$

where

$$\tilde{Q} = \begin{pmatrix} -Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

All the diagonal elements of this diagonal matrix $\tilde{Q}$ are positive. In view of (4-1), (4-42) and (4-43) we obtain the relation

$$\tilde{N}(i\omega)+\tilde{N}(i\omega)^* = P(i\omega)^*\{N(i\omega)+N(i\omega)^*\}P(i\omega) \tag{4-44}$$

where

$$P(i\omega) = \begin{pmatrix} (E_1+K_1W_{11})^{-1}, & K_1(E_1+K_1W_{11})^{-1}W_{12} \\ 0, & -E_2 \end{pmatrix}.$$

The matrix $P(i\omega)$ is nonsingular for all $\omega\geq 0$.
Thus, when the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is positive definite, also the Hermitian matrix $\frac{1}{2}\{\tilde{N}_\varepsilon(i\omega)+\tilde{N}(i\omega)^*\}$ is positive definite. Since all the diagonal elements of the diagonal matrix $\tilde{Q}$ are positive, the system (4–41) is absolutely stable in $[O, K]$, provided that the Hermitian matrix $\frac{1}{2}\{\tilde{N}(i\omega)+\tilde{N}(i\omega)^*\}$ is positive definite. Consequently if the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is positive definite, the system (2–1) is absolutely stable in $[0, K]$.

Now let us consider the case in which the condition of Theorem 3

$$\frac{1}{2}\{N(i\omega)+N(i\omega)^*\} \geq \delta E \qquad \text{for all} \quad \omega\geq 0 \tag{4-45}$$

is satisfied for the diagonal matrix Q such that some of its diagonal elements are zero i.e.

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} q_{r+1} & & 0 \\ & \ddots & \\ 0 & & q_m \end{pmatrix}, \quad q_{r+1}, \cdots, q_m \neq 0. \tag{4-46}$$

In this case

$$N(i\omega) = \begin{pmatrix} K_1^{-1}+W_1, & W_{12} \\ (E_2+i\omega Q_2)W_{21}, & K_2^{-1}+(E_2+i\omega Q_2)W_{22} \end{pmatrix}.$$

Let ε to be sufficiently small number and let

$$Q_\varepsilon = \begin{pmatrix} \varepsilon E_1 & 0 \\ 0 & Q_2 \end{pmatrix} \tag{4-47}$$

where E_1 is $r\times r$ unit matrix, and $N_\varepsilon(i\omega)=K^{-1}+(E+i\omega Q_\varepsilon)W(i\omega)$, then we get

$$\frac{1}{2}\{N(i\omega)+N(i\omega)^*\} \geq \delta_\varepsilon E \qquad \text{for all } \omega\geq 0,$$

from (4–45) and (4–37), where δ_ε (>0) is slightly different from δ in (4–45).

Thus, the system (2–1) is absolutely stable in $[0, K]$. This completes the proof of the Theorem 3.

(ii) The case where some of $k_1, \cdots, k_m$ are infinite.

Let us consider the case in which $k_1, \cdots, k_l$ are infinite and $k_{l+1}, \cdots, k_m$ are finite positive numbers. Concerning the absolute stability in this case the following theorem holds.

Theorem 4: Let the constant matrix A to be asymptotically stable and $W(i\omega)$ to be the frequency characteristic of the linear parts of the system, whose input y and output $-\sigma$. Let us define the $m\times m$ matrix

$$N(i\omega)=\begin{pmatrix}0 & 0\\ 0 & K_1^{-2}\end{pmatrix}+(E+i\omega Q)W(i\omega), \tag{4-48}$$

where

$$K_2=\begin{pmatrix}k_{l+1} & & 0\\ & \ddots & \\ 0 & & k_m\end{pmatrix}.$$

The system (2-1) is absolutely stable in $[0, K]$, if there exists a real diagonal matrix $Q=\begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}$ such that the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is positive definite for all $\omega\geq 0$ and for $\omega\to\infty$, where $K=\begin{pmatrix}\infty & & & & 0\\ & \ddots & & & \\ & & \infty & & \\ & & & k_{l+1} & \\ & & & & \ddots \\ 0 & & & & & k_m\end{pmatrix}$. For the absolute stability of the system (2-1) in $[0, K]$ it is necessary that the linear system obtained by putting $y_j=h_j\sigma_j$ in (2-1) is asymptotically stable for any h_j such that $0\leq h_1,\cdots,h_l$ and $0\leq h_j\leq k_j$ $(j=l+1,\cdots,m)$. Differently from Theorem 3 all the diagonal elements of the diagonal matrix Q_1 must be non-negative.

Theorem 4 can be proved analogously to Theorem 3. A few differences between them are the followings. First, throughout the proof the matrix K^{-1} is replaced by the matrix $\begin{pmatrix}0 & 0\\ 0 & K_2^{-1}\end{pmatrix}$. Second, in the inequality (4-29) of the estimation of $\|x(t)\|$ the inequality $\|\varphi[\sigma(t)]\|\leq\|K\|\|\sigma(t)\|$ was used. But in our case where $K=\begin{pmatrix}\infty & & & & 0\\ & \ddots & & & \\ & & \infty & & \\ & & & k_{l+1} & \\ & & & & \ddots \\ 0 & & & & & k_m\end{pmatrix}$, this inequality is useless for the estimation. However, from the fact that the functions $\varphi_j(\sigma_j)$, $(j=1,\cdots,m)$ are defined for all σ_j and in view of (4-26) we obtain the estimation

$$\|\varphi[\sigma(t)]\|\leq M_9\|\sigma(t)\|\leq M_9M_4 \qquad \text{for all} \quad t>0$$

where M_9 is a finite positive constant depending on the fundtion $\varphi(\sigma)$. Thus, we can estimate $\|x(t)\|$ analogously as in (4-29). Finally, when $k_1=\cdots k_l=\infty$, it is imposible to transform the variable y such as (4-40) for $r\leq l$. Then, $q_1,\cdots,q_l$ can not take negative values.

(iii) Discussion.

In particular, the Popov's sufficient condition for the absolute stability of the system (2-1) with one nonlinear characteristic is such that

$$Re\,(1+i\omega q)W(i\omega)+\frac{1}{k}\geq\delta>0 \tag{4-49}$$

for all $\omega\geq 0$ and for some real q. Putting

$$Re\, W(i\omega) = X(\omega)\,, \quad \omega\, Im\, W(i\omega) = Y(\omega)$$

the inequality (4-49) is rewritten as follows

$$X(\omega) - qY(\omega) + \frac{1}{k} \geq \delta > 0$$

for all $\omega \geq 0$ and for some real q.

In other words, the system (2-2) with one nonlinear characteristic is absolutely stable in $[0, k]$, if we can take real q such that the vector locus of the modified frequency characteristics $\overline{W}(i\omega) = X(\omega) + iY(\omega)$ is laid strictly right from the line $X - qY + \frac{1}{k} = 0$. In this way we can give simple geometrical interpretation of the Popov's sufficient condition (4-49) for the absolute stability of the system (2-1) with one nonlinear characteristic. But it seems difficult to interpret geometrically the condition: $\frac{1}{2}\{N(i\omega) + N(i\omega)^*\} \geq \delta E$ for all $\omega \geq 0$, for the system (2-1) with m nonlinear characteristics.

5. Relation between the Problem of Existence of the Lyapunov Functions and the Sufficient Condition for the Absolute Stability in Terms of Frequency Characteristic.

In the paragraphs 3 and 4 we made use of the two different methods for the investigation of the absolute stability of the system (2-1). In this paragraph we shall discuss the relation between the results obtained by those methods.

For the system with one nonlinear characteristic it was proved that the necessary and sufficient condition for the existence of the Lyapunov function of the type "a quadratic form plus the integral of the nonlinear function" under the restriction of the S-process is the Popov's condition (4-49)[12].

For the system (2-1) with many nonlinear characteristics only the following proposition is proved. If there exist the Lyapunov functions of the type "a quadratic form plus integrals of the nonlinear function" under the restriction of the S-process and the $m \times m$ symmetric matrix H is positive definite, then there exists a diagonal matrix Q such that the Hermitian matrix $\frac{1}{2}\{N(i\omega) + N(i\omega)^*\}$ is positive definite for all $\omega \geq 0$ and for $\omega \to \infty$. The condition of the above proposition is equivalent to positive definiteness of the $(m+n) \times (m+n)$ symmetric matrix

$$\tilde{G} = \begin{pmatrix} G & -a' \\ -a & H \end{pmatrix} \tag{5-1}$$

where

$$\alpha = B'L+\frac{1}{2}\beta C'A+\frac{1}{2}C', \tag{5-2}$$

$$H = K^{-1}-\frac{1}{2}(\beta C'B+B'C\beta). \tag{5-3}$$

If we put $Q=\beta$, from the definition of the matrix H and from (4-4) clearly

$$H = \operatorname*{line}_{\omega\to\infty}\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}. \tag{5-4}$$

Let us define the matrix

$$A_\omega = A-i\omega E, \tag{5-5}$$

Since the matrix L is obtained by operating the Lyapunov's operatior $\mathcal{L}$ on the symmetric matrix G, the relation

$$A'L+LA = -G \tag{5-6}$$

takes place. In view of (5-5) and (5-6) we obtain the formula

$$A_\omega^*L+LA_\omega = -G. \tag{5-7}$$

Defining the matrix B_ω as follows

$$B_\omega = A_\omega^{-1}B = (A-i\omega E)^{-1}B \tag{5-8}$$

we obtain the relation

$$\begin{aligned} B_\omega^*NB_\omega &= -B_\omega^*(A_\omega^*L+LA_\omega)B_\omega \\ &= -\{(B'L)B_\omega+B_\omega^*(B'L)'\}. \end{aligned}$$

From (5-2)

$$B'L = \alpha-\frac{1}{2}\beta C'A-\frac{1}{2}C'.$$

Therefore

$$\begin{aligned} B_\omega^*GB_\omega &= -\left[\left(\alpha-\frac{1}{2}\beta C'A-\frac{1}{2}C'\right)B_\omega+B_\omega^*\left(\alpha-\frac{1}{2}\beta C'A-\frac{1}{2}C'\right)\right] \\ &= -(\alpha B_\omega+B_\omega^*\alpha')+\frac{1}{2}\{(\beta C'A)B_\omega+B_\omega^*(\beta C'A)'\} \\ &\quad +\frac{1}{2}(C'B_\omega+B_\omega^*C). \end{aligned}$$

On the other hand from the definition of the frequency characteristic $W(i\omega)$ of the linear parts of the system, whose input is y and output $-\sigma$, we obtain

$$C'B_\omega = C'(A-i\omega E)^{-1}B = W(i\omega). \tag{5-9}$$

As we put $Q=\beta$, from the definition of the matrix $N(i\omega)$, we obtain

$$\begin{aligned} B_\omega^*GB_\omega = &-(\alpha B_\omega+B_\omega^*\alpha)+\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}-K^{-1} \\ &+\{\beta C'AB_\omega+B_\omega^*(\beta C'A)'\}-\frac{1}{2}\{i\omega QW(i\omega)+W(i\omega)^*(i\omega Q)^*\}. \end{aligned}$$

In view of (5-7) and (5-9)

$$\frac{1}{2}\{\beta C'AB_\omega + B_\omega^*(\beta C'A)\} - \frac{1}{2}\{i\omega\beta W(i\omega) + (i\omega\beta W(i\omega))^*\}$$
$$= \frac{1}{2}(\beta C'B + B'C\beta).$$

Thus, we obtain

$$\frac{1}{2}\{N(i\omega) + N(i\omega)^*\} = B_\omega^*GB_\omega + (aB_\omega + B_\omega^*a) + H. \tag{5-10}$$

As the matrix $\tilde{G}$ is positive definite, for non-zero $(m+n)$-dimensional vector $\begin{pmatrix} B_\omega y \\ -y \end{pmatrix}$ we obtain the inequality

$$\begin{pmatrix} B_\omega y \\ -y \end{pmatrix}^* \tilde{G} \begin{pmatrix} B_\omega y \\ -y \end{pmatrix} = y^*\{B_\omega^*GB_\omega + (aB_\omega + B_\omega^*a') + H\}y > 0 \tag{5-11}$$

where m-dimensional vector y is non zero. From (5-10) and (5-11) the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ is positive definite for all $\omega \geq 0$. Thus, for the system (2-1) with m nonlinear characteristics, existence of the Lyapunov functions of the above type with the positive definite matrix H under the restriction of the S-process guarantees the positive definiteness of the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ for all $\omega \geq 0$ and for $\omega \to \infty$, where $Q=\beta$. This completes the proof of our proposition.

Now, the region of the absolute stability in the parameters space, which is obtained by Theorem 1, depends upon the choice of the positive definite matrix P and the real numbers $\beta_1, \cdots, \beta_m$. But the proof of our proposition in this paragraph is independent of the matrix P. Thus, the region of the absolute stability obtained by Theorem 1 for all possible P and for all possible $\beta_1, \cdots, \beta_m$ is contained by the region of the absolute stability obtained by Theorem 3.

6. Some Simpe Examples

Now let us consider a simple example. A simple loop system with two nonliner elements in series is shown in Fig. 3. If the gains of $W_1(s)$ and

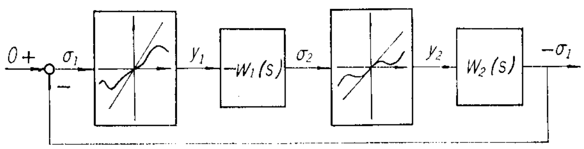

Fig. 3.

$W_2(s)$ are both positive, this system is a positive feedback system. The response of the system is represented by the following equations.

$$
\begin{aligned}
&\dot{x}_1 = A_1 x_1 + b_1 y_1, \qquad y_1 = \varphi_1(\sigma_1), \\
&\dot{x}_2 = A_2 x_2 + b_2 y_2, \qquad y_2 = \varphi_2(\sigma_2), \\
&\sigma_1 = c_2' \cdot x_2, \qquad \sigma_2 = c_1' \cdot x_1 \\
&W_j(s) = -c_j'(sE - A_j)^{-1} b_j \qquad (j = 1, 2)
\end{aligned}
\tag{6-1}
$$

where x_j is n_j-dimensional state vector of the element $W_j(s)$, b_j and c_j are n_j-dimensional constant vectors and A_j is $n_j \times n_j$ constant asymptotically stable matrix. Introducing the notations

$$
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_2 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
$$

$$
B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & c_2' \\ c_1' & 0 \end{pmatrix}, \quad W(i\omega) = \begin{pmatrix} 0 & W_2(i\omega) \\ W_1(i\omega) & 0 \end{pmatrix}
$$

the system (6–1) becomes a particular case of (2–1).

Applying Theorem 3 to this system, we get the sufficient condition for the absolute stability of the system as follows

$$
\frac{4}{k_1 k_2} > \{u_1(\omega) + u_2(\omega) - q_2 v_1(\omega) - q_1 v_2(\omega)\}^2 + \left\{\omega(q_2 u_1(\omega) - q_1 u_2(\omega)) + \frac{1}{\omega}(v_1(\omega) - v_2(\omega))\right\}^2, \tag{6-2}
$$

where

$$
\begin{aligned}
u_j(\omega) &= Re\, W_j(i\omega) \\
v_j(\omega) &= \omega\, Im\, W_j(i\omega) \qquad (j = 1, 2).
\end{aligned}
$$

It is interesting that k_1 and k_2 are contained only as a product of them. When $W_1(s) = \frac{1}{1+T_1 s}$, $W_2(s) = \frac{1}{1+T_2 s}$ (T_1, $T_2 > 0$) (6–2) becomes

$$
\frac{4}{k_1 k_2} > \left\{\frac{1+q_2 T_1 \omega^2}{1+T_1^2\omega^2} + \frac{1+q_1 T_2 \omega^2}{1+T_2^2\omega^2}\right\}^2 + \left\{\frac{(q_2 - T_1)\omega}{1+T_1^2\omega^2} - \frac{(q_1 - T_2)\omega}{1+T_2^2\omega^2}\right\}^2. \tag{6-3}
$$

It is easily seen that

$$
\operatorname*{Max}_{\omega \geq 0} \frac{1+q_j T_k \omega^2}{1+T_k^2\omega^2} = \begin{cases} \dfrac{q_j}{T_k} & \text{when} \quad q_j > T_k \\ 1 & \text{when} \quad q_j \leq T_k. \end{cases}
$$

Therefore, if q_1 and q_2 are taken as $q_1 = T_2$, $q_2 = T_1$, the maximum value of the right-hand side of (6–3) with ω takes the minimum value. Thus, the sufficient condition for the absolute stability becomes

$$
k_1 k_2 < 1. \tag{6-4}
$$

Now, let us replace the nolinear elements by linear elements $\varphi_j(\sigma_j) = h_j \sigma_j$, $0 \leq h_j \leq k_j$, $(j = 1, 2)$. Then, the characteristic equation for this linearized system is

$$T_1T_2\lambda^2+(T_1+T_2)\lambda+(1-h_1h_2) = 0\,.$$

In order that the linearized system is asymptotically stable for all h_1 and h_2 such that $0\leq h_j\leq k_j$ ($j=1, 2$), it is necessary and sufficient that $k_1k_2<1$. This condition coincides with (6–4) exactly. This means that Theorem 3 gave not only the sufficient but also the necessary condition for the absolute stability for this case.

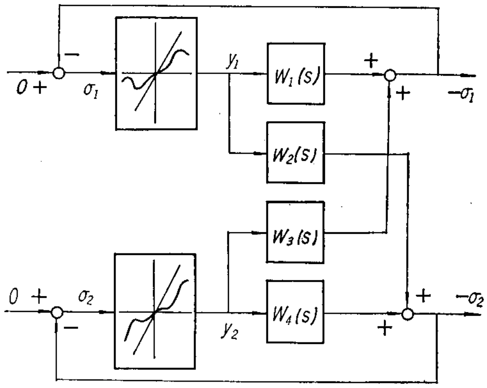

Fig. 4.

Next, let us consider an example of a multi-variable control system shown in Fig. 4. The response of the system is given by

$$\begin{aligned}
\dot{x}_1 &= A_1x_1+b_1y_1\,, \\
\dot{x}_2 &= A_2x_2+b_2y_1\,, \qquad y_1 = \varphi_1(\sigma_1)\,, \\
\dot{x}_3 &= A_3x_3+b_3y_2\,, \\
\dot{x}_4 &= A_4x_4+b_4y_2\,, \qquad y_2 = \varphi_2(\sigma_2)\,, \\
\sigma_1 &= c_1'x_1+c_3'x_3\,, \qquad \sigma_2 = c_2'x_2+c_4'x_4\,, \\
W_j(s) &= -C_j'(sE-A)^{-1}b_j\,, \qquad (j = 1, \cdots, 4)\,.
\end{aligned} \tag{6–5}$$

Introducing the notations

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1, & 0 \\ b_2, & 0 \\ 0, & b_3 \\ 0, & b_4 \end{pmatrix}, \quad C = \begin{pmatrix} c_1', & 0, & c_3', & 0 \\ 0, & c_2', & 0, & c_4' \end{pmatrix},$$

the system (6–5) becomes a particular case of (2–1). The positive definiteness of the Hermitian matrix $\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ of Theorme 3 is written as follows

$$\begin{aligned}
&u_1(\omega)-q_1v_1(\omega)+\frac{1}{k_1} > 0\,, \\
&\left\{u_1(\omega)-q_1v_1(\omega)+\frac{1}{k_1}\right\}\left\{u_4(\omega)-q_2v_4(\omega)+\frac{1}{k_2}\right\} \\
&\quad > \{u_2(\omega)+u_3(\omega)-q_1v_3(\omega)-q_2v_2(\omega)\}^2 \\
&\quad +\left\{\omega(q_1u_3(\omega)-q_2u_2(\omega))+\frac{1}{\omega}(v_3(\omega)-v_2(\omega))\right\}^2,
\end{aligned} \tag{6–6}$$

where

$$u_j(\omega) = Re\,W_j(i\omega)\,, \quad v_j(\omega) = \omega\, Im\, W_j(i\omega)\,.$$

Now, let us consider the case where

$$W_1(s)=\frac{\kappa_1}{1+Ts},\quad W_2(s)=W_3(s)=\frac{1}{1+Ts},$$

$$W_4(s)=\frac{\kappa_4}{1+Ts}\qquad (\kappa_1,\kappa_4,T>0).$$

Then, the first inequality of (6–6) always holds. The last inequality

$$4\kappa_1\kappa_4\left\{\left(1+\frac{1}{k_1\kappa_1}\right)+T\left(q_1+\frac{T}{k_1\kappa_1}\right)\omega^2\right\}\left\{\left(1+\frac{1}{k_2\kappa_4}\right)+T\left(q_2+\frac{T}{k_2\kappa_4}\right)\omega^2\right\}$$
$$>\{2+T(q_1+q_2)\omega^2\}^2+(q_1-q_2)^2\omega^2. \tag{6–7}$$

If $k_1=\infty$ and $k_2=\infty$ (6–7) yields

$$4(\kappa_1\kappa_4-1)(1+q_1T\omega^2)(1+q_2T\omega^2)-(q_1-q_2)^2(\omega^2+T^2\omega^4)>0. \tag{6–8}$$

When

$$\kappa_1\kappa_4>1, \tag{6–9}$$

(6–8) holds for all $\omega\geq0$, if we select q_1 and q_2 as $q_1=q_2$. When $\kappa_1\kappa_4\leq1$, there does not exist real positive q_1 and q_2 which make (6–8) to be true for all $\omega\geq0$. Therefore, the system is absolutely stable in $\left[0,\begin{pmatrix}\infty & 0\\ 0 & \infty\end{pmatrix}\right]$, if (6–9) holds.

If both k_1 and k_2 are not infinite, (6–7) is written down as follows

$$a_0\omega^4+a_1\omega^2+a_2>0, \tag{6–10}$$

where

$$a_0=4\kappa_1\kappa_4T^2\left(q_1+\frac{T}{k_1\kappa_1}\right)\left(q_2+\frac{T}{k_2\kappa_4}\right)-T^2(q_1+q_2)^2,$$

$$a_1=4\kappa_1\kappa_4T\left\{\left(1+\frac{1}{k_1\kappa_1}\right)\left(q_2+\frac{T}{k_2\kappa_4}\right)+\left(1+\frac{1}{k_2\kappa_4}\right)\left(q_1+\frac{T}{k_1\kappa_1}\right)\right.$$
$$-4T(q_1+q_2)-(q_1-q_2)^2,$$

$$a_2=4\kappa_1\kappa_4\left(1+\frac{1}{k_1\kappa_1}\right)\left(1+\frac{1}{k_2\kappa_4}\right)-4.$$

Thus, when

$$\kappa_1\kappa_4\left(1+\frac{1}{k_1\kappa_1}\right)\left(1+\frac{1}{k_2\kappa_4}\right)>1, \tag{6–11}$$

the inequality (6–10) holds for all $\omega\geq0$, if we select q_1 and q_2 small. That is, the system is absolutely stable in $\left[0,\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}\right]$, if the κ_1 and κ_4 satisfy (5–12).

Now, let us replace the nonlinear elements in Fig. 4 by the linear elements as follows

$$\varphi_1(\sigma_1)=h_1\sigma_1\quad\text{and}\quad\varphi_2(\sigma_2)=h_2\sigma_2.$$

The characteristic equation for this linearized system is

$$T^2\lambda^2+(2+h_1\kappa_1+h_2\kappa_4)Ts+\{1+h_1\kappa_1+h_2\kappa_4+h_1h_2(\kappa_1\kappa_4-1)\}=0.$$

Therefore, in order that the system is asymptotically stable for all h_1 and h_2 such $0 \leq h_1$ and $0 \leq h_2$, it is necessary and sufficient that

$$\kappa_1 \kappa_4 > 1 . \qquad (6\text{-}12)$$

This coincides with (6-9) exactly.

If the system should be assymptotically stable for all h_1 and h_2 such that

$$0 \leq h_1 \leq k_1 < \infty \quad \text{and} \quad 0 \leq h_2 \leq k_2 < \infty ,$$

it is necessary and sufficient that

$$1 + k_1\kappa_1 + k_2\kappa_4 + k_1k_2(\kappa_1\kappa_4 - 1) > 0 .$$

Again, this coincides with (6-11) exactly.

7. Conclusions

In the previous paragraphs the problem of the absolute stability of the control system with many nonilinear characteristics have been discussed.

In the paragraph 3 by means of the Lyapunov's direct method a system of the quadratic equations was obtained, which is the extension of the case where the control system contains only one nonlinear characteristic, and it was shown that if the system of quadratic equations has real solutions, then the system (2-1) is absolutely stable. In the paper[14] the same system of quadratic equations was obtained and the real numbers $\beta_1, \cdots, \beta_m$ were all positive. But as shown in the paragraph 3 of this paper, it is not necessary that all the $\beta_1, \cdots, \beta_m$ be positive.

In the paragraph 4 analogously to the Popov's criteria of the absolute stability of the control system with one nonlinear characteristic, a sufficient condition for the sbsolute stability of the control system (2-1) with m nonlinear characteristics in terms of the frequency characteristic was obtained.

For the system with one nonlinear characteristic, in the case where the scalar quantity corresponding to $\lim_{\omega \to \infty} \frac{1}{2}\{N(i\omega) + N(i\omega)^*\}$ is non-negative, the system is absolutely stable if the Popov's condition (4-49) is satisfied. But for the system with many nonlinear characteristics in Theorem 3 we assumed that the $m \times m$ symmetric matrix $\lim_{\omega \to \infty} \frac{1}{2}\{N(i\omega) + N(i\omega)^*\}$ was positive definite.

In the paragraph 5 relations between the problem of existence of Lyapunov functions and the sufficient condition for the absolute stability in terms of the frequency characteristic are discussed. For the control system with one nonlinear characteristic existence of the Lyapunov function of the type "a quadratic form plus integral of nonlinear function", under the restriction of the S-process is equivalent to the Popov's sufficient condition for the absolute stability. Buf for the system with m nonlinear characteristics it is shown only that the former is a sufficient condition of the later.

From begining to end the constant matrix A has been assumed to be asymptotically stable. It seems interesting to discuss the case where some of characteristic roots of the matrix A are on the imaginary axis of complex plane and the other are laid in the left half plane.

Appendix I (Lyapunov's theorem)

Let us consider the differential equation

$$\frac{dx}{dt} = Ax \tag{I-1}$$

and the quadratic form $V(x)=x'Lx$. The derivative $\frac{dV}{dt}$ along the trajectories of the equation is

$$\frac{dV}{dt} = x'(A'L+LA)x.$$

Setting

$$A'L+LA = -G \tag{I-2}$$

we see that

$$G' = -(A'L+LA)' = -(LA+A'L) = G,$$

so that G is tha matrix of a quadratic form.

If the real matrix A is asymptotically stable, then for positive definite matrix G the unique solution L of (I-2) exists and is positive definite.

Conversely, if for every positive definite matrix G there exists a positive definite matrix L, then the real matrix A is asymptotically stable.

This theorem is proved in the reference[15].

Appendix II

Let the real vector functions

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_m(t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_m(t) \end{pmatrix}, \quad k(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix} \tag{II-1}$$

satisfy the conditions

$$\begin{aligned} \|f(t)\| &\le \alpha_1 \exp(-\beta_1 t), \\ \|g(t)\| &\le \alpha_2 \exp(-\beta_2 t), \\ \|k(t)\| &\le \alpha_3 \exp(-\beta_3 t), \end{aligned} \tag{II-2}$$

where α_j, β_j $(j=1, 2, 3)$ are positive constants.

The Fourier transforms of these functions exist

$$F(i\omega)=\begin{pmatrix}F_1(i\omega)\\ \vdots\\ F_m(i\omega)\end{pmatrix},\quad G(i\omega)=\begin{pmatrix}G_1(i\omega)\\ \vdots\\ G_m(i\omega)\end{pmatrix},\quad K(i\omega)=\begin{pmatrix}K_1(i\omega)\\ \vdots\\ K_m(i\omega)\end{pmatrix},\tag{II-3}$$

where

$$F_j(i\omega)=\int_0^\infty f_j(t)e^{-i\omega t}dt\,,\quad G_j(i\omega)=\int_0^\infty g_j(t)e^{-i\omega t}dt\,,\quad K_j(i\omega)=\int_0^\infty k_j(t)e^{-i\omega t}dt\,.$$

Let us assume the existence of $N(i\omega)$ satisfying the relation

$$F(i\omega)=N(i\omega)K(i\omega)+G(i\omega)\tag{II-4}$$

and also assume that for the Hermitian matrix $L(i\omega)=\frac{1}{2}\{N(i\omega)+N(i\omega)^*\}$ the inequality

$$L(i\omega)\geq\delta E\tag{II-5}$$

holds for all $\omega\geq 0$, where δ is a positive constant independent of ω.

Then the inequality

$$-\int_0^\infty f(t)'\cdot k(t)dt\leq C\tag{II-6}$$

takes place, where C is a constant such that

$$C=\frac{1}{8\pi\delta}\int_{-\infty}^\infty G(i\omega)^*G(i\omega)d\omega\,.\tag{II-7}$$

Proof of this proposition.

By the Parseval formula the following equality takes place

$$\int_0^\infty f(t)'k(t)dt=\frac{1}{2\pi}\int_{-\infty}^\infty F(i\omega)^*F(i\omega)d\omega\tag{II-8}$$

Substituting (II-4) into (II-8) we obtain

$$\int_0^\infty f(t)'k(t)dt=\frac{1}{2\pi}\int_{-\infty}^\infty\{N(i\omega)K(i\omega)+G(i\omega)\}^*K(i\omega)d\omega\,.$$

Since the left-hand side of this equality takes real value, so does the right-hand side. Thus,

$$\begin{aligned}\int_0^\infty f'(t)k(t)dt&=\frac{1}{2\pi}\int_{-\infty}^\infty K(i\omega)^*L(i\omega)d\omega+\frac{1}{4\pi}\int_{-\infty}^\infty\{G(i\omega)^*K(i\omega)+K(i\omega)^*G(i\omega)\}\,d\omega\\&=\frac{1}{2\pi}\int_{-\infty}^\infty\Big[(K(i\omega)^*L(i\omega)^{1/2})(L(i\omega)^{1/2}K(i\omega))+\frac{1}{2}\{G(i\omega)^*K(i\omega)+K(i\omega)^*G(i\omega)\}\,d\omega\Big]\\&=\frac{1}{2\pi}\int_{-\infty}^\infty\Big\{L(i\omega)^{1/2}K(i\omega)+\frac{1}{2}L(i\omega)^{-1/2*}G(i\omega)\Big\}^*\Big\{L(i\omega)^{1/2}K(i\omega)+\frac{1}{2}L(i\omega)^{-1/2*}G(i\omega)\Big\}d\omega\\&\quad-\frac{1}{8\pi}\int_{-\infty}^\infty G(i\omega)^*L(i\omega)^{-1}G(i\omega)d\omega\\&\geqq-\frac{1}{8\pi}\int_{-\infty}^\infty G(i\omega)^*L(i\omega)^{-1}G(i\omega)d\omega\,.\end{aligned}$$

From the inequality (II-5) we obtain

$$\frac{1}{\delta}E \geq L(i\omega)^{-1}.$$

Thus,

$$-\frac{1}{8\pi}\int_{-\infty}^{\infty} G(i\omega)^* L(i\omega)^{-1} G(i\omega)d\omega \geq -\frac{1}{8\pi}\int_{-\infty}^{\infty} G(i\omega)^* \frac{E}{\delta} G(i\omega)d\omega.$$

Consequently

$$\int_0^{\infty} f(t)'k(t)dt \geq -\frac{1}{8\pi\delta}\int_{-\infty}^{\infty} G(i\omega)^* G(i\omega)d\omega.$$

Appendix III

If 1) a continuous function $f(t)$ and its derivative $\frac{df(t)}{dt}$ are bounded for $t \geq 0$, 2) a continuous function $G(x) \geq 0$ for any $x \neq 0$, $G(o)=0$ and 3) $\int_0^{\infty} G[f(t)]dt < \infty$, then $\lim_{t\to\infty} f(t)=0$.

This proposition is proved in the reference[13].

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