Seepage from a Canal into Ground with a Shallow Water Depth

By

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The groundwater surface at the seepage from a canal into the deep groundwater table is proposed by Kozeny, but the problem with respect to the shallow groundwater surface is not presented.

This paper explicitly proposes equations of the groundwater surface caused by the seepage from a canal built on the ground with symmetrically disposed collectors at some depth under the earth surface and also the seepage from a canal constructed near the lake. The equations are, however, based on some assumptions, one of which is that there exists no water depth in the canal. The experimental examination is then carried out about the shape of groundwater surface, the rate of recharge and so on. As a result, considerable amount of additional rise of the water table, almost equal to 20~30 per cent of the height calculated from the analytically proposed equation, must be taken into account in the calculation of the actual groundwater level.

1. Introduction

When the lake works as a reservoir to prevent flood disaster, the level of the lake cannot be controlled at an extremely high or low level, so that the usual groundwater level near the lake rises or falls to some degree. A proposal can be made that the level of lake is made to descend before the flood comes and the groundwater table near the lake is made to rise by recharging water into a canal constructed near the lake. In this study the groundwater table at the seepage from canal is sought and additional experimental considerations are carried out to establish a method of calculating the groundwater surface.

2. Analysis of the Seepage from a Canal

Seepage water from a canal usually flows down into the groundwater table at depth T_0 below the earth surface. It is of interest how the groundwater surface at the seepage from the canal changes as the decrease in T_0,

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i.e., as the rise of the groundwater level. In Fig. 1 water percolates from canal and spreads along the horizontal surface AH, as if the line of AH were the asymptote of the free surface. In such a case the steady flow may not be accomplished because the groundwater table varies gradually as time passes due to no boundary condition. So we consider the flow domain shown in Fig. 1.

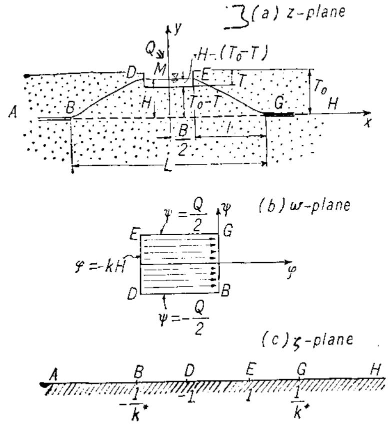

Fig. 1. (a) z-plane, (b) ω-plane and (c) ζ-half plane of the seepage domain from a canal to symmetrically disposed collectors.

The canal has a single form of horizontal segment and the draining operators are horizontal segments AB and GH. When we designate the height of the water depth in the canal from the groundwater level by H and the depth of canal by T, the depth of water in canal is equal to $H-(T_0-T)$. Now, let's consider the case of no water depth in canal, *i.e.*, the case that the rate of recharge into canal balances to the seepage rate of discharge into the ground.

The flow region is ADEH whose upper boundary line DE is an equipotential line ; $\varphi=-kH$ where k is the coefficient of permeability of the ground. Seepage water percolates from the plane DE into the space A~H. The lines EG and DB are free surfaces and yet stream lines, which means that such stream-function along each line takes the conditions, respectively ;

$$\psi = Q/2, \qquad \psi = -Q/2$$

If we take these conditions into account and transform the flow region of z-plane into the complex potential plane, ω-plane ($\omega=\varphi+i\psi$) in a rectangle, the flow in a rectangular ω-plane is parallel one. Transforming the ω-plane into ζ-plane by Schwarz-Christoffel formula, we obtain the relation

$$\omega = \frac{Qi}{2K}\int_0^{\zeta}\frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^{*2}\zeta^2)}}-kH \tag{1}$$

where K is the complete elliptical integral of the first kind with modulus k^*.

To deduce the relation between the rate of discharge and the properties of the ground, we use the condition at the point G ;

$$\varphi = 0, \qquad \psi = \frac{Q}{2}, \qquad \zeta = \frac{1}{k^*}$$

in Eq. (1) and then obtain the relation

$$Q = \frac{2kHK}{K'} \tag{2}$$

in which K' denotes the complete elliptical integral of the first kind with modulus $k^{*\prime}$, where $k^{*\prime}=\sqrt{1-k^{*2}}$.

If the Zhoukovsky's function

$$\left.\begin{array}{l} \Theta = \Theta_1+i\Theta_2 = \omega-ikz\,, \quad z = x+iy \\ \Theta_1 = \varphi+ky\,, \quad \Theta_2 = \psi-kx \end{array}\right\} \tag{3}$$

is introduced in order to seek for the correlation between ω-plane and z-plane, the expression

$$\Theta = a\zeta+b = \omega-ikz \tag{4}$$

holds between Θ-plane and ζ-plane, where a and b are complex constants. If we use the conditions at the point M;

$$\zeta = 0\,, \quad z = iH\,, \quad \omega+kH = 0$$

into Eq. (4), we can show that $b=0$. If we also use the conditions at the point E;

$$\zeta = 1, \quad z = \frac{B}{2}+iH\,, \quad \omega = -kH+i\cdot\frac{Q}{2}$$

into Eq. (4), designating the width of the canal by B, we obtain

$$a = \frac{1}{2}(Q-kB)i$$

and therefore

$$\omega-ikz = \frac{i}{2}(Q-kB)\zeta\,. \tag{5}$$

If the length between B and G is designated by L and the conditions at G;

$$\omega = i\cdot\frac{Q}{2}\,, \quad z = \frac{L}{2}\,, \quad \zeta = \frac{1}{k^*}$$

are substituted into Eq. (5), we have

$$Q = \frac{k}{1-k^*}\cdot(B-k^*\cdot L) \tag{6}$$

By combining Eq. (6) with Eq. (2), we obtain the equation

$$\frac{Q(1-k^*)}{kH} = 2(1-k^*)\cdot\frac{K}{K'} = \frac{B}{H}-k^*\cdot\frac{L}{H} \tag{7}$$

If the length of EG or BD is taken as $L=2l+B$, we obtain

$$Q-kB = -\frac{2kk^*\cdot l}{1-k^*} \tag{8}$$

and we can rewrite Eq. (5) in the form;

$$\zeta = \frac{1-k^*}{k^*}\left(\frac{z}{l}+\frac{\omega}{kl}\cdot i\right) \tag{9}$$

On the other hand, Eq. (1) can be transformed into

$$F(\arcsin \zeta,\, k^*) = -K'\left(\frac{\omega}{kH}+1\right)\cdot i \tag{10}$$

in which

$$F(\arcsin\zeta,\, k^*) = \int_0^\zeta \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^{*2}\zeta^2)}}.$$

Using the inverse function of the elliptical integral of the first kind, namely, Jacobi's elliptical integral sn into Eq. (10), we obtain

$$\zeta = -\mathrm{sn}\left\{K'\left(\frac{\omega}{kH}+1\right)i,\, k^*\right\} = \frac{-1}{k^*\,\mathrm{sn}\left(\frac{K'\omega}{kH}i,\, k^*\right)} \tag{11}$$

By equalizing Eq. (9) to Eq. (11), we find the solution

$$\frac{z}{l} = -\frac{\omega i}{kl}-\frac{1}{1-k^*}\cdot\frac{1}{\mathrm{sn}\left(\frac{K'\omega}{kH}i,\, k^*\right)} \tag{12}$$

which expresses the equation of the flow net and the relation between z-plane as the real flow domain and ω-plane as the complex potential domain. Taking the conditions of the free surface into account;

$$\varphi+ky = 0, \quad \psi = \frac{Q}{2} = \frac{kHK}{K'},$$

we obtain the equation of free surface

$$\frac{z}{l} = \frac{KH}{K'l}+\frac{y}{l}i+\frac{\mathrm{dn}\left(\frac{K'y}{H},\, k^{*\prime}\right)}{1-k^*} \tag{13}$$

or

$$\frac{x}{l} = \frac{KH}{K'l}+\frac{\mathrm{dn}\left(\frac{K'y}{H},\, k^{*\prime}\right)}{1-k^*} \tag{14}$$

in which dn-function is defined as

$$k^{*2}\,\mathrm{sn}^2(u,\, k^*)+\mathrm{dn}^2(u,\, k^*) = 1\,.$$

When Q is substituted for H in Eq. (14), we obtain

$$\frac{x}{l} = \frac{Q}{2kl}+\frac{1}{1-k^*}\cdot\mathrm{dn}\left(\frac{2kKy}{Q},\, k^*\right). \tag{15}$$

The results obtained are summarized as follows.

The rate of recharge is in proportion to the coefficient of permeability of the ground, the head loss of the groundwater level from the water level in canal and the ratio K/K' determined by the shape of ground. The shape of the free groundwater surface is expressed as the Jacobi's elliptical function by Eq. (14). Under the condition that the depth of water in the canal is zero (naught), the shape of free surface is calculated only by three dimensions, *i.e.*, the width of the canal B, the head loss H of the groundwater surface from the water level in canal to the original groundwater level and the length L from the canal to the hydraulic boundary.

For example, the computed free surface is drawn in the case that $H=3$ m,

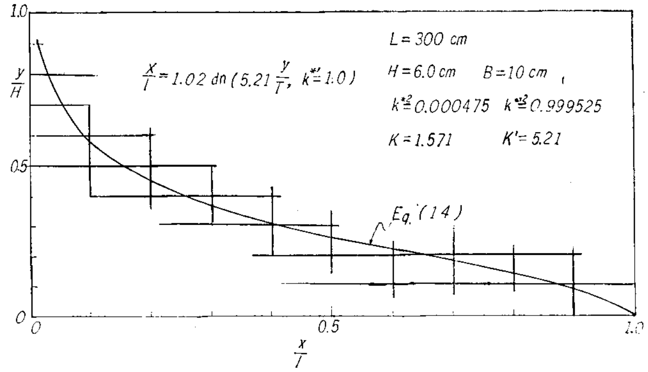

Fig. 2. Groundwater surface at the seepage from canal to symmetrically disposed collectors.

$B=5$ m and $L=150$ m in Fig. 2. According to the trial computation, the shape of free surface changes with a slight variation of the value of B or L.

The above considerations treat of the symmetrical flow pattern. The actual flow phenomena are often non-symmetrical like the seepage from canal constructed along the lake or sea. The model of such a case is shown in Fig. 3. The groudwater level controlled at one side is kept constant by the filter segment and that of the other side spreads so as to be constant at infinity. It is

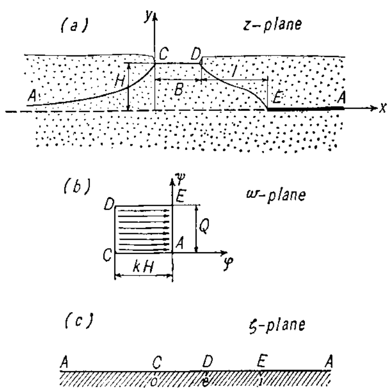

Fig. 3. (a) z-plane, (b) ω-plane and (c) ζ-half plane of the seepage domain from a canal to collector on one side.

also assumed in this case that the coefficient of permeability does not vary throughout the ground and the depth of the canal water is zero. In such a case, using the conditions;

$$\text{CD}\,;\ \varphi=-kH,\quad \text{CA}\,;\ \varphi+ky=0,\quad \psi=0$$
$$\text{DE}\,;\ \varphi+ky=0,\quad \psi=Q,$$

the real flow region (ω-plane) is transformed into the ζ-half plane by

$$\omega=\frac{Qi}{K}\int_0^t\frac{dt}{\sqrt{(1-t^2)(1-k^{*2}t^2)}}-kH \tag{16}$$

in which $\zeta=et^2$. If the condition at E is used in Eq. (16) ($\zeta=1=k^{*2}t^2$), we obtain

$$Q=\frac{kHK}{K'} \tag{17}$$

We also introduce the Zhoukovsky's function

$$\Theta=a\zeta=ak^{*2}t^2=\omega-ikz \tag{18}$$

and consider the condition at D;

$$z=B+iH,\quad \zeta=e,\quad \varphi=-kH,\quad \psi=Q$$

and then we find the relationship

$$\omega-ikz=\frac{Q-kB}{k^{*2}}\cdot\zeta i=(Q-kB)t^2i \tag{19}$$

and therefore

$$Q=kB-k\cdot\frac{k^{*2}}{k^{*\prime 2}}\cdot l. \tag{20}$$

From Eq. (17) and Eq. (20), we deduce the equation

$$\frac{Q}{kH}=\frac{B}{H}-\frac{k^{*2}}{k^{*\prime 2}}\cdot\frac{l}{H}=\frac{K}{K'} \tag{21}$$

which is regarded as the requirements to decide the modulus k^*.

Combining the equation

$$t=\text{sn}\left\{\frac{K}{Qi}(\omega+kH),k^*\right\} \tag{22}$$

obtained from Eq. (16) with Eq. (19), we have

$$\omega-ikz=ak^{*2}t^2=(Q-kB)i\cdot\text{sn}^2\left\{\frac{K}{Qi}(\omega+kH),k^*\right\}$$

or

$$z=-\frac{\omega}{k}i+\frac{k^{*2}}{k^{*\prime 2}}l\cdot\text{sn}^2\left\{K'i\left(\frac{\omega}{kH}+1\right),k^*\right\}. \tag{23}$$

This is the correlation between z-plane and ω-plane.

Again, using the relation

$$\operatorname{sn}\left(K'\frac{\omega}{kH}i+K'i,\,k^*\right)=\frac{1}{k^*\operatorname{sn}\left\{-\frac{K'\psi}{kH}+\frac{K'\varphi}{kH}i,\,k^*\right\}}$$

and the conditions along the free surface AC; $\varphi=-ky$, $\psi=0$, we obtain

$$\operatorname{sn}\left\{\frac{K'\omega}{kH}i+K'i,\,k^*\right\}=\frac{-i}{k^*}\cdot\frac{\operatorname{cn}\left(\frac{K'y}{H},\,k^{*\prime}\right)}{\operatorname{sn}\left(\frac{K'y}{H},\,k^{*\prime}\right)}. \tag{24}$$

We also obtain along the free surface DE

$$\operatorname{sn}\left\{\frac{K'\omega}{kH}i+K'i,\,k^*\right\}=-\frac{1}{k^*}\cdot\operatorname{dn}\left\{\frac{K'y}{H},\,k^{*\prime}\right\} \tag{25}$$

If we put Eq. (24) into Eq. (23), we have

$$z+\frac{\omega i}{k}=-\frac{l}{k^{*\prime 2}}\cdot\left\{\frac{1}{\operatorname{sn}^2\left(\frac{K'y}{H},\,k^{*\prime}\right)}-1\right\} \tag{26}$$

and from the real part of Eq. (26)

$$x=-\frac{l}{k^{*\prime 2}}\cdot\left\{\frac{1}{\operatorname{sn}^2\left(\frac{K'y}{H},\,k^{*\prime}\right)}-1\right\} \tag{27}$$

In the same way we find the following relation along DE

$$z+\frac{\omega}{k}i=\frac{l}{k^{*\prime 2}}\cdot\operatorname{dn}^2\left\{\frac{K'y}{H},\,k^{*\prime}\right\} \tag{28}$$

or

$$x=\frac{K}{K'}H+\frac{l}{k^{*\prime 2}}\cdot\operatorname{dn}^2\left\{\frac{K'y}{H},\,k^{*\prime}\right\} \tag{29}$$

Some calculated results of Eqs. (27) and (29) are shown in Fig. 4 in case of $B=3$ m, $H=3$ m and $l=200$ m (or 300 m, 600 m). Judging from this figure, the rise of groundwater table is about 1 m at the distance 100 m from the canal, about 50 cm. at 500 m, and about 30 cm. at 1000 m. Therefore, we cannot expect

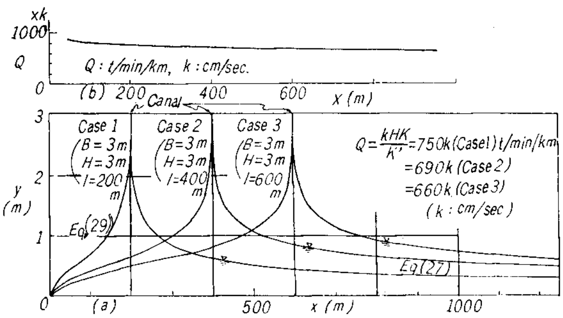

Fig. 4. An example of the groundwater surface at the seepage from a canal to collector on one side calculated.

the extreme rise of groundwater table by the seepage from canal. The rate of recharge into canal is calculated as $Q=3.5$ t/min/km taking $k=5.0\times10^{-3}$ cm/sec. In Fig. 4(b) it is clear that the rate of recharge does not decrease so much even if the distance x between canal and the lakeside increases.

3. Experimental Results

3.1 Experimental apparatus and procedure

Rectangular sand mass (23 cm. × 45 cm. × 315 cm.) is formed in a steel water channel which has a dimension of about 400 cm. in length, 50 cm. in height and 25 cm. in width. The front face of the channel has a glass window so that the seepage flow can be observed. The equipment by which the water level as a boundary head is controlled is two drainage pipes which can be made to rise and drop. The variation in water head due to the seepage flow in the sand mass is measured by the vynyl pipe lead to the observation points and the water level in canal by the point gauge. The grain-size curve of uniform sand in the model is shown in Fig. 5. The uniformity coefficient of the sand 1.78, the dry density 1.56 g/cm³, the void ratio of the sand 0.69, the coefficient of permeability $k=2.38\times10^{-2}$ cm/sec, and the capillary rise in sand $h_c\fallingdotseq20$ cm. are measured.

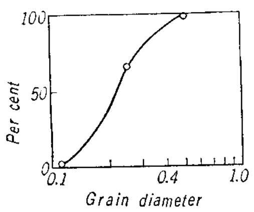

Fig. 5. Grain-size curve.

The canal is made on the sand model at the center (Position I), at the intermediate point (Position II) and at the edge (Position III), respectively, as shown in Fig. 6. The shape of the canal section is adopted by five kinds as indicated in Fig. 7. In the actual ground, there exist many strata and in general, there is often observed a permeable layer under the top stratum which

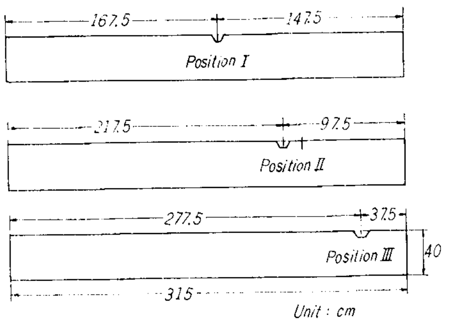

Fig. 6. Canal position.

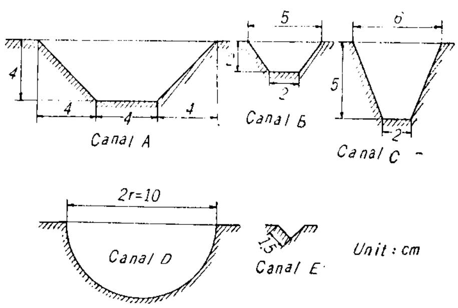

Fig. 7. Shape of canal section.

has rather low permeability. Therefore, we consider a two-layered system of the stratified ground and compare it with homogeneous one. The top layer used in this study consists of the silty sand with the coefficient of permeability $k=3.30\times10^{-4}$ cm/sec and with 5 cm. in thickness, the lower one being the above-mentioned sand with 40 cm. in thickness.

3.2 Experimental results

First test results in the case of the seepage from a canal into the homogeneous ground are stated. Fig. 8 shows the correlation between the

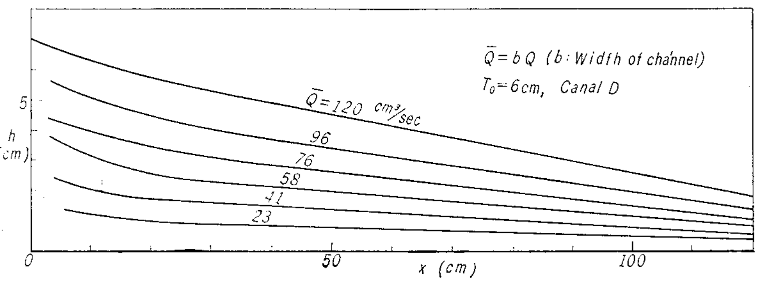

Fig. 8. Correlation between h and x as a parameter $\bar{Q}$ in experiment. ($\bar{Q}=bQ$, b; width of a channel)

rise of water table h and the distance from the canal x when the rate of recharge Q is poured in the canal. It is clear that as the recharge Q increases the rise of water table h also becomes greater, which is more clearly recognized in Fig. 9, and the correlation between h and Q is found to be linear one. The above fact leads to the thought that the correlation between h/Q and x can be expressed by one curve, which is proved to be valid for the different depth T_0 as shown in Fig. 10(a). Some results in the case where the original groundwater table has a slight slope are also shown in Fig. 10(b), taking x as the abscissa and h/Q as the ordinate, where h is the rise from the original water level at each point. Such a little gradient of the original water table has no influence on the $h/Q\sim x$ relation.

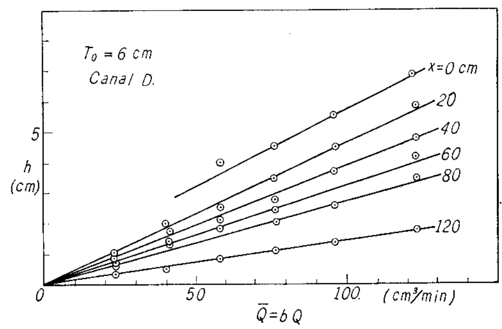

Fig. 9. Correlation between h and $\bar{Q}$ as a parameter x in experiment.

The influence of the difference of canal section on the rise of water table

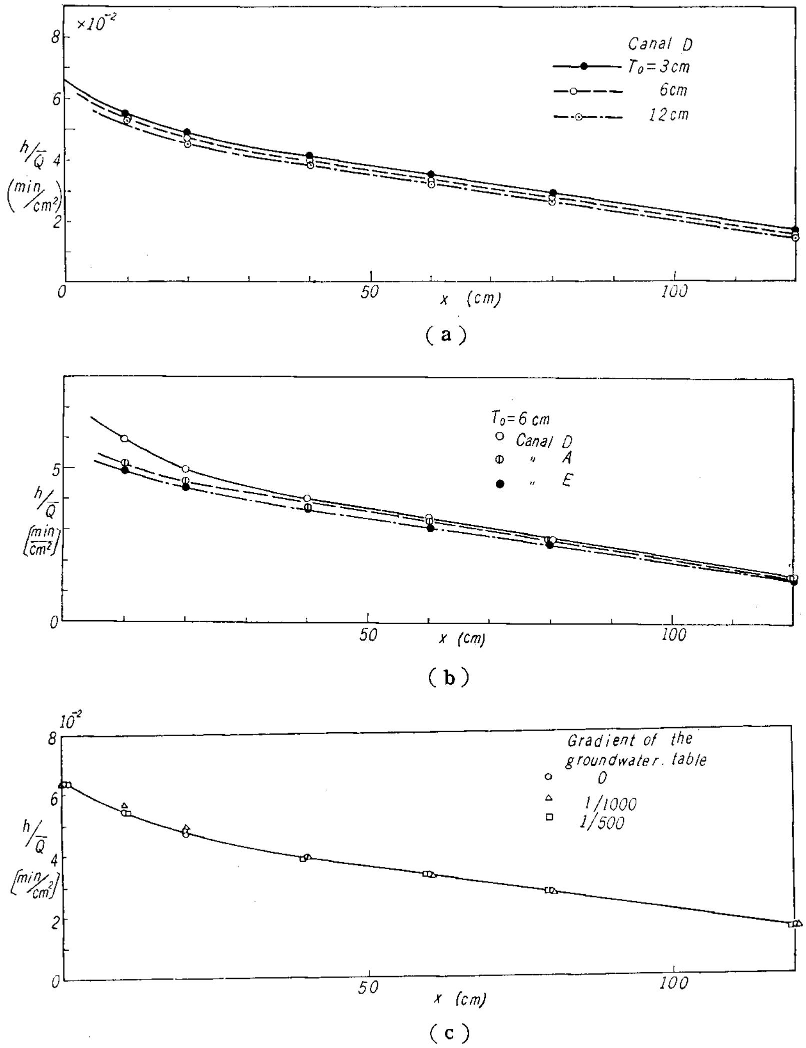

Fig. 10. Correlation between $h/\bar{Q}$ and x.

in the ground is also recognized as small. This can be judged from Fig. 10(c). From these results it can be said that a proportional correlation happens between h and H, considering a linear relation between Q and h in experiment and that between Q and H in Eq. (2) or Eq. (17). The character of free surface is well grasped by arranging the data in terms of h/H instead of h/Q. Fig. 11(a) shows the case of canal positions I, Fig. 11(b) the canal position II and Fig. 11(c) the canal position III in the case of the homogeneous ground. The experimental curve appears considerably higher than the theoretical one, Eq. (14), which represents the free surface of the symmetrical flow from the

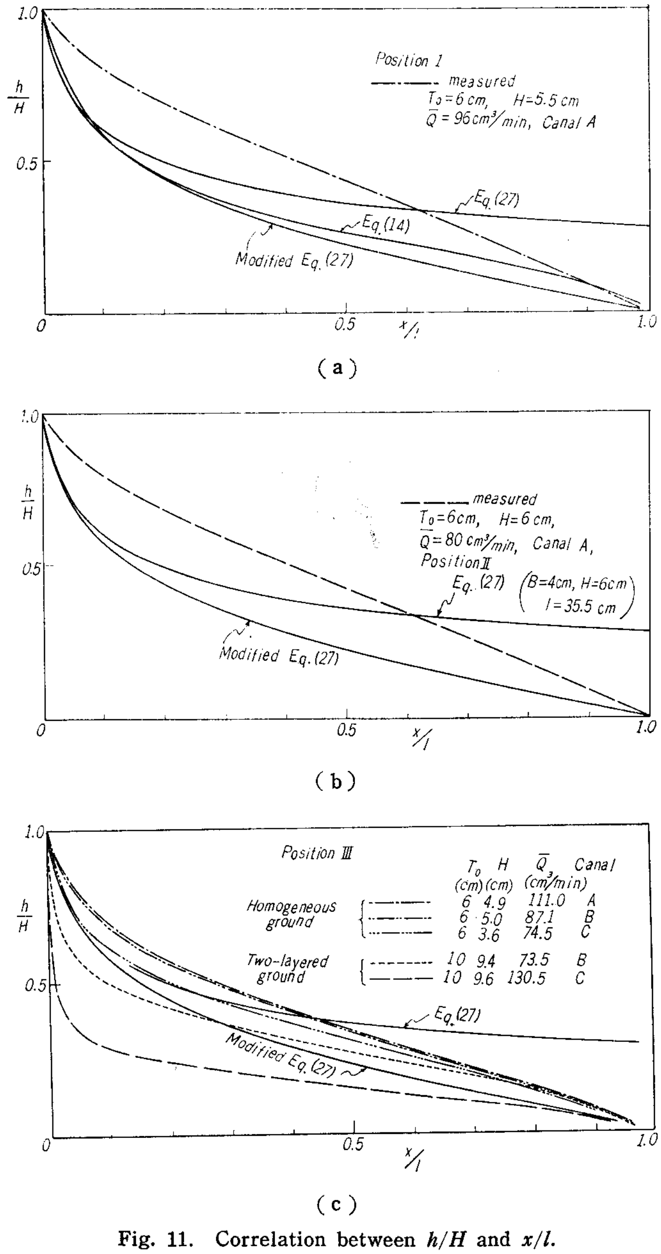

Fig. 11. Correlation between h/H and x/l.

canal. The discrepancy would be due to the existence of water in canal, the capillary zone and the impermeable channel base. The discrepancy seems to slightly decrease as the canal position leaves the center of the sand mass. Applying the method of image at the hydraulic boundary and crossing across the water surface represented by Eq. (14), the theoretical equation (27) is modified.

The groundwater surface with the canal in the position III approaches to that of the theoretical curve. Comparing three data in the homogeneous ground in Fig. 11(c) with each other, the groundwater surface in the case of canal B is the lowest, because of the greater magnitude of H with the small section of canal. Broadly speaking, we conclude that the actual free surface should be regarded as the height added by 20~30 per cent to the theoretical height. The proportional correlation between H and Q is also proved in Fig. 12. From this figure it can be concluded that the increase in the water depth in the canal makes a considerable influence on the rate of recharge.

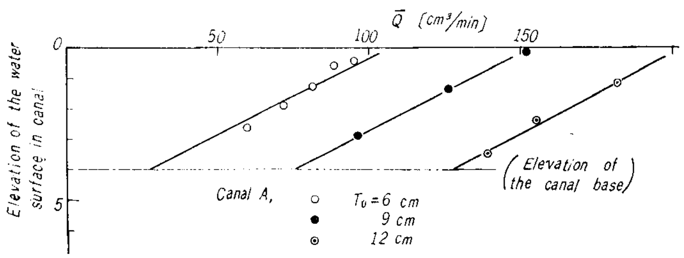

Fig. 12. Correlation between $\bar{Q}$ and the elevation of the water in a canal.

In the case of two-layered ground the characteristics of the water table is of peculiar tendency. Two layers used in the experiment are of constant permeability $k=3.30\times10^{-4}$ cm/sec in the upper layer and $k=3.20\times10^{-2}$ cm/sec in the lower one. The groundwater surface in the case of the homogeneous ground and that of two-layered ground are compared in Fig. 13, taking h/Q as the ordinate. Rearrangement of Fig. 13 in terms of h/H instead of h/Q is also drawn in Fig. 11(c). The groundwater surface by the seepage from a canal formed on two-layered ground falls very rapidly

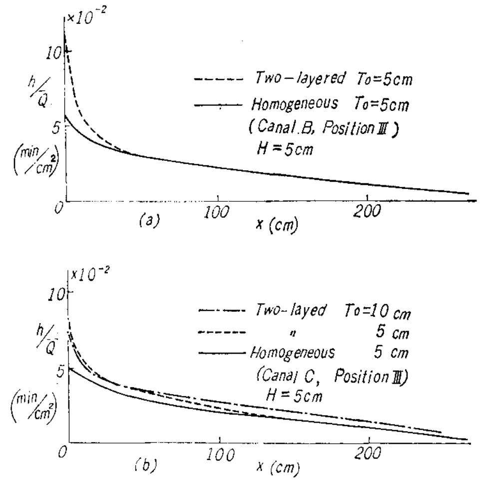

Fig. 13. Comparison of the results in the homogeneous ground with that in the two-layered one.

with distance from the canal compared with the case of homogeneous ground. As the actual ground consists of many kinds of layers, considerations on the stratification of the ground are very important. The height of groundwater surface by the seepage from canal in two-layered ground becomes only half of that in the homogeneous ground in this experimental case. In other words, only half of the rate of recharge into the canal can be permitted in two-layered ground. It is judged from the $h/Q \sim x$ relation in Fig. 13 and the $h/H \sim x$ relation in Fig. 11(c).

The depth of the canal in two-layered ground is of much importance because the permeability of the upper layer plays a big role in the seepage. Let's compare the rise of water table in the cases of the shallow and deep canal, and we find the distinguished tendency; the groundwater table from the deep canal is much higher than the one from the shallow one. This means that the deeper the depth of canal is, the greater the influence on the horizontal velocity of flow becomes. In the case of shallow canal, the rate of recharge is smaller and the water flow mainly consists of vertical flow through the upper layer in which water seeps in the state of a waterdrop. As for the rate of recharge, however, it is necessary to pour much more quantity into the deep canal than into the shallow one, because seepage water permeates into the layer having a high permeability mainly, and because the deep canal penetrates into the lower high permeable layer. We therefore know that, though the rate of recharge into the deep canal is greater, the groundwater table becomes higher in two-layered ground.

4. Conclusion

This paper proposes the equations of the groundwater surface at the seepage from the canal which is made on the ground with symmetrically disposed collectors at some depth under the earth surface and also from that canal which is constructed near the edge of the ground, for example, near the lake or seaside. The equations are deduced from the following assumptions;

1) The amount of discharge percolating into the ground through the canal is equal to that of the recharge poured into the canal; there is no water in canal.
2) The ground is homogeneous.
3) The ground has a constant permeability.
4) There is little influence of the capillarity on the seepage flow, as is usually treated in the potential theory.

The influence of water in the canal on the groundwater surface from canal cannot be made sufficiently clear owing to the complexity of the boundary conditions of the flow region.

The results from this study may be summarized as follows;

1) Considerable amount of additional rise of water table, almost equal to 20～30 per cent of the height calculated from the analytically proposed equation of water table, must be taken into account in the calculation of the actual groundwater surface.

2) As for the shape of canal section and the gradient of the original groundwater slope, the influence on the shape of groundwater table is not recognized so much.

3) The rise of groundwater table h is in proportion to the rate of recharge Q when the ground consists of the homogeneous and isotropic soils.

4) Where the ground consists of two layers, the rise of groundwater surface becomes considerably small.

The author wishes to search for the complete solution by considering the influence of water depth in the canal and the capillarity of the soil on the character of the groundwater surface.

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