A Method for the Calculation of the Turbulent Boundary Layer with Pressure Gradient

By

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(Received June 29, 1965)

A method of calculating the turbulent boundary layer is explained in this paper. The major assumption of this method is that the rate of increase of the boundary layer thickness is proportional to the turbulent velocity fluctuation due to the vorticity contained in the layer. The velocity profile which consists of two parts is assumed, then an equation is obtained. With this equation and the momentum integral equation, the boundary layer thickness and the velocity profile are determined.

1. Introduction

The assumption that the rate of increase of the width of the wake or jet is proportional to the velocity fluctuation perpendicular to the direction of mean flow gives satisfactory result. The same assumption is applied to the case of the turbulent boundary layer and it is assumed that the rate of increase of the boundary layer thickness is proportional to the velocity fluctuation due to the vorticity contained in the layer.

2. Fundamental Relations

Take x-axis along the wall surface, y-axis perpendicular to it and let u be the velocity, u_0 be the volocity outside the boundary layer, v' be the velocity fluctuation in y-direction, δ be the boundary layer thickness and t be the time, then the present assumption can be expressed as follows,

$$\frac{d\delta}{dt} = u_0 \frac{d\delta}{dx} \backsim v'. \tag{1}$$

Also it is assumed that v' is proportional to the vorticity contained in the boundary layer, i.e.

$$v' \backsim \delta \frac{du}{dy}. \tag{2}$$

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From eqs. (1) and (2)

$$\frac{d\delta}{dx} = c\frac{\delta}{u_0}\left(\frac{du}{dy}\right)_C, \tag{3}$$

where $(du/dy)_C$ is the value of du/dy at a properly chosen point in the layer and c is a constant. As the effect of the pressure gradient is included in the value of $(du/dy)_C$, so it can be seen that the constant c is independent of the pressure gradient.

Let θ be the momentum thickness, δ^* be the displacement thickness, $H=\delta^*/\theta$ and also let

$$\varphi_1 = \frac{\theta}{\delta} \quad \text{and} \quad \varphi_2 = c\frac{\delta}{u_0}\left(\frac{du}{dy}\right)_C, \tag{4}$$

then by eq. (3)

$$\frac{\theta}{\varphi_1}\cdot\frac{d\varphi_1}{dx} = \frac{d\theta}{dx} - \varphi_1\varphi_2. \tag{5}$$

The momentum integral equation of the boundary layer is

$$\frac{d\theta}{dx} + \frac{\theta}{u_0}\cdot\frac{du_0}{dx}(H+2) = \zeta^2, \tag{6}$$

where $\zeta=\sqrt{c_f/2}$, c_f being the local skin friction coefficient.

By eq. (6), eq. (5) can be also expressed as follows,

$$\frac{1}{\varphi_1}\cdot\frac{d\varphi_1}{dx} = -\frac{1}{u_0}\cdot\frac{du_0}{dx}(H+2) + \frac{\zeta^2-\varphi_1\varphi_2}{\theta}. \tag{7}$$

The values of φ_1, φ_2, H and ζ are functions of Reynolds number and also depend on the form of velocity profile. Let them be known, then, as solving eqs. (5) and (6) or eqs. (6) and (7), the momentum thickness, velocity profile and other quantities are determined.

3. Velocity Profile and Local Skin Friction Coefficient

In this calculation, the velocity profile is divided into two parts, the one due to the frictional resistance on the wall surface and the other due to the pressure gradient, and it is assumed as follows[1],

$$\frac{u}{u_0} = f(\eta) + ah(\eta), \tag{8}$$

where $\quad f(\eta) = (1-a) + 2.5\zeta\log\eta, \quad h(\eta) = \frac{4}{3}\eta - \frac{1}{3}\eta^4,$

$\eta=y/\delta$ and a being a parameter which is zero when there is no pressure gradient or $u_0=$const.

Then from eq. (8)

$$\frac{\delta^*}{\delta} = \frac{5}{2}\zeta + \frac{2}{5}a, \tag{9}$$

$$\varphi_1 = \frac{\theta}{\delta} = \frac{5}{2}\zeta + \frac{2}{5}a - \frac{25}{2}\zeta^2 - \frac{17}{5}a\zeta - \frac{104}{405}a^2. \tag{10}$$

Let ν be the kinematic coefficient of viscosity and $u^* = \zeta u_0$, the friction velocity, then in the neighbourhood of the wall surface,

$$\frac{u}{u^*} = 5.5 + 2.5\log\eta + 2.5\log\frac{u^*\delta}{\nu},$$

hence, from eq. (10)

$$\frac{u_0\theta}{\nu} = \frac{1}{\zeta}\left(\frac{5}{2}\zeta + \frac{2}{5}a - \frac{25}{2}\zeta^2 - \frac{17}{5}a\zeta - \frac{104}{405}a^2\right)\exp\left(\frac{0.4(1-a)}{\zeta} - 2.2\right). \tag{11}$$

Equation (11) gives the relation between the local skin friction coefficient and Reynolds number $R_\theta = u_0\theta/\nu$ and this is shown in Fig. 1. For the sake of

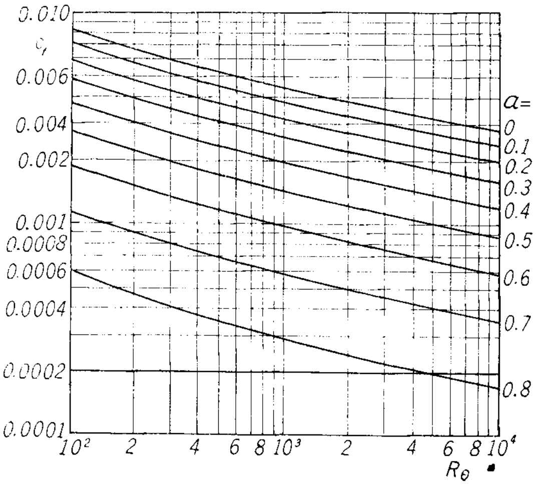

Fig. 1. Coefficient of local skin friction.

simplicity, this local skin friction coefficient is expressed approximately by the following equations.

$$\left.\begin{array}{ll} c_{f_0} = 0.0172\,R_\theta^{-1/5}, & 10^2 < R_\theta < 10^4, \\ \dfrac{c_f}{c_{f_0}} = 1 - 1.38a + 0.527a^5, & 0 < a < 0.8, \end{array}\right\} \tag{12}$$

where c_{f_0} is the coefficient when $a=0$.

Figure 2 shows $\varphi_1 = \theta/\delta$ for several Reynolds numbers calculated by eqs. (10) and (12).

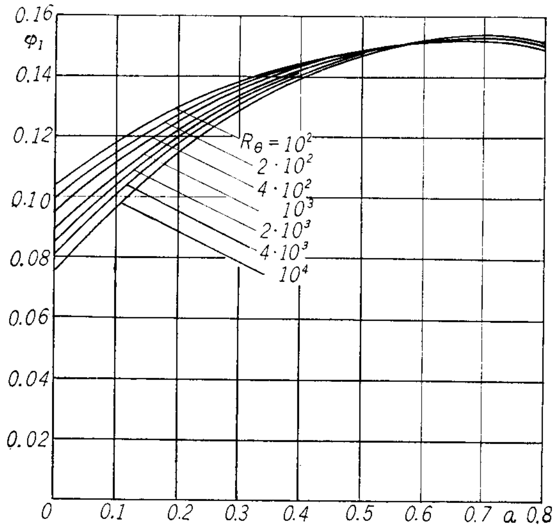

Fig. 2. Relation between $\varphi_1 = \theta/\delta$ and parameter a.

4. The Function φ_2

The strength of the vorticity du/dy varies with y. To determine appropriate position of y to represent the strength of vorticity in the boundary layer, the center of the vorticity distribution due to function $h(\eta)$ is taken. This point is given by $\eta = \int_0^1 (dh/d\eta)\eta d\eta$ and $\eta = 0.4$ in the present case. Hence φ_2 is expressed as follows,

$$\varphi_2 = c\left[\frac{d(u/u_0)}{d\eta}\right]_{\eta=0.4}, \tag{13}$$

where
$$\left[\frac{d(u/u_0)}{d\eta}\right]_{\eta=0.4} = 6.25\zeta + 1.248a.$$

Now, in the case of flow without pressure gradient or $u_0 = \text{const.}$, a must be zero. Using the approximate expression of eq. (12) i.e.

$$\sqrt{c_{f_0}/2} = \zeta_0 = 0.0927\,(\nu/u_0\theta)^{1/10}, \tag{14}$$

the following relations are obtained from eqs. (10) and (14) when $a = 0$,

$$\frac{d\varphi_1}{dx} = \left(\frac{5}{2} - 25\zeta_0\right)\frac{d\zeta_0}{dx} \quad \text{and} \quad \frac{d\zeta_0}{dx} = -\frac{\zeta_0}{10\theta}\cdot\frac{d\theta}{dx}.$$

Then by eq. (6) i.e. $d\theta/dx = \zeta_0^2$,

$$\frac{d\varphi_1}{dx} = -\frac{(1-10\zeta_0)\zeta_0^3}{4\theta},$$

and putting this into eq. (7)

$$\varphi_2 = \frac{(11-60\zeta_0)\zeta_0}{25(1-5\zeta_0)^2}.$$

Comparing this equation with eq. (13) i.e. $\varphi_2 = 6.25c\zeta_0$, the constant c becomes as follows,

$$c = \frac{0.0064(11-60\zeta_0)}{(1-5\zeta_0)^2}. \tag{15}$$

The calculated values of the constant c is given in the accompanying table.

R_θ	10^2	10^3	10^4
c	0.0958	0.0892	0.0846

Corresponding value of c in the wake lies between 0.19~0.27 after Schlichting's experiment[2] and, for example, in the case of a flat plate the value of c in the wake must be at least twice the value in the boundary layer, so it seems that the value given by eq. (15) is adequate.

Using this constant, φ_2 becomes as follows,

$$\varphi_2 = \frac{11-60\zeta_0}{25(1-5\zeta_0)^2}(\zeta + 0.1997a). \tag{16}$$

Figure 3 shows the product $\varphi_1\varphi_2$ for several Reynolds numbers.

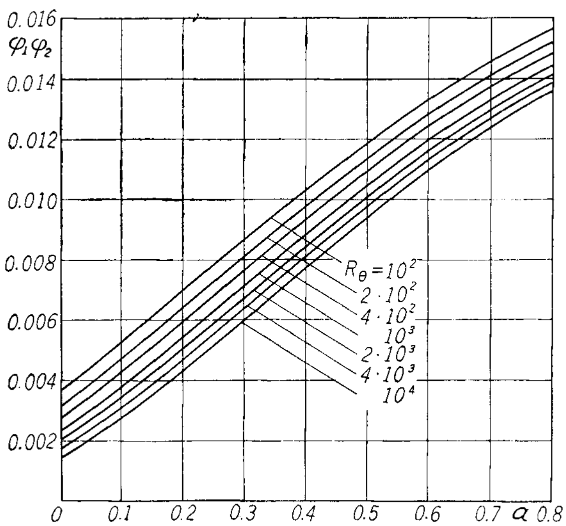

Fig. 3. Relation between $\varphi_1\varphi_2$ and parameter a.

5. Transformation of Equation (5)

Let $\zeta=\xi\zeta_0$, then from eq. (10)

$$\frac{d\varphi_1}{dx}=k_1\frac{da}{dx}-k_2\left(\frac{1}{u_0}\cdot\frac{du_0}{dx}+\frac{1}{\theta}\cdot\frac{d\theta}{dx}\right),$$

where

$$k_1=(0.4-0.5136a-3.4\zeta)+(2.5-3.4a-25\zeta)\zeta_0\frac{d\xi}{da},$$

$$k_2=(0.25-0.34a-2.5\zeta)\zeta\,.$$

Hence, by eq. (6)

$$\frac{k_1}{\varphi_1}\cdot\frac{da}{dx}=-\frac{b_0}{u_0}\cdot\frac{du_0}{dx}-\frac{c_0}{\theta}, \tag{17}$$

where

$$b_0=\left(\frac{k_2}{\varphi_1}+1\right)(H+2)-\frac{k_2}{\varphi_1},$$

$$c_0=\varphi_1\varphi_2-\left(\frac{k_2}{\varphi_1}+1\right)\zeta^2.$$

Figures 4, 5 and 6 show k_1/φ_1, b_0 and c_0 respectively. The value of k_1/φ_1

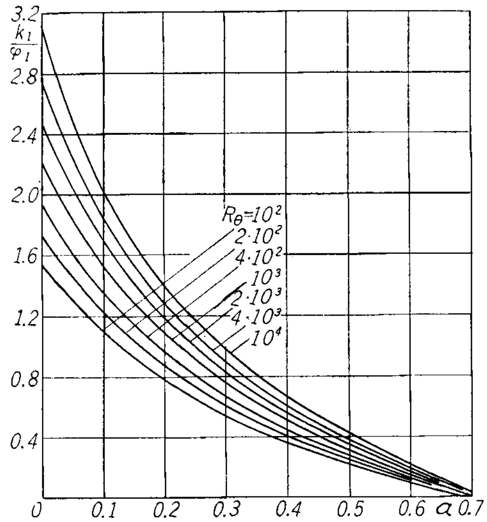

Fig. 4. Values of k_1/φ_1 in eq. (17).

decreases with a and becomes zero at $a\approx0.7$, while b_0 and c_0 are both positive. Hence, in this method, the separation point is given by $k_1=d\varphi_1/da=0$ unless $(\theta/u_0)du_0/dx=-c_0/b_0$. It is noticed that the condition $k_1=0$ depends on the form of the assumed velocity profile and Reynolds number, but independent of the value of φ_2.

Again, after some calculations, eq. (17) is transformed into the following equation,

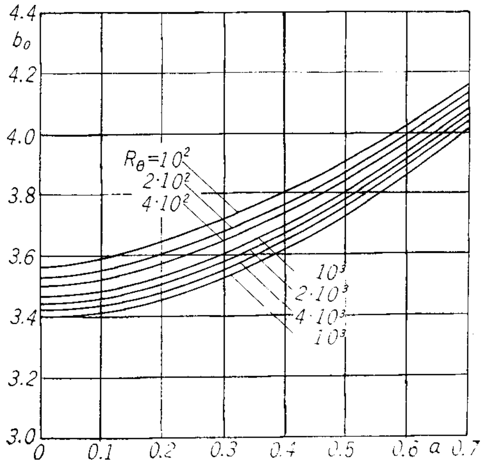

Fig. 5. Values of b_0 in eq. (17).

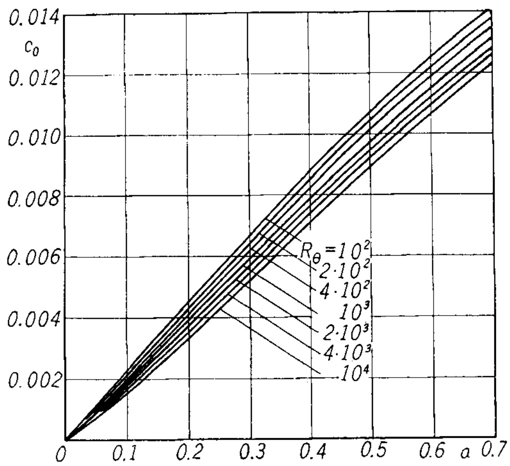

Fig. 6. Values of c_0 in eq. (17).

$$\theta\frac{dH}{dx} = -\frac{b_H\theta}{u_0}\cdot\frac{du_0}{dx} - c_H, \tag{18}$$

where

$$b_H = \frac{\varphi_1 b_0}{k_1}\left(H_1 + H_2\zeta_0\frac{d\xi}{da}\right) + 0.1\zeta H_2(H+1),$$

$$c_H = \frac{\varphi_1 c_0}{k_1}\left(H_1 + H_2\zeta_0\frac{d\xi}{da}\right) + 0.1\zeta^3 H_2,$$

$$H_1 = \frac{0.4\varphi_1 - (2.5\zeta + 0.4a)(0.4 - 3.4\zeta - 0.5136a)}{\varphi_1^2},$$

$$H_2 = \frac{2.5\varphi_1 - (2.5\zeta + 0.4a)(2.5 - 25\zeta - 3.4a)}{\varphi_1^2}.$$

Examples of the values of b_H and c_H are shown in Fig. 7. Figure 8 shows the relation between H and $-(\theta/u_0)du_0/dx$ for equilibrium layer i.e. $dH/dx=0$[3], the points are the experimental result of F. H. Clauser. The broken lines show the same relations taken from the author's previous paper[4].

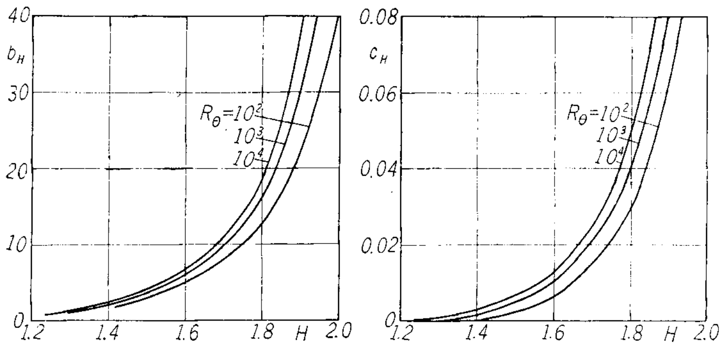

Fig. 7. Values of b_H and c_H in eq. (18).

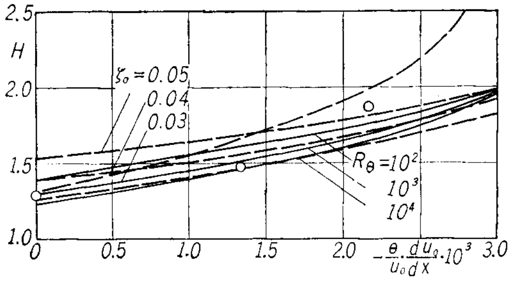

Fig. 8. Relation between H and $-(\theta_0/u_0)du_0/dx$ when $dH/dx=0$.

6. Practical Method of Calculation and Examples of Numerical Calculation

In the practical calculation, the momentum thickness can be estimated by the following equation.

$$\left(\frac{u_0\theta}{\nu}\right)^{1/5}\theta = \frac{1}{u_0^4}\left\{0.0103\int u_0^4 dx + \text{const.}\right\}, \quad R_\theta < 10^4 .$$

Let φ_1 and θ at $x=x_0$ be φ_{10} and θ_0 respectively, then from eq. (5)

$$\frac{\varphi_1}{\varphi_{10}} = \frac{\theta}{\theta_0}\exp\left(-\int_{x_0}^{x}\frac{\varphi_1\varphi_2}{\theta}dx\right),$$

or let $x-x_0=\Delta x$, Δx being small, then

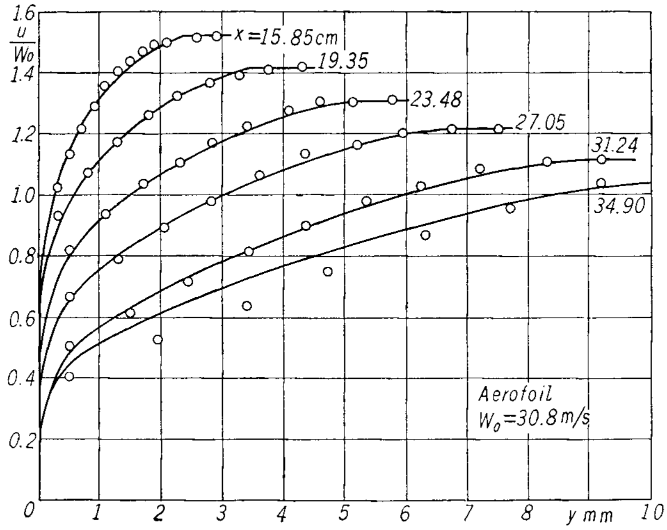

Fig. 9. Comparison of the experimental and the calculated velocity profiles.

$$\frac{\varphi_1}{\varphi_{10}} = \frac{\theta}{\theta_0} \exp\left(-\frac{\varphi_1\varphi_2\Delta x}{\theta}\right),$$

and this relation is convenient for successive calculation to estimate φ_1.

Figures 9, 10 and 11 show three examples of calculation compared with the experimental results of Gruschwitz[5], the same examples which were used

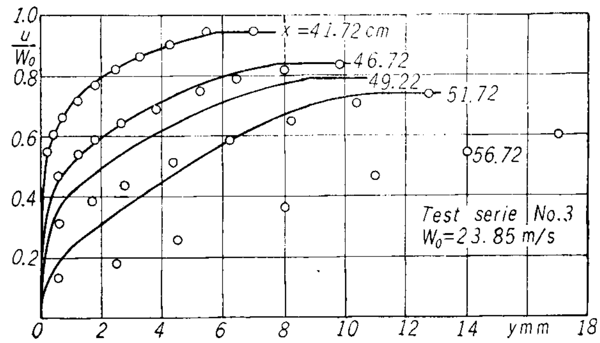

Fig. 10. Comparison of the experimental and the calculated velocity profiles.

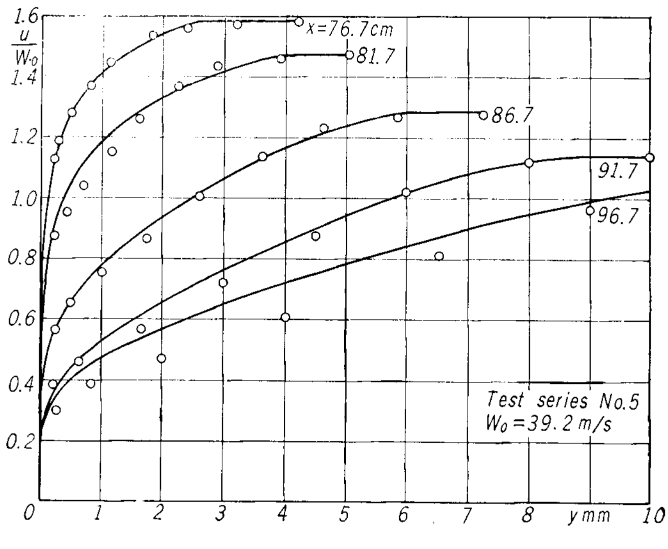

Fig. 11. Comparison of the experimental and the calculated velocity profiles.

in the previous papers[6]. In the cases of Figs. 9 and 11 satisfactory results are obtained, but unfortunately, in the case of Fig. 10, the boundary layer separates near $x=51.72$ cm while the experimental result show that the separation takes place at $x=56.72$ cm. In this respect, the accompanying table

	Test Series No. 3		Test Series No. 5	
x cm	51.72	56.72	91.7	96.7
$-\frac{\theta}{u_0}\cdot\frac{du_0}{dx}\cdot 10^3$	4.6	8.0	3.3	2.1

which shows the result of calculation, will explain the above-mentioned discrepancy.

7. Conclusion

In this method, it is assumed that the rate of increase of the boundary layer thickness is proportional to the vorticity contained in the layer. So the method of the calculation consists of one dynamical equation i.e. the momentum integral equation and one kinematical relation eq. (3). There is a doubt whether c in eq. (3) is a constant or varies with the velocity profile, still the present method gives rather satisfactory results except premature separation of the layer in some cases.

References

1) B. Hudimoto: This memoirs, Vol. 24, Part 1, 1962.
2) H. Schlichting: Ing.-Archiv, Bd. 1, p. 537, 1930.
3) J. C. Rotta: In the book "Progress in Aeronautical Sciences" Vol. 2 (Pergamon Press), p. 202, 1962.
4) B. Hudimoto: This memoirs, Vol. 13, No. 4, 1951.
5) E. Gruschwitz: Ing.-Archiv, Bd. 2, p. 321, 1931.
6) B. Hudimoto: ibid.