On the Sub-Interval Optimization Technique for Final-Value Control Systems with Magnitude Constraint

By

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A method of designing optimal final value-control systems with magnitude constraint is presented in this paper. A second-order linear dynamical system is considered here as a controlled plant. The magnitude constraint is pre-assigned on the control variable without assigning any penalties on the performance criterion. The optimal final-value control problem is formulated as a two point boundary value problem, and a physically meaningful solution is obtained by introducing a new concept of sub-interval optimization technique which avoids the direct solution of the two point boundary value problem. The method presented here, can be applied to the other types of optimal control problems with magnitude constraint.

List of Principal Symbols

t and τ: time variable and reversed time variable respectively

$x(t)$ and $\theta(t)$: controlled variable and control variable respectively

J_f: performance functional

$\phi(x_1, x_2; \tau)$: value of performance index when the system starts from the states x_1 and x_2 at the time τ, where x_1 and x_2 denote the state variables of controlled plant respectively

$k_i(\tau)$ $(i=0, 1, \cdots, 5)$: coefficients in expansion of $\phi(x_1, x_2; \tau)$

$\bar{\theta}(t)$: optimum control variable which minimizes the performance functional J_f

k, a and b: parameters of a controlled plant

L and T: pre-assigned constraint on magnitude of the control variable and the final instant of control operation respectively

c_1 and c_2: initial values corresponding to the state variables of the controlled plant x_1 and x_2 respectively

x_{1d} and x_{2d}: desired values of the state variables x_1 and x_2 of the controlled

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plant at the final instant

μ: non-negative constant expressing a weighting factor in the performance functional

1. Introduction

In recent years, there has been a growing interest in the development of design techniques on optimal control systems. Various analytical techniques as well as computer approaches have been reported by many authors[1-5]. In these investigations it is a well-known fact that the solution of a two point boundary value problem is inevitable to obtain the optimal control variable subjected to a constraint on the magnitude. Although some approximate techniques for solving the two point boundary value problem have been published in literature[6-8], there remain many problems to be examined from engineering viewpoints.

The authors have already proposed an approximate method of designing the optimum control system with magnitude constraint which minimizes the integral-error-squared criterion from practical viewpoints[9]. In this paper, the previous method is extended to the case of final value-control problems considering a magnitude constraint. Linear second-order dynamical systems are considered as the controlled plant. The magnitude constraint on the control variable is attacked directly without assigning any penalties on the performance criterion. The optimal control problem is formulated as a two point boundary value problem by using the principle of Dynamic Progrmming[10]. Our present attention is directed to obtaining a physically meaningful solution without solving the two point boundary value problem.

2. Definition of Newly Introduced Terminologies

Assumption realizing the Sub-Optimal Control[9]: In designing the optimal control of a linear second order controlled system subjected to a constraint on its control variable, the final instant of control operation T is assumed to be well given beforehand so that the optimal control might be performed by no more than one switching action within the control interval $(0, T)$, which depends upon both initial conditions and system parameters.

Sub-Optimal Control[9]: Optimal control for the restricted sub-interval which satisfies the assumption mentioned above.

Optimization Interval by the Sub-Optimal Control: The restricted control interval which satisfies the assumption mentioned above.

Switching Function of Sub-Optimal Control: The switching function which provides the control law for the realization of sub-optimal control.

Sub-Optimal Control Problems: Optimal control problems subjected to the constraint on the control variable, in which the optimization may be carried out by the sub-optimal control law.

Sub-Interval Optimization Technique: General concept introduced to provide a pseudo-optimizotion technique for the control interval which does not satisfy the assumption realizing the sub-optimal control.

3. Statement of the Problem

We consider a linear second order controlled plant as shown in Fig. 1, of which the dynamical characteristic is represented by a transfer function of the from $k/(s+a)(s+b)$ between the control variable $\theta(t)$ and the controlled variable $x(t)$. The control variable is subjected to a constraint on its magnitude as

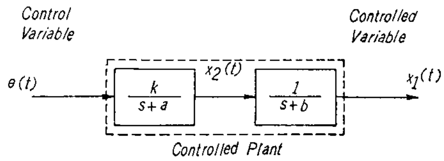

Fig. 1. Block Diagram of the controlled Plant to be considered here.

$$|\theta(t)| \leq L, \tag{3.1}$$

where L is a pre-assigned positive constant. By using the state variables of the controlled plant, the plant dynamics can be expressed as

$$\left.\begin{aligned} \dot{x}_1(t) &= -bx_1(t)+x_2(t), \quad x_1(0) = c_1 \\ \dot{x}_2(t) &= -ax_2(t)+k\theta(t), \quad x_2(0) = c_2 \end{aligned}\right\}. \tag{3.2}$$

where the symbol "·" expresses the differentiation with respect to a time variable. In Eq. (3.2), c_1 and c_2 denote the initial states of the controlled variable respectively.

The problem considered here is to design the controller which minimizes the performance functional,

$$J_f = [x_{1d}-x_1(T)]^2+\mu[x_{2d}-x_2(T)]^2, \tag{3.3}$$

for any initial states of the controlled plant at the time $t=0$. In Eq. (3.3), x_{1d} and x_{2d} respectively express the desired values corresponding to the state variables $x_1(t)$ and $x_2(t)$ of the controlled plant at $t=T$, where T is also a pre-assigned constant which expresses the final instant of control operation. For the convenience of our discussion, we assume $x_{1d}=x_{2d}=0$ in Eq. (3.3) without any loss of generally.

The design problem is therefore to find the control variable $\theta(t)$ which minimizes Eq. (3.3) under the constraints given by Eqs. (3.1) and (3.2).

Let us define the new variables as*

$$\left.\begin{aligned} \hat{t} &= at \\ \hat{x}_1 &= a^2x_1/kL \\ \hat{x}_2 &= ax_2/kl \end{aligned}\right\}. \tag{3.4}$$

Eqs. (3.1), (3.2) and (3.3) become

$$|\hat{\theta}(\hat{t})| \leq 1. \tag{3.5}$$

$$J_f = (kL)^2/a^4 \cdot \{[\hat{x}_{1d} - \hat{x}_1(\hat{T})]^2 + \hat{\mu}[\hat{x}_{2d} - \hat{x}_2(\hat{T})]^2\}, \tag{3.6}$$

and

$$\left.\begin{aligned} \dot{\hat{x}}_1(\hat{t}) &= -\hat{b}\hat{x}_1(\hat{t}) + \hat{x}_2(\hat{t}), \quad \hat{x}_1(0) = \hat{c}_1 \equiv a^2c_1/kL \\ \dot{\hat{x}}_2(\hat{t}) &= -\hat{x}_2(\hat{t}) + \hat{\theta}(\hat{t}), \quad \hat{x}_2(0) = \hat{c}_2 \equiv ac_2/kL \end{aligned}\right\}. \tag{3.7}$$

where $\hat{\theta} \equiv \theta/L$, $\hat{x}_{1d} \equiv a^2x_{1d}/kL = 0$, $\hat{x}_{2d} \equiv ax_{2d}/kL = 0$, $\hat{\mu} \equiv a^2\mu$, $\hat{b} \equiv b/a$ and $\hat{T} \equiv aT$.

From Eq. (3.6), we obtain a relation;

$$\min_{\substack{\theta \\ |\theta| \leq L}} J_f = (kL)^2/a^4 \cdot \min_{\substack{\hat{\theta} \\ |\hat{\theta}| \leq 1}} \hat{J}_f. \tag{3.8}$$

where

$$\hat{J}_f \equiv \hat{x}_1(\hat{T})^2 + \hat{\mu}\hat{x}_2(\hat{T})^2. \tag{3.9}$$

The problem is therefore reduced to the one which minimizes the functional given by Eq. (3.9), taking the constraints shown by Eqs. (3.5) and (3.7) into account. For simplicity of the present description we shall omit the chapeau "∧" henceforth, unless it is necessary.

4. Configuration of the Optimal Final-Value Control System

By applying the well known concept of Dynamic Programming[10], the problem is reduced to solve the following partial differential equation;

$$\frac{\partial \phi}{\partial \tau} = \min_{\substack{\theta \\ |\theta| \leq 1}} \left\{(-bx_1 + x_2)\frac{\partial \phi}{\partial x_1} + (-x_2 + \theta)\frac{\partial \phi}{\partial x_2}\right\} \quad (\tau > 0). \tag{4.1}$$

with the initial condition;

$$\phi = x_1(T)^2 + \mu x_2(T)^2 \quad (\tau = 0), \tag{4.2}$$

where $\phi = \phi(x_1, x_2; \tau)$ is defined by

$$\phi(x_1, x_2; \tau) = \min_{\substack{\theta \\ |\theta| \leq 1}} \{x_1(T)^2 + \mu x_2(T)^2\}, \tag{4.3}$$

and $\tau = T - t$ denotes an auxiliary time variable which is introduced for the

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* This transformation ceases to be valid in the case where the plant parameters a and b become zero simultaneously. It is necessary to treat the case where $a = b = 0$ separately. The treatment will be presented in Appendix-I.

convenience of the later description and is called the reversed time variable.

From Eq. (4.1) the optimum control variable $\bar{\theta}(t)$ can be obtained as*

$$\bar{\theta}(t) = -\operatorname{sgn}[\bar{z}(\tau)], \tag{4.4$_1$}$$

and

$$\bar{z}(\tau) \equiv \partial\phi/\partial x_2, \tag{4.4$_2$}$$

where the function ϕ is the solution of the non-linear partial differential equation;

$$\frac{\partial\phi}{\partial\tau} = -(bx_1 - x_2)\frac{\partial\phi}{\partial x_1} - x_2\frac{\partial\phi}{\partial x_2} - \frac{\partial\phi}{\partial x_2}\operatorname{sgn}\left[\frac{\partial\phi}{\partial x_2}\right], \quad (\tau > 0). \tag{4.5}$$

with the initial condition given by Eq. (4.2).

Thus the optimum configuration of control system subjected to the constraint on its control variable becomes a Bang-Bang control system with an optimum switching function as shown in Fig. 2.

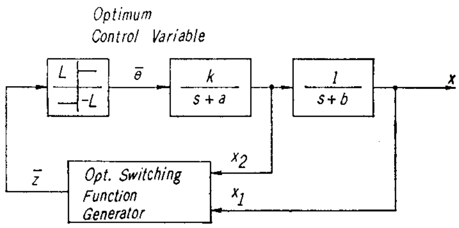

Fig. 2. Optimum Configuration of the Control System with Magnitude Constraint on the Control Variable.

In Eq. (4.5), if we assume the solution is of a quadratic form;

$$\phi = k_0(\tau) + k_1(\tau)x_1 + k_2(\tau)x_2 + k_3(\tau)x_1x_2 + k_4(\tau)x_1^2 + k_5(\tau)x_2^2. \tag{4.6}$$

then the following set of non-linear simultaneous differential equations can be derived as

$$\left.\begin{aligned} k_0'(\tau) &= -k_2(\tau)\operatorname{sgn}[\bar{z}(\tau)], \\ k_1'(\tau) &= -bk_1(\tau) - k_3(\tau)\operatorname{sgn}[\bar{z}(\tau)], \\ k_2'(\tau) &= k_1(\tau) - k_2(\tau) - 2k_5(\tau)\operatorname{sgn}[\bar{z}(\tau)], \\ k_3'(\tau) &= 2k_4(\tau) - (b+1)k_3(\tau), \\ k_4'(\tau) &= -2bk_4(\tau) \\ k_5'(\tau) &= k_3(\tau) - 2k_5(\tau), \end{aligned}\right\}, \tag{4.7$_1$}$$

with the initial condition;

$$k_0(0) = 0, \quad k_1(0) = 0, \quad k_2(0) = 0, \quad k_3(0) = 0, \quad k_4(0) = 1 \quad \text{and} \quad k_5(0) = \mu. \tag{4.7$_2$}$$

where "′" expresses the differentiation with respect to the reversed time variable τ and the switching function $\bar{z}(\tau)$ is expressed by

$$\bar{z}(\tau) \equiv k_2(\tau) + k_3(\tau)x_1 + 2k_5(\tau)x_2. \tag{4.8}$$

* $\operatorname{sgn}[\bar{z}] = 1\ (\bar{z} > 0),\ -1\ (\bar{z} < 0)$.

On the other hand, replacing the variable t by the reversed time variable τ and substituting $\bar{\theta}(t)$ for $\theta(t)$ into Eq. (3.7), we have

$$\left.\begin{aligned} x_1'(\tau) &= bx_1(\tau)-x_2(\tau)\,, & x_1(T) &= c_1 \\ x_2'(\tau) &= x_2(\tau)+\operatorname{sgn}\left[k_2(\tau)+k_3(\tau)x_1(\tau)+2k_5(\tau)x_2(\tau)\right], & x_2(T) &= c_2 \end{aligned}\right\}. \tag{4.9}$$

The determination of a switching function is therefore reduced to solve the two point boundary value problem which consists of Eqs. (4.7) and (4.9).

5. Determination of Switching Functions realizing the Sub-Optimal Control

Assuming that the final instant of control operation T satisfies the assumption realizing the sub-optimal control[9], the coefficients necessary for the switching function of sub-optimal control can be obtained from the following set of equations;

$$\left.\begin{aligned} k_1'^{+}(\tau) &= -bk_1^{+}(\tau)-k_3(\tau)\,, & k_1^{+}(0) &= 0 \\ k_2'^{+}(\tau) &= k_1^{+}(\tau)-k_2^{+}(\tau)-2k_5(\tau)\,, & k_2^{+}(0) &= 0 \end{aligned}\right\}, \tag{5.1$_1$}$$

for the positive relay output at $t=0$, and for the negative relay output at $t=0$,

$$\left.\begin{aligned} k_1'^{-}(\tau) &= -bk_1^{-}(\tau)+k_3(\tau)\,, & k_1^{-}(0) &= 0 \\ k_2'^{-}(\tau) &= k_1^{-}(\tau)-k_2^{-}(\tau)+2k_5(\tau)\,, & k_2^{-}(0) &= 0 \end{aligned}\right\}, \tag{5.1$_2$}$$

where the functions $k_3(\tau)$ and $k_5(\tau)$ are respectively derived from the last three equations in Eq. (4.7) as

$$\left.\begin{aligned} k_3(\tau) &= \frac{2}{1-b}\{\exp(-2b\tau)-\exp[-(b+1)\tau]\} \\ k_5(\tau) &= \frac{1}{(1-b)^2}\{\exp(-\tau)-\exp(-b\tau)\}^2+\mu\exp(-2\tau) \end{aligned}\right\}. \tag{5.2}$$

From Eqs. (5.1)$_1$, (5.2)$_2$ and (5.2), the switching functions of the sub-optimal control can be obtained as

$$z_{\mathrm{SUB}}^{\pm} \equiv k_2^{\pm}(\tau)+k_3(\tau)x_1+2k_5(\tau)x_2\,, \tag{5.3}$$

where

$$\begin{aligned} k_2^{\pm}(\tau) = &\mp\frac{1}{b(1-b)^2}\{\exp(-b\tau)-\exp(-\tau)[\{1-\exp(-b\tau)\}-b\{1-\exp(-\tau)\}]\\ &\mp 2\mu\exp(-\tau)\{1-\exp(-\tau)\}\,. \end{aligned} \tag{5.4}$$

In Eq. (5.3), $z_{\mathrm{SUB}}^{+}(\tau)$ expresses the switching function for the positive trajectories corresponding to the positive sign of the relay output at $t=0$, and $z_{\mathrm{SUB}}^{-}(\tau)$ denotes the one for negative trajectories. The detailed procedure of derivation is presented in Appendix-I, where the double-integrator plant is treated for the simplicity of description.

6. Sub-Interval Optimization Technique

6.1. Basic Concept of the Sub-Interval Optimization Technique

As the authors have already stated, since, there are no items to be mentioned here in the case where the sub-optimal control can be realized during the time interval $(0, T)$, then the technique introduced here is to provide a pseudo optimization strategy for the control interval which does not satisfy the assumption realizing the sub-optimal control. The fundamental idea may be stated along the illustration of Fig. 3 as follows:

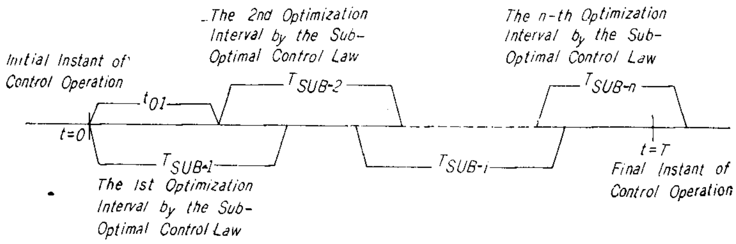

Fig. 3. Illustration of the Sub-Interval (Fictitious Optimization Interval by the Sub-Optimal Control Law).

(1) By taking account of both the initial conditions and the system parameters, determine the fictitious final instant T_{SUB-I} of sub-interval which satisfies the assumption realizing the sub-optimal control.

(2) Solve the fictitious sub-optimal control problem where T_{SUB-I} is tentatively considered as the final instant and decide the control law for the optimization of the first fictitious sub-interval.

(3) Compute the system states at the terminus of the first fictitious interval T_{SUB-I} and examine whether the rest interval satisfies the assumption with respect to the computed system states, or not.

(4) If the terminus of the first fictitious interval T_{SUB-I} does not satisfy the assumption, then compute the value of x_1-coordinate corresponding to the $x_2=0$ after the first change over. Determine the second fictitious final instant T_{SUB-II} by considering the values of x_1 calculated above and $x_2=0$ as the initial conditions and decide the control law for the second fictitious interval.

(5) Continue the iterative procedure listed above until the n-th extension involves the final instant of control operation T.

Fig. 4 shows the flow chart of the pseudo-optimization procedure. Since the necessary condition realizing the sub-optimal control is expressed as a inequality which is described by system initial conditions, system parameters and the final instant of control operation, in Fig. 4, the relation $f(c_1, c_2, b, T)\leqq 0$ gives the necessary condition realizing the sub-optimal control. However, it is in general difficult to derive the relation $T\leqq g(c_1, c_2, b)$ from $f(c_1, c_2, b, T)\leqq 0$ because it is a transcendental inequality. So the following graphical method is applied to determine the final instant of fictitious optimization interval without using the function $f(c_1, c_2, b, T)$.

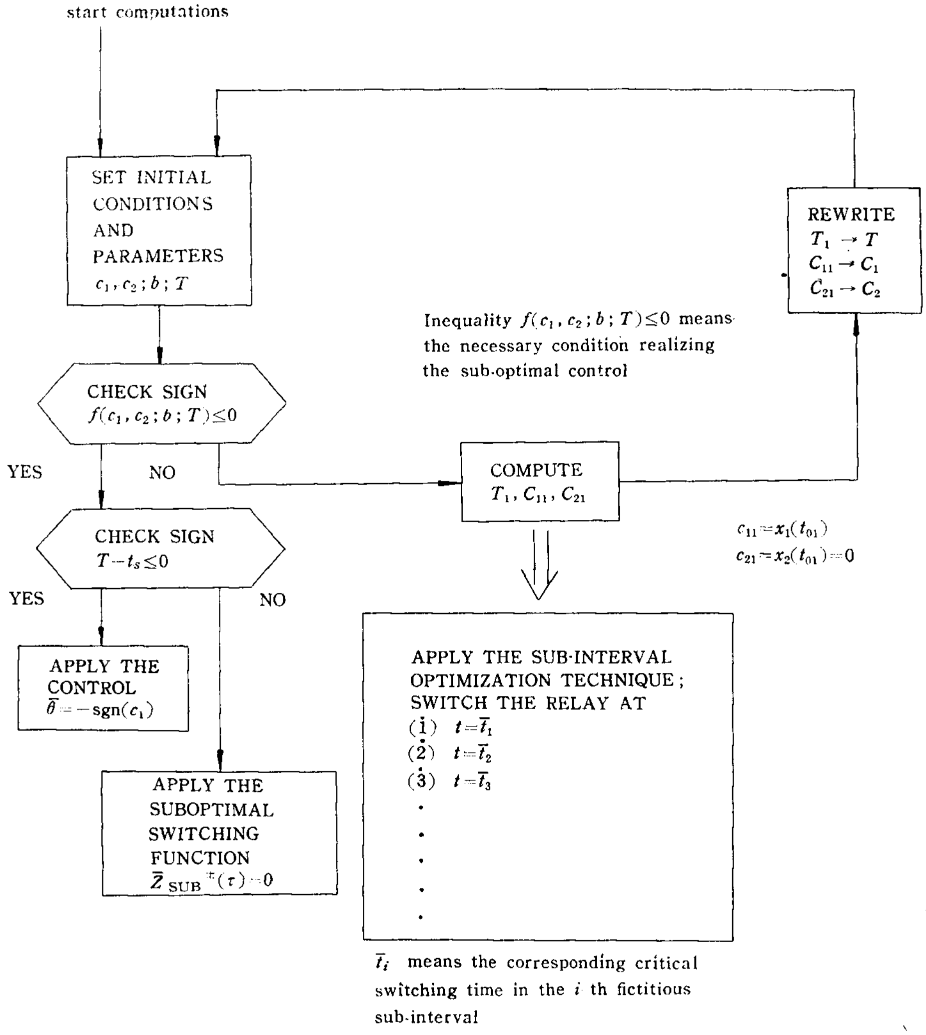

Fig. 4. Basic Concept of the Sub-Interval Optimization Technique.

6.2. Sub-Optimal Switching Lines in the case where $\mu=0$

As a preliminary consideration, let us calculate numerically intersections of the system response trajectories with the switching line expressed as $\hat{z}^{\pm}_{SUB}=0$ in the phase plane*. For simplicity, we consider the limiting case where the system parameters a and b in Eq. (3.2), become zero. The switching line of sub-optimal control for the negative trajectories in this case is derived with the help of Appendix-I as

$$\hat{z}^{-}_{SUB}(\tau)\Big|_{b=a=0} \equiv \tau^3+2\tau\hat{x}_1+2\tau^2\hat{x}_2+2\mu(\tau+\hat{x}_2)=0\,, \tag{6.1}$$

where both $\hat{x}_1$ and $\hat{x}_2$ are the normalized state variables respectively defined by

$$\left.\begin{aligned}\hat{x}_1 &= x_1/kL\\ \hat{x}_2 &= x_2/kL\end{aligned}\right\}. \tag{6.2}$$

On the other hand, by substituting the condition $a=b=0$ into Eq. (3.2) and integrating with respect to the reversed time, the system responses corresponding to the negative relay output at $t=0$ are obtained as

$$\left.\begin{aligned}\hat{x}_1(\tau) &= \hat{c}_1+\hat{c}_2(T-\tau)-(T-\tau)^2/2\\ \hat{x}_2(\tau) &= \hat{c}_2-(T-\tau)\end{aligned}\right\}. \tag{6.3}$$

By using Eqs. (6.1) and (6.3), and considering the value of T as a parameter we calculate the intersections. The results are plotted as the solid curves in Figs. 5-(a) and 5-(b), in which we can set the initial condition c_2 to be zero without loss of generality. Fig. 5-(a) shows the loci of intersections with respect to both the initial condition c_1 and the control interval T, by using Eqs. (6.3) and (6.1) where $\mu=0$. On the other hand, Fig. 5-(b) shows the effect of changing the value of weighting factor μ on the

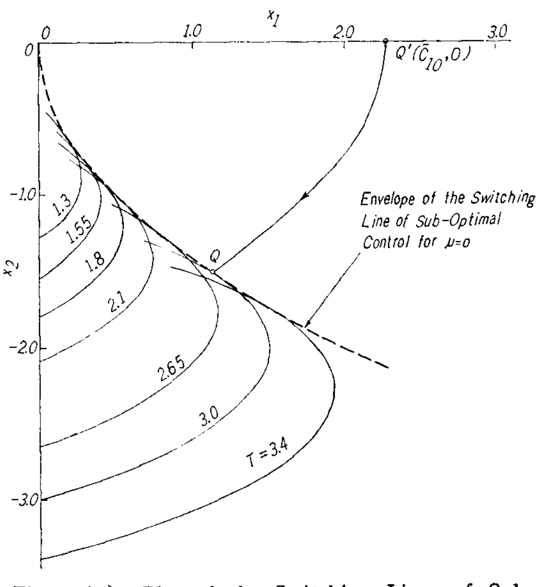

Fig. 5-(a). Plot of the Switching Lines of Sub-Optimal Control with respect to the Final Instant of Control Operation T ($\mu=0$, $kL=1.0$).

* Strictly speaking, the two-dimensional state space must be used for this terminology.

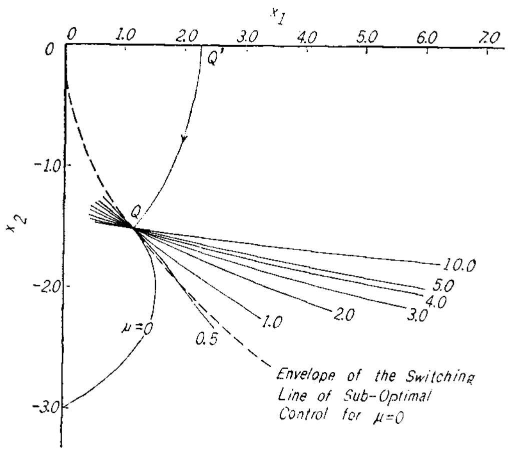

Fig. 5-(b). Plot of the Switching Lines of Sub-Optimal Control with respect to the Weighting Factor μ ($T=3.0$, $kL=1.0$).

loci of intersections, where the control interval in this case is fixed to be $T=3.0$.

In order to explain the role of sub-optimal switching in detail, we consider the particular value of $T=3.0$ and show schematically the figure of switching line as shown in Fig. 6 by extracting the curve corresponding to $T=3.0$ from Fig. 5-(a). In Figs. 5-(a) and 6, the broken line expresses the envelope of the plot of the intersections, which indicates the right boundary re-

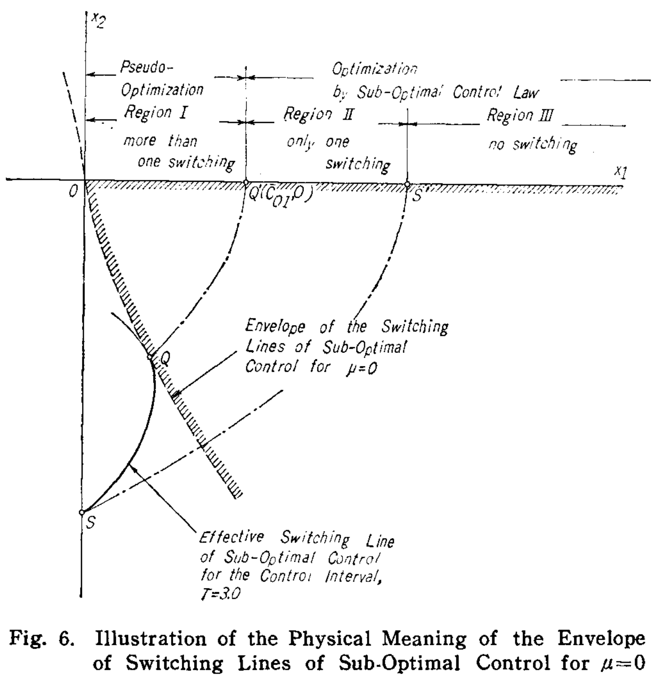

Fig. 6. Illustration of the Physical Meaning of the Envelope of Switching Lines of Sub-Optimal Control for $\mu=0$ ($T=3.0$, $kL=1.0$).

gion of the sub-optimal switching. In other word, this means that the plots of the point of sub-optimal switching do not locate in the hatched area as shown in Fig. 6. The left-boundary region is obtained as the x_2-axis from the consideration of taking the limit of the variable τ to zero in Eq. (6.1). Furthermore, from the definition of the sub-optimal control, it is evident that the effective (optimal) switching line of sub-optimal control becomes the part of the curve running downward from the point, Q, of tanget to the envelope, if we fix the final instant of control operation T to be 3.0. Since we can draw a negative trajectory so as to run across at the point Q, we express the point of intersection between negative trajectory and the x_1-axis by Q'. Since we can also calculate the x_1-coordinate of the intersection Q', if we represent this by the symbol $\bar{c}_{10}$, then the relation $c_1 \geq \bar{c}_{10}$ has the equivalent physical meaning of the necessary condition relalizing the sub-optimal control in the case where $T=3.0$, $\mu=0$ in Eq. (3.2) and $c_2=0$, $a=b=0$ in Eq. (3.2). Since the envelope means a set of the critical point for the fixed T like Q, it is concluded that the envelope gives the boundary switching line of the sub-optimal control. This conclusion is also valid to the case of $\mu \neq 0$ because the switching lines of sub-optimal control for $\mu \neq 0$ have the intersection on the envelope at the same point as the critical point for the fixed T as shown in Fig. 5-(b). Mathematical verification of this fact is carried out in Appendix-II. Fig. 5-(b) illustrates an example of the situation where the final instant T is fixed to be 3.0. It is revealed from the discussions presented above that, by obtaining the envelope of the sub-optimal switching line on the phase plane through the graphical method, we can get the necssary condition realizing the sub-optimal control in a geometrical sense.

As the authors have already mentioned, the basic concept of the sub-interval optimization technique is introduced to give a pseudo-optimization rule for the control interval which does not satisfy the assumption realizing the sub-optimal control. This means that the concept provides a pseudo switching rule for a smaller initial condition c_1 than $\bar{c}_{10}$ in the case of a fixed control interval.

6.3. Importance of the Sub-Interval Optimization Technique in the Final-Value Control Problem

In this paragraph, let us consider the geometrical and physical meaning of sub-interval optimization technique applied to the final-value control problem treated in this paper to determine an optimal non-linear switching line.

Fig. 7 shows schematically the basic concept of sub-interval optimization technique from the graphical point of view. In Fig. 7, let us consider the optimization problem with respect to the initial state $F(c_1, c_2)$ from which the response trajectory will across the x_1-axis at the point R' nearer to the origin than the point Q'. In this case the response trajectory meets the boundary

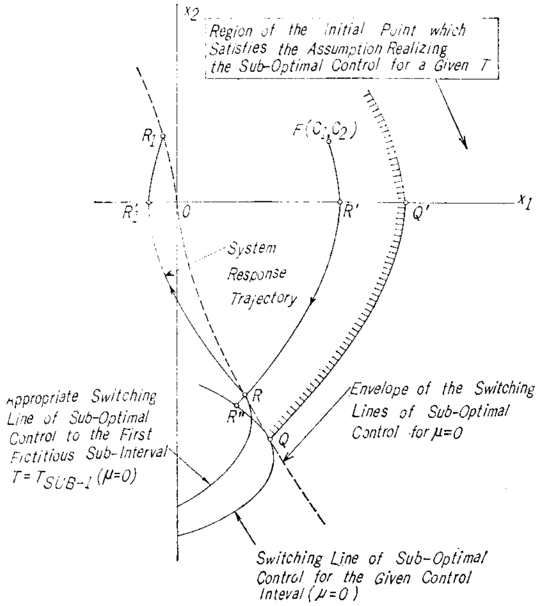

Fig. 7. Geometrical Meaning of the Sub-Interval Optimization Technique.

line for the realization of sub-optimal control at the point R. The intersection of the response trajectory and the sub-optimal switching line derived for the given control interval will be, on the other hand, occurred at the point R''. Then we must introduce the sub-interval optimization technique for this problem. The procedures stated in 6.1 can be interpreted as follows: Firstly we consider the fictitious sub-interval $(0, T_{SUB\text{-}I})$ of which the critical point of sub-optimal switching will be occurred at R. Then the response trajectory switched at R will run toward the point R_1'. Since the time duration t_{01} which the response changes the state from these of the point F to these of R_1' can easily be obtained, then we can calculate both the rest of control interval $T_1 = T - t_{01}$ and the x_1-coordinate of the point R_1'. Secondly, we must examine if the rest of control interval satisfies the assumption realizing the sub-optimal control regarding to the newly computed system state, i.e., the x_1-coordinate of R_1'. By considering the results of examination mentioned above, we may carry out either the determination of switching time by using the sub-optimal switching rule, or the further computation deriving the second fictitious sub-interval. The procedures presented above have to be continued until the final step.

From discussions stated in the previous paragraph, since it turns out that the envelope of the sub-optimal switching line for the case where $\mu=0$ gives the necessary condition realizing the sub-optimal control for $\mu=0$ and $\mu\neq0$, then our interest is directed to deriving the equation of envelope in the phase plane.

By putting $\mu=0$ in Eq. (6.1), the sub-optimal switching line for the negative trajectories becomes

$$\tau^2 + 2\hat{x}_1 + 2\tau\hat{x}_2 = 0 . \tag{6.4}$$

Differentiating the both side of Eq. (6.4) with respect to a variable τ we get the relation as

$$\tau+\hat{x}_2=0. \tag{6.5}$$

From both Eqs. (6.4) and (6.5), by eliminating the variable τ, we can obtain the equation expressing the envelope of the sub-optimal switching line for negative trajectories in the case where $\mu=0$ as follows:

$$2\hat{x}_1-\hat{x}_2^2=0. \tag{6.6}$$

It is evident from Eq. (6.6) that the equation expressing the envelope coincides with the time-minimal switching line[2]. Then in this case, this fact means that the point corresponding to R_1' in Fig. 7 becomes the origin. In general, as concerns a relay control system with time-minimum switching function, it can be considered that the origin becomes a stable equilibrium point. Then the response trajectory once reached the origin should be stayed there during the rest of control interval. The sub-interval optimization technique can, therefore, be realized by the non-linear switching line which is synthesized by combining the time-minimal switching line with the effective sub-optimal switching line for the given control interval on the phase plane. Fig. 8 shows an illustrative example of this procedure, where both the case of $\mu=0$ and $\mu=1.0$ are simultaneously shown with respect to the optimization interval $T=3.0$.* This is the geomatrical meaning of the sub-interval optimization technique applied to the final-value control problems. It is emphasized that the sub-interval optimization technique in this case does not provide a pseudo-switch-

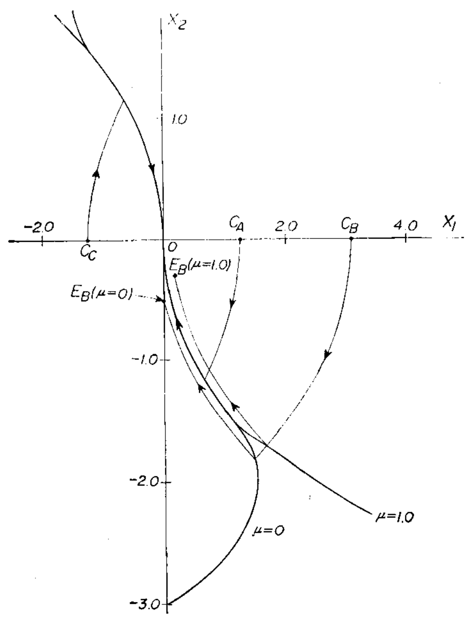

Fig. 8. Illstrative Examples of the Optimum Non-Linear Switching Line derived by the Method in This Paper ($T=3.0$, $a=b=0$, $kL=1.0$).

* Since the switching lines for $\mu\neq0$ as shown in Figs. 5(b) are meaningful only for negative trajectories, then there never occurs a sliding or chattering mode along the switching line of sub-optimal control.

ing rule but an optimal switching rule. It is also examined that the properties of the sub-interval optimization technique applied to the final-value control problem are independent of the mathematical form of controlled systems of second-order with real poles.

7. Conclusions

In this paper, a method of designing the optimal final-value control system with magnitude constraint on the control variable is presented. The second-order linear dynamical systems with the magnitude constraint on the control variable are considered directly, without assigning penalties on the performance criterion. The optimal control problem is formulated as a two point boundary value problem, and a physically meaningful solution of optimal control is obtained by applying the new concept of sub-interval optimization technique which is introduced to provide a pseudo control rule without solving the direct solution of the two point boundary value problem.

It turns out that the concept of sub-interval optimization technique introduced in this paper gives a powerful tool for the solution of such problems, and it will be expected that the method as presently employed can be extended and applied to other types of optimal control problems with magnitude constraint.

Appendix-I: Derivation of the Optimum Switching Function of Sub-Optimal Control for the Double-Integrator Plant

In this appendix, we shall derive the optimum switching function of sub-optimal control for the double integrator plant where both system parameters a and b in Eq. (3.2) are equal to zero. Derivations for the other cases will be performed by a similar procedure.

Substituting $a=b=0$ in Eq. (3.2) and applying a similar method based on Dynamic Programming as stated in Sec. 4 to the minimization procedure, we get the following set of non-linear simultaneous differential equations corresponding to Eqs. (4.7);

$$\left.\begin{array}{ll}
k_0'(\tau) = -k_2(\tau)kL \operatorname{sgn}[\bar{z}(\tau)], & k_0(0)=0 \\
k_1'(\tau) = -k_3(\tau)kL \operatorname{sgn}[\bar{z}(\tau)], & k_1(0)=0 \\
k_2'(\tau) = k_1(\tau)-2k_5(\tau)kL \operatorname{sgn}\bar{z}(\tau), & k_2(0)=0 \\
k_3'(\tau) = 2k_4(\tau), & k_3(0)=0 \\
k_4'(\tau) = 0, & k_4(0)=1 \\
k_5'(\tau) = k_3(\tau), & k_5(0)=\mu
\end{array}\right\} \tag{I-1}$$

We assume that the final instant of control operation T satisfies the assumption realizing the sub-optimal control and that the instant of switching in the reversed time is expressed by $\tau=\tau_s$, which is equivalent to $t=t_s$ in the real time. Furthermore, assuming that the sign of relay output at $t=0$ is positive, we get the following pair of equations;

$$\left.\begin{aligned} k_1'^{+}(\tau) &= -kLk_3(\tau), & k_1^{+}(0) &= 0 \\ k_2'^{+}(\tau) &= k_1^{+}(\tau)-2kLk_5(\tau), & k_2^{+}(0) &= 0 \end{aligned}\right\}, \tag{I-2$_1$}$$

and

$$\left.\begin{aligned} k_1'^{-}(\tau) &= kLk_3(\tau), & k_1^{-}(\tau_s) &= k_1^{+}(\tau_s) \\ k_2'^{-}(\tau) &= k_1^{-}(\tau)+2kLk_5(\tau), & k_2^{-}(\tau_s) &= k_2^{+}(\tau_s) \end{aligned}\right\}, \tag{I-2$_2$}$$

where

$$k_3(\tau) = 2\tau, \quad \text{and} \quad k_5(\tau) = \tau^2+\mu. \tag{I-3}$$

In Eq. (I-2)$_2$, the initial conditions are derived by assuming the continuity of the solution surface for the partial differential equation on the switching boundary. From Eqs. (I-2), the coefficients $k_2^{+}(\tau)$ and $k_2^{-}(\tau)$ are obtained as

$$k_2^{+}(\tau) = \{-\tau^3+2\mu\tau\}kL \tag{I-4$_1$}$$

and

$$\begin{aligned} k_2^{-}(\tau) &= \{\tau^3-2(\tau^2-\mu)\tau-4\mu\tau_s\}kL \\ &= k_2^{+}(\tau)+2(\tau-\tau_s)(\tau_s\tau+2\mu)kL. \end{aligned} \tag{I-4$_2$}$$

By considering the assumption on the switching time mentioned above, we can derive the following from Eq. (4.8) as;

$$z^{+}(\tau) \equiv k_2^{+}(\tau)+k_5(\tau)x_1+2k_3(\tau)x_2 > 0, \quad 0 \le \tau < \tau_s, \tag{I-5$_1$}$$

$$z^{-}(\tau) \equiv k_2^{-}(\tau)+k_3(\tau)x_1+2k_5(\tau)x_2 < 0, \quad \tau_s < \tau \le T. \tag{I-5$_2$}$$

By substituting Eq. (I-4)$_2$ into the function $z^{-}(\tau)$ we get the relation as

$$z^{-}(\tau) = z^{+}(\tau)+2(\tau-\tau_s)(\tau^2+\tau_s\tau+2\mu)kL. \tag{I-6}$$

By considering that the second term in the right hand side of Eq. (I-6) is a non-negative function with respect to $\tau \ge \tau_s$ and that continuity of the solution surface on the switching boundary is assumed, we can derive the following necessary condition which must be satisfied at the time of switching, i.e., $\tau=\tau_s$;

$$z^{+}(\tau)\Big|_{\tau=\tau_s-0} \equiv k_2^{+}(\tau)\Big|_{\tau=\tau_s-0}+k_3(\tau_s)x_1+2k_5(\tau_s)x_2 = 0. \tag{I-7}$$

By substituting Eqs. (I-3) and (I-4)$_1$ into Eq. (1-7), we have the switching function of sub-optimal control for the positive relay output at $t=0$ as follows:

$$z_{SUB}^{+} \equiv -\tau^3+2\tau x_1/kL+2\tau^2 x_2/kL+2\mu(-\tau+x_2/kL). \tag{I-8}$$

On the other hand, the switching function for the negative relay output at $t=0$, can be derived by a similar way as

$$\bar{z}_{SUB} \equiv \tau^3+2\tau x_1/kL+2\tau^2 x_2/kL+2\mu(\tau+x_2/kL). \qquad \text{(I-9)}$$

Eqs. (I-8) and (I-9) are the final results of this appendix, which are the relations in the case where $a=b=0$ corresponding to Eq. (4.8).

Appendix-II: Derivation of the μ-free Relation of Eq. (6.1)

In this appendix, we show the fact that the envelope of the switching lines of the sub-optimal control for $\mu=0$ gives the boundary switching line of the sub-optimal control. The μ-free relations of Eq. (6.1) can be obtained by eliminating the parameter, μ, from Eq. (6.1) as

$$\left.\begin{aligned} &\tau^3+2\tau\hat{x}_1+2\tau^2\hat{x}_2=0 \\ &\tau+\hat{x}_2=0 \end{aligned}\right\}. \qquad \text{(II-1)}$$

The elimination of the variable τ from Eq. (II-1) gives us the relation;

$$2\hat{x}_1-\hat{x}_2^2=0. \qquad \text{(II-2)}$$

Since Eq. (II-1) coincides with the relation expressing the envelope of switching line of the sub-optimal control for $\mu=0$, which is derived in 6.3, then no further presentation is necessary for the verification.

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