Optimum Sampled-Data Control System Design by Dynamic Programming Technique

By

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In this paper, the authors deal with an application of dynamic programming technique to an optimum sampled-data control system design.

Since the optimum sampled-data control problem may be treated as an n-stage decision process, the determination of the optimum control law is carried out by means of the dynamic programming technique. Optimum control policies are derived to fulfill the minimum integral squared error for the deterministic case and the minimum expected value of integral squared error for the stochastic case. It is shown that the control signal of the optimum system consists of a linear combination of system variables. The over-all optimum control system is a time-varying system. However, the quasi-optimum control can be achieved by feeding back all the state variables through appropriate constant multipliers and the quasi-optimum control system can be considered as a good approximation of the optimum system.

1. Introduction

In recent years, modern control technology has made very rapid progress. The core of modern approaches to the control system design rests upon the determination of a control law so as to minimize or maximize a set of performance criteria. In contrast with these approaches, the essence of classical approaches to the design of control systems lies in the determination of a compensator to fulfill a set of arbitrary requirements; and the configuration of the system is more or less fixed ahead of the system design. However, modern approaches will yield an optimum configuration as well as an optimum controller for the system.

The optimum system design problem is essentially formulated as a variational problem and these methods which have been used successfully in many applications are the classical calculus of variation, the Maximum Principle of Pontriyagin[1] and Dynamic Programming of Bellman[2]. Recently

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a number of papers applying the techniques of dynamic programming to the treatment of control problems have appeared in literature. In these papers, some authors have achieved essential results in the optimum design of sampled-data control systems[3),4)].

However, these investigations are restricted to the case optimizing the criterion functions which are expressed by the function of state variables only at sampling instants and does not include the behaivors over the sampling intervals. Forthermore, the discussions for the realization of the optimum control system and the quasi-optimum control system which has nearly the same performance as the optimum one are few. This paper deals with the optimum design of linear sampled-data control systems with a deterministic input and a stochastic input, respectively.

2. Dynamic Programming Approach

Consider the control system shown in Fig. 2.1. The reference input $\boldsymbol{r}$, the state variable $\boldsymbol{c}$ (characterizing the controlled element at any time) and the control signal $\boldsymbol{y}$ are vector quantities and T is a sampling period. The design problem is to make the control signal by the computer so as to optimize the performance criteria or criterion functions (integral squared error, for example). Let $\boldsymbol{x}$ be the value of the state variable at any sampling instant; $\boldsymbol{x}$ is called a initial value of a state variable for any sampling period. If we consider a control signal as a decision and a initial value of a state variable as a state, this control process can be viewed as a discrete multi-stage decision process.

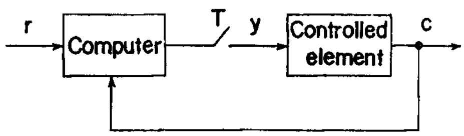

Fig. 2.1. Sampled-data control system.

In this section, we deal only with the optimum design for the deterministic input. Fig. 2.2 shows the relation between time and system variables. The criterion function over the interval $(n-k)T \leq t \leq (n-k+1)T$ may be expressed as a function of the control

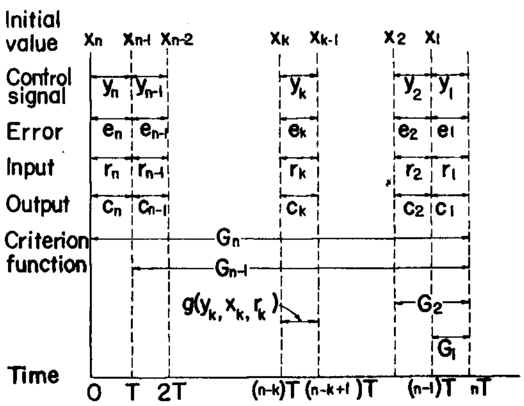

Fig. 2.2. Relation between time and system variables.

signal y_k, the reference input r_k and the inital value x_k. Then, it is written as

$$g(y_k, x_k, r_k) \tag{2.1}$$

If we consider such a control process that the criterion function over the total interval $0 \leq t \leq nT$ is a sum of criterion functions for each interval, the criterion function over the total interval, G_n is given by

$$G_n = g(y_n, x_n, r_n) + g(y_{n-1}, x_{n-1}, r_{n-1}) + \cdots + g(y_1, x_1, r_1) \tag{2.2}$$

Criterion functions of this type arise naturally in the study of control processes.

The dynamic behaivor of a controlled element is usually described by a set of differential equations. Under such process, the initial value x_k is given by

$$x_k = h(x_{k+1}, y_{k+1}) \qquad (k = n-1, \cdots, 1) \tag{2.3}$$

From Eqs. (2.2) and (2.3), the criterion function G_n may be written as

$$G_n(y_n, y_{n-1}, \cdots, y_1, x_n, r_n, r_{n-1}, \cdots, r_1) \tag{2.4}$$

For a deterministic control process, the optimum design problem can be regarded as the minimization problem of a function of n-variables. The sequence of control signals, $(y_n, y_{n-1}, \cdots, y_1)$ which yields the minimum value of criterion function is referred to as an optimum policy. The usual approach to the solution of the n-dimentional minimization problem yields a solution in the n-dimentional form. Then we cannot apply the routine technique of setting partial derivative equal to zero, unless G_n is a function of particularly simple form. As will be shown, the dynamic programming approach will yield a sequence of solutions; first, the choice of y_1, then the choice of y_2, and so on.

First consider a single-stage decision process. The criterion function over the interval $(n-1)T \leq t \leq nT$ is given by

$$G_1(y_1, x_1, r_1) = g(y_1, x_1, r_1) \tag{2.5}$$

It is easy to obtain the control signal y_1 in order to minimize Eq. (2.5). Since the minimum value of the criterion function is a function of the initial value x_1 and the reference input r_1, it may be written as

$$\begin{aligned} f_1(x_1, r_1) &= \min_{y_1}[G_1(y_1, x_1, r_1)] \\ &= \min_{y_1}[g(y_1, x_1, r_1)] \end{aligned} \tag{2.6}$$

Next, consider a two-stage decision process. From Eq. (2.2) the criterion function over the interval $(n-2)T \leq t \leq nT$ is given by

$$G_2(\boldsymbol{y}_2, \boldsymbol{y}_1, \boldsymbol{x}_2, \boldsymbol{r}_2, \boldsymbol{r}_1) = g(\boldsymbol{y}_2, \boldsymbol{x}_2, \boldsymbol{r}_2) + g(\boldsymbol{y}_1, \boldsymbol{x}_1, \boldsymbol{r}_1) \tag{2.7}$$

The optimum design problem is to choose a sequence of allowable control signals $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ in order to minimize Eq. (2.7). From Eq. (2.3), $\boldsymbol{x}_1$ is expressed by a function of the control signal $\boldsymbol{y}_2$ and the initial value $\boldsymbol{x}_2$. Then, the minimum value of Eq. (2.7) can be written as

$$\begin{aligned} f_2(\boldsymbol{y}_2, \boldsymbol{r}_2, \boldsymbol{r}_1) &= \min_{\boldsymbol{y}_2 \cdot \boldsymbol{y}_1} [g(\boldsymbol{y}_2, \boldsymbol{x}_2, \boldsymbol{r}_2) + g(\boldsymbol{y}_1, \boldsymbol{x}_1, \boldsymbol{r}_1)] \\ &= \min_{\boldsymbol{y}_2} [g(\boldsymbol{y}_2, \boldsymbol{x}_2, \boldsymbol{r}_2) + \min_{\boldsymbol{y}_1} \{g(\boldsymbol{y}_1, \boldsymbol{x}_1, \boldsymbol{r}_1)\}] \end{aligned} \tag{2.8}$$

Using Eqs. (2.6) and (2.3), the above equation yields

$$\begin{aligned} f_2(\boldsymbol{x}_2, \boldsymbol{r}_2, \boldsymbol{r}_1) &= \min_{\boldsymbol{y}_2} [g(\boldsymbol{y}_2, \boldsymbol{x}_2, \boldsymbol{r}_1) + f_1(\boldsymbol{x}_1, \boldsymbol{r}_1)] \\ &= \min_{\boldsymbol{y}_2} [g(\boldsymbol{y}_2, \boldsymbol{x}_2, \boldsymbol{r}_2) + f_1\{h(\boldsymbol{y}_2, \boldsymbol{x}_2), \boldsymbol{r}_1\}] \end{aligned} \tag{2.9}$$

In the same manner, the minimum value of the criterion function over the interval $0 \leq t \leq nT$ may be written as

$$f_n(\boldsymbol{x}_n, \boldsymbol{r}_n, \cdots, \boldsymbol{r}_1) = \min_{\boldsymbol{y}_n} [g(\boldsymbol{y}_n, \boldsymbol{x}_n, \boldsymbol{r}_n) + f_{n-1}(\boldsymbol{x}_{n-1}, \boldsymbol{r}_{n-1}, \cdots, \boldsymbol{r}_1)] \tag{2.10}$$

where

$$\boldsymbol{r}_{n-1} = h(\boldsymbol{y}_n, \boldsymbol{x}_n) \tag{2.11}$$

for $n=2, 3, \cdots$.

The term, $g(\boldsymbol{y}_n, \boldsymbol{x}_n, \boldsymbol{r}_n)$, on the right-hand side of Eq. (2.10) is the criterion function over the interval $0 \leq t \leq T$ and the term, $f_{n-1}(\boldsymbol{x}_{n-1}, \boldsymbol{r}_{n-1}, \cdots, \boldsymbol{r}_1)$ represents the minimum value of the criterion function for the final $(n-1)$ stages. And Eq. (2.10) represnts the concept of the principle of optimality[2]. Eq. (2.10) shows that the n-stage decision process is reduced to a sequence of n-single-stage decision processes.

Hence, it is shown that the optimization problem can be solved in an iterative manner. From the recurrence relations of Eqs. (2.10), (2.11) and (2.6), we can inductively obtain the sequence of minimum criterion functions

$$f_1(\boldsymbol{x}_1, \boldsymbol{r}_1), f_2(\boldsymbol{x}_2, \boldsymbol{r}_2, \boldsymbol{r}_1), \cdots, f_n(\boldsymbol{x}_n, \boldsymbol{r}_n, \cdots, \boldsymbol{r}_1) \tag{2.12}$$

and the sequence of optimum control signals, i.e., the optimum policy for the n-stage decision process.

$$\bar{\boldsymbol{y}}_1(\boldsymbol{x}_1, \boldsymbol{r}_1), \bar{\boldsymbol{y}}_2(\boldsymbol{x}_2, \boldsymbol{r}_2, \boldsymbol{r}_1), \cdots, \bar{\boldsymbol{y}}_n(\boldsymbol{x}_n, \boldsymbol{r}_n, \cdots, \boldsymbol{r}_1) \tag{2.13}$$

Using Eq. (2.3), $\bar{\boldsymbol{y}}_n, \bar{\boldsymbol{y}}_{n-1}, \cdots, \bar{\boldsymbol{y}}_1$ are functions of the initial value $\boldsymbol{x}_n$ at $t=0$ and the sampled values $\boldsymbol{r}_k$'s of the reference input. Then, if the reference input is known, the sequence of optimum control signals is determined from Eqs. (2.3) and (2.13).

3. Optimum Design for the Deterministic Case[7]

In this section, for simplicity a design procedure is discussed for a single-variable control system. The reference input is assumed to be a step function with the magnitude r. It is assumed that the controlled element is preceded by a sampler and a zero-order hold circuit shown in Fig. 3.1 so that the

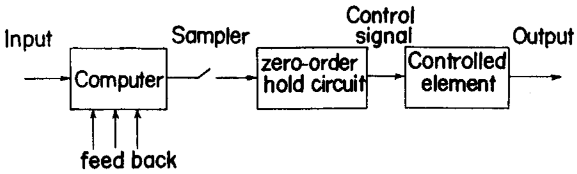

Fig. 3.1. Sampled-data control system with hold circuit.

input into the controlled element over one sampling perod is constant and equal to the magnitude of the control signal at a sampling instant. Therefore, the criterion functions over the interval $0 \leq t \leq nT$ and the interval $(n-k)T \leq t \leq (n-k+1)T$, respectively, may be written as

$$G_n(y_n, y_{n-1}, \cdots, y_1, \boldsymbol{x}_n, r) = \sum_{k=1}^{n} \int_0^T \{r - c_k^{(0)}(\tau)\}^2 d\tau$$

$$\equiv \sum_{k=1}^{n} \int_0^T \{e_k(\tau)\}^2 d\tau \tag{3.1}$$

and

$$g(y_k, \boldsymbol{x}_k, r) = \int_0^T \{r - c_k^{(0)}(\tau)\}^2 d\tau$$

$$\equiv \int_0^T \{e_k(\tau)\}^2 d\tau \qquad (k = n, n-1, \cdots, 1) \tag{3.2}$$

where $e_k(\tau)$ and $c_k^{(0)}(\tau)$, respectively, are the error and the output over the interval $(n-k)T \leq \tau \leq (n-k+1)T$ and τ is measured from the beginning of every sampling period. The controlled element is assumed to be described by an mth order differential equation with constant coefficients, i.e., the controlled element is assumed to be of mth order. Usually, the output of the controlled element, $c^{(0)}(\tau)$, the first derivative of the output, $c^{(1)}(\tau), \cdots$ and the $(m-1)$th derivative of the output, $c^{(m-1)}(\tau)$, are selected as the m components of the state variable $\boldsymbol{c}(\tau)$, and the corresponding initial values can be expressed by $x^{(i)}$'s. Then, the system output $c_k^{(0)}(\tau)$ can be written as

$$c_k^{(0)}(\tau) = a(\tau) y_k + b_0(\tau) x_k^{(0)} + b_1(\tau) x_k^{(1)} + \cdots + b_{m-1}(\tau) x_k^{(m-1)} \tag{3.3}$$

where τ is measured from the beginning of the sampling instant. Substituting Eq. (3.3) into Eq. (3.2), the criterion function $g(y_k, \boldsymbol{x}_k, r)$ yields

$$g(y_k, \boldsymbol{x}_k, r) = A_1(\boldsymbol{x}_k, r) - 2B_1(\boldsymbol{x}_k, r) y_k + C_1 y_k^2 \tag{3.3}$$

where $A_1(\boldsymbol{x}_k, r)$ is a quadratic form of r and $x_k^{(i)}$'s, $B_1(\boldsymbol{x}_k, r)$ is expressed by a linear combination of r and $x_k^{(i)}$'s, and C_1 is a positive constant. The initial values $x_k^{(i)}$'s are given by

$$x_k^{(i)} = [c_{k+1}^{(i)}(\tau)]_{\tau=T} \qquad (i = 0, 1, \cdots, n-1) \tag{3.5}$$

and by differenciating Eq. (3.3) successively, $c_k^{(i)}(\tau)$'s can be obtained, then $c_k^{(i)}(\tau)$ is also expressed by the linear combination of y_k and $x_k^{(i)}$'s.

Now, consider a single-stage decision process, that is, the interval $(n-1)T \leq t \leq nT$. The criterion function is given by

$$g(y_1, \boldsymbol{x}_1, r) = A_1(\boldsymbol{x}_1, r) - 2B_1(\boldsymbol{x}_1, r)y_1 + C_1 y_1^2 \tag{3.6}$$

which is a quadratic equation of y_1 and C_1 is a positive constant, then the control signal $\bar{y}_1$ minimizing Eq. (3.6) and the minimum value of Eq. (3.6) become, respectively,

$$\bar{y}_1 = B_1(\boldsymbol{x}_1, r)/C_1 \tag{3.7}$$

and

$$\begin{aligned} f_1(\boldsymbol{x}_1, r) &= g(\bar{y}_1, \boldsymbol{x}_1, r) \\ &= A_1(\boldsymbol{x}_1, r) - \{B(\boldsymbol{x}_1, r)\}^2/C_1 \end{aligned} \tag{3.8}$$

Next, consider the criterion function over the interval $(n-2)T \leq t \leq nT$. From Eqs. (2.9), (3.4) and (3.8), the minimum value of the criterion function can be written as

$$\begin{aligned} f_2(\boldsymbol{x}_2, r) &= \min_{y_2} [g(y_2, \boldsymbol{x}_2, r) + f_1(\boldsymbol{x}_1, r)] \\ &= \min_{y_2} [A_1(\boldsymbol{x}_2, r) - 2B_1(\boldsymbol{x}_2, r)y_2 + C_1 y_2^2 + A_1(\boldsymbol{x}_1, r) - \{B_1(\boldsymbol{x}_1, r)\}^2/C_1] \end{aligned} \tag{3.9}$$

In Eq. (3.9), the components of the initial value $\boldsymbol{x}_1$ are expressed by a linear combination of y_2 and $x_2^{(i)}$'s from Eqs. (3.3) and (3.5), then Eq. (3.9) can be written as

$$f_2(\boldsymbol{x}_2, r) = \min_{y_2} [A_2(\boldsymbol{x}_2, r) - 2A_2(\boldsymbol{x}_2, r)y_2 + C_2 y_2^2] \tag{3.10}$$

where $A_2(\boldsymbol{x}_2, r)$ is a quadratic form of r and $x_2^{(i)}$'s, $B_2(\boldsymbol{x}_2, r)$ is expressed by a linear combination of r and $x_2^{(i)}$'s and C_2 is a positive constant. From Eq. (3.10), we can obtain the optimum control signal $\bar{y}_2$ and the minimum criterion function $f_2(\boldsymbol{x}_2, r)$ as follows.

$$\bar{y}_2 = B_2(\boldsymbol{x}_2, r)/C_2 \tag{3.11}$$

and

$$f_2(\boldsymbol{x}_2, r) = A_2(\boldsymbol{x}_2, r) - \{B_2(\boldsymbol{x}_2, r)\}^2/C_2 \tag{3.12}$$

In the same manner, the recurrence formula for the n-stage decision process may be written as

$$\begin{aligned} f_n(x_n, r) &= \min_{y_n} [g(y_n, x_n, r) + f_{n-1}(x_{n-1}, r)] \\ &= \min_{y_n} [A_1(x_n, r) - 2B_1(x_n, r)y_n + C_1 y_n^2 \\ &\qquad + A_{n-1}(x_{n-1}, r) - \{B_{n-1}(x_{n-1}, r)\}^2/C_{n-1}] \\ &= \min_{y_n} [A_n(x_n, r) - 2B_n(x_n, r)y_n + C_n y_n^2] \end{aligned} \tag{3.13}$$

where $A_n(x, r)$, $B_n(x, r)$ and C_n are expressed by the same forms as $A_1(x, r)$, $B_1(x, r)$ and C_1, respectively. Then, the optimum control signal $\bar{y}_n$ becomes

$$\bar{y}_n = B_n(x_n, r)/C_n \tag{3.14}$$

From Eq. (3.14), the optimum control signal $\bar{y}_n$ is given by the linear combination of the reference input r and the components of a initial value, $x_n^{(i)}$'s.

From the above discussion, we can obtain the sequence of optimum control signals

$$\bar{y}_n, \bar{y}_{n-1}, \cdots, \bar{y}_2, \bar{y}_1 \tag{3.15}$$

where

$$\left.\begin{aligned} &\bar{y}_n = B_n(x_n, r)/C_n \\ &\bar{y}_{n-1} = B_{n-1}(x_{n-1}, r)/C_{n-1} = B_{n-1}\{h(\bar{y}_n, x_n), r\}/C_{n-1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\bar{y}_1 = B_1(x_1, r)/C_1 = B_1\{h(\bar{y}_2, x_2), r\}/C_1 \end{aligned}\right\} \tag{3.16}$$

It is shown that optimum control signals given by Eq. (3.15) can be expressed by a linear combination of the magnitude of the reference input r and the components of the initial value at $t=0$, $x_n^{(i)}$'s (see Appendix). But as seen above, for example, in Eq. (3.11) or Eq. (3.14), the optimum control signal at any sampling instant consists of a linear combination of system variables at that instant and the reference input. These facts suggest the configuration of the optimum controller. Therefore, it is more important to determine $B_k(x, r)/C_k$, $(k=1, 2, \cdots, n)$ given by Eq. (3.14) than to get the control signals $\bar{y}_k$'s expressed by the function of the reference input r and the initial value at $t=0$, x_n. The optimum controller can be designed as shown in Fig. 3.2,

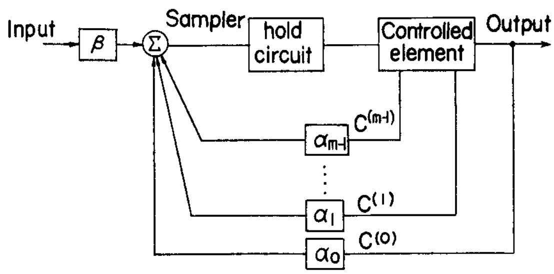

Fig. 3.2. Block diagram of the optimum system.

in which the coefficients a_k and β are constant over every sampling interval and vary at every sampling instant. Then, the optimum control system is a time-varying system and this result is the same as derivcd by some other authors[5),6)].

4. Optimum Design for the Stochastic Case[8)]

In the previous section, the input signal has been considered as a deterministic, or a known function of the time. In many cases of practical interest, this is not the case and the input signal must be considered to be a random function of the time. In this section, assume that the distribution function for the sampled values, r_k's of the reference input signal is known, and for simplicity, consider the staircase random input $r(t)$ as a reference input for the design in place of the continuous random input $R(t)$. As shown in Fig. 4.1 the staircase random input $r(t)$ is taken as the output of zero-order hold circuit, when the continuous random input $R(t)$ applies to the sample-hold circuit. Fig. 4.2 shows the relation between these random inputs. In order to make the analogy with the previous discussion of the deterministic control process, let us parallel the route followed in the previous section.

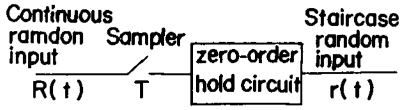

Fig. 4.1. Generation of the staircase random input.

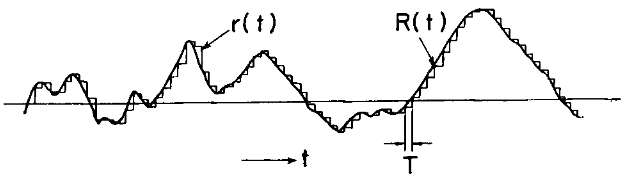

Fig. 4.2. Continuous random input $R(t)$ and staircase random input $r(t)$.

Now, for simplicity of descriptions, by defining

$$X=[r\,x']'=[r\,x^{(0)}x^{(1)}\cdots x^{(m-1)}\mu]' \tag{4.1}$$

as the state variable, the criterion function over the interval $(n-1)T\leq t\leq nT$ given by Eq. (3.6) can be written as

$$g(y_1,X_1)=X_1'A_1X_1-2B_1X_1y_1+C_1y_1^2 \tag{4.2}$$

where μ is the mean value of sampled values of the random reference input, "$'$" expresses the symbol representing the transposed matrix and A_1, B_1 and C_1 are an $(m+2)\times(m+2)$ matrix, an $m+2$ dimensional column vector and a real number, respectively.

Consider the single decision process. Since we can assume that the sampled value of the input at $t=(n-1)T$, r_1 is known, the control signal $\bar{y}_1$ minimizing Eq. (4.2) and the minimum value of Eq. (4.2), respectively, can written as

$$\bar{y}_1 = \boldsymbol{B}_1\boldsymbol{X}_1/C_1 \tag{4.3}$$

and

$$\begin{aligned} f_1(\boldsymbol{X}_1) &= g(\bar{y}_1, \boldsymbol{X}_1) = \boldsymbol{X}_1'[\boldsymbol{A}_1-(\boldsymbol{B}_1'\boldsymbol{B}_1)/C_1]\boldsymbol{X}_1 \\ &= [r_1\ \boldsymbol{x}_1'] \begin{bmatrix} a_1 & \boldsymbol{b}_1 \\ \boldsymbol{b}_1' & \boldsymbol{d}_1 \end{bmatrix} \begin{bmatrix} r_1 \\ \boldsymbol{x}_1 \end{bmatrix} \\ &= r_1^2 a_1 + 2r_1\boldsymbol{b}_1\boldsymbol{x}_1 + \boldsymbol{x}_1'\boldsymbol{d}_1\boldsymbol{x}_1 \end{aligned} \tag{4.4}$$

where a_1, $\boldsymbol{b}_1$ and $\boldsymbol{d}_1$ are a real number, an $m+1$ dimensional column vector and an $(m+1)\times(m+1)$ matrix, respectively. Let $\varphi(r_1)$ be the probability distribution function for sampled values of the reference input r_1 and if the mean value and the variance of the input r_1 are given by μ and σ^2, the expected value of Eq. (4.4) becomes

$$\begin{aligned} F_1(\boldsymbol{x}_1) &= \int_{-\infty}^{\infty} f_1(\boldsymbol{X}_1)\varphi(r_1)dr_1 = \underset{r_1}{E}[f_1(\boldsymbol{X}_1)] \\ &= (\mu^2+\sigma^2)a_1 + 2\mu\boldsymbol{b}_1\boldsymbol{x}_1 + \boldsymbol{x}_1'\boldsymbol{d}_1\boldsymbol{x}_1 \\ &= \sigma^2 a_1 + \boldsymbol{x}_1'\boldsymbol{H}_1\boldsymbol{x}_1 \end{aligned} \tag{4.5}$$

where $E[\ ]$ expresses the symbol representing the expected value and $\boldsymbol{H}_1$ is an $(m+1)\times(m+1)$ matrix and is given by

$$\boldsymbol{H}_1 = \begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & a_1 \end{bmatrix} + \begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ & \boldsymbol{b}_1 & \end{bmatrix} + \begin{bmatrix} 0 & \cdots\cdots & 0 & \\ \vdots & & \vdots & \boldsymbol{b}_1' \\ 0 & \cdots\cdots & 0 & \end{bmatrix} + \boldsymbol{d}_1 \tag{4.6}$$

From Eqs. (3.3) and (3.5), the initial value at $t=(n-1)T$, $\boldsymbol{x}_1$ can be written as

$$\boldsymbol{x}_1 = \boldsymbol{h}\begin{bmatrix} \boldsymbol{x}_2 \\ y_2 \end{bmatrix} \tag{4.7}$$

where $\boldsymbol{h}$ is called the state transition matrix and depends only upon the dynamic behaivor of the controlled element. By using Eq. (4.7), the second term of the right-hand side of Eq. (4.5) becomes

$$\begin{aligned} \boldsymbol{x}_1'\boldsymbol{H}_1\boldsymbol{x}_1 &= [\boldsymbol{x}_2'\ y_2]\boldsymbol{h}'\boldsymbol{H}_1\boldsymbol{h}\begin{bmatrix} \boldsymbol{x}_2 \\ y_2 \end{bmatrix} \\ &= [\boldsymbol{X}_2'\ y_2]\begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ \vdots & \boldsymbol{h}'\boldsymbol{H}_1\boldsymbol{h} & \\ 0 & & \end{bmatrix}\begin{bmatrix} \boldsymbol{X}_2 \\ y_2 \end{bmatrix} \\ &= [\boldsymbol{X}_2'\ y_2]\begin{bmatrix} \boldsymbol{H}_{1p} & \boldsymbol{H}_{1q}' \\ \boldsymbol{H}_{1q} & H_{1r} \end{bmatrix}\begin{bmatrix} \boldsymbol{X}_2 \\ y_2 \end{bmatrix} \end{aligned} \tag{4.8}$$

where $\boldsymbol{h}$, $\boldsymbol{H}_{1p}$, $\boldsymbol{H}_{1q}$ and H_{1r} are an $(m+1)\times(m+2)$ matrix, an $(m+2)\times(m+2)$ matrix, an $m+2$ dimensional column vector and a real number, respectively. Using Eqs. (4.5) and (4.8), we shall obtain the minimum expected value of integral squared error over the interval $(n-2)T\leq t\leq nT$ as

$$\begin{aligned} F_2(\boldsymbol{X}_2) &= \underset{r_2}{E}[\underset{y_2}{\min}\{g(y_2, \boldsymbol{X}_2)+F_1(\boldsymbol{x}_1)\}] \\ &= \underset{r_2}{E}[\underset{y_2}{\min}\{\sigma^2 a_1+\boldsymbol{X}_2'\boldsymbol{A}_2\boldsymbol{X}_2-2\boldsymbol{B}_2\boldsymbol{X}_2 y_2+C_2 y_2^2\}] \end{aligned} \tag{4.9}$$

where

$$\left.\begin{aligned} \boldsymbol{A}_2 &= \boldsymbol{A}_1+\boldsymbol{H}_{1p} \\ \boldsymbol{B}_2 &= \boldsymbol{B}_1-\boldsymbol{H}_{1q} \\ C_2 &= C_1+H_{1r} \end{aligned}\right\} \tag{4.10}$$

If we assume that the reference input at $t=(n-2)T$, r_2 is known, the optimum control signal $\bar{y}_2$ can be otained as

$$\bar{y}_2 = \boldsymbol{B}_2\boldsymbol{X}_2/C_2 \tag{4.11}$$

In the same manner, consider the n-stage control process. The recurrence formula may be written as

$$\begin{aligned} F_n(\boldsymbol{x}_n) &= \underset{r_n}{E}[f_n(\boldsymbol{X}_n)] \\ &= \underset{r_n}{E}[\min\{g(y_n, \boldsymbol{X}_n)+F_{n-1}(\boldsymbol{x}_{n-1})\}] \end{aligned} \tag{4.12}$$

Therefore, by the same method as mentioned above, the minimum expected value of integral squared error over the total interval can be obtained as

$$\begin{aligned} F_n(x_n) &= \underset{r_n}{E}[\underset{y_n}{\min}\{\sigma^2\sum_{k=1}^{n-1}a_k+\boldsymbol{X}_n'\boldsymbol{A}_n\boldsymbol{X}_n-2\boldsymbol{B}_n\boldsymbol{X}_n y_n+C_n y_n^2\}] \\ &= \underset{r_n}{E}[\sigma^2\sum_{k=1}^{n-1}a_k+r_n^2 a_n+2r_n\boldsymbol{b}_n\boldsymbol{x}_n+\boldsymbol{x}_n'\boldsymbol{d}_n\boldsymbol{x}_n] \\ &= \sigma^2\sum_{k=1}^{n-1}a_k+\boldsymbol{x}_n'\boldsymbol{H}_n\boldsymbol{x}_n \end{aligned} \tag{4.13}$$

where $\boldsymbol{A}_n$, $\boldsymbol{B}_n$, C_n etc. are given by the recurrence formula in Eq. (4.17). And the optimum control signal $\bar{y}_n$ can also obtained as

$$\bar{y}_n = \boldsymbol{B}_n\boldsymbol{X}_n/C_n \tag{4.14}$$

From the above discussion, the sequence of optimum control signals, i.e., the optimum policy for the n-stage control process can be written as

$$\bar{y}_n, \bar{y}_{n-1}, \cdots, \bar{y}_2, \bar{y}_1 \tag{4.15}$$

where

$$\bar{y}_k = \boldsymbol{B}_k\boldsymbol{X}_k/C_k \qquad (k = n, n-1, \cdots, 1) \tag{4.16}$$

In Eq. (4.16), $\boldsymbol{B}_k$ and C_k can be obtained from the following recurrence formulas:

$$\left.\begin{array}{l}
\begin{bmatrix} a_{k-1} & \boldsymbol{b}_{k-1} \\ \boldsymbol{b}'_{k-1} & \boldsymbol{d}_{k-1} \end{bmatrix} = [\boldsymbol{A}_{k-1}-(\boldsymbol{B}'_{k-1}\boldsymbol{B}_{k-1})/C_{k-1}] \\
\boldsymbol{H}_{k-1} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a_{k-1} \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ & \boldsymbol{b}_{k-1} & \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & \\ \vdots & & \vdots & \boldsymbol{b}'_{k-1} \\ 0 & \cdots & 0 & \end{bmatrix} + \boldsymbol{d}_{k-1} \\
\begin{bmatrix} \boldsymbol{H}_{k-1,p} & \boldsymbol{H}'_{k-1,q} \\ \boldsymbol{H}_{k-1,q} & H_{k-1,r} \end{bmatrix} = \begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & \boldsymbol{h}'\boldsymbol{H}_{k-1}\boldsymbol{h} & \\ 0 & & \end{bmatrix} \\
\boldsymbol{A}_k = \boldsymbol{A}_1 + \boldsymbol{H}_{k-1,p} \\
\boldsymbol{B}_k = \boldsymbol{B}_1 - \boldsymbol{H}_{k-1,q} \\
C_k = C_1 + H_{k-1,r} \\
\qquad\qquad (k = n,\ n-1, \cdots, 1)
\end{array}\right\} \tag{4.17}$$

where $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, C_1 and $\boldsymbol{h}$ depend only upon the dynamics of the controlled element.

From Eq. (4.17), $\boldsymbol{B}_1, C_1$; $\boldsymbol{B}_2, C_2$; $\cdots$ can be successively evaluated. It is observed that the optimum control signal $\bar{y}_k$ given by Eq. (4.16) can be expressed by a linear combination of r_k, μ and $x_k^{(i)}$'s. Therefore, the controller can be realized by the similar configuration as shown in Fig. 3.2. The difference in Eq. (4.16) from Eq. (3.14) is to add the term for the mean value of a random input, μ. If the mean values of random inputs are equal to each other, the optimum controllers are same regardless of the type of a probability distribution function of the input. Forthermore, it is shown that the function $\boldsymbol{B}_k\boldsymbol{X}_k/C_k$ converges to any function having the form, $\boldsymbol{BX}/C$, as k approaches infinity. If the control system operate for a long time, the optimum control signal can be approximately represented by

$$\bar{y}_k = \boldsymbol{B}\boldsymbol{X}_k/C \qquad (k = n,\ n-1, \cdots, 1) \tag{4.18}$$

In this case, the control system is called a quasi-optimum control system and time-invariant. As will be shown in the following section, it can be considered as a good approximation of the optimum system.

5. Examples and Discussions

(a) Design for a first order controlled element

Consider the control system shown in Fig. 5.1. First, we assume that the reference input is a step function. The dynamic characterization of the controlled element over any sampling interval is described by the first order differential equation

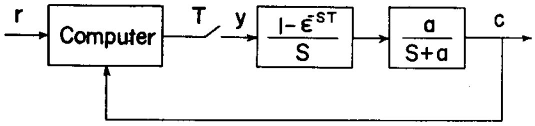

Fig. 5.1. Sampled-data control system with a first order controlled element.

$$\frac{dc(\tau)}{d\tau}+ac(\tau)=ay \tag{5.1}$$

where a and y are constants. The solution of Eq. (5.1) can be obtained as

$$c(\tau)=(1-\varepsilon^{-a\tau})y+\varepsilon^{-a\tau}x \tag{5.2}$$

where x is a initial value at $\tau=0$. By using Eq. (5.2), the integral squared error over the interval $0\leq\tau\leq T$ is obtained as

$$g(y, x, r)=A_1(x, r)-2B_1(x, r)y+C_1y^2 \tag{5.3}$$

where

$$\left.\begin{aligned} A_1(x, r) &= \{aTr^2-2(1-\varepsilon^{-aT})xr+(1-\varepsilon^{-2aT})x^2/2\}/a \\ B_1(x, r) &= \{(aT+\varepsilon^{-aT}-1)r-(1-\varepsilon^{-aT})^2x/2\}/a \\ C_1 &= \{(aT+\varepsilon^{-aT}-1)-(1-\varepsilon^{-aT})^2/2\}/a \end{aligned}\right\} \tag{5.4}$$

Now, consider a single-stage decision process. From Eqs. (5.3) and (5.4), the optimum control signal $\bar{y}_1$ minimizing the integral squard error over the interval $(n-1)T\leq t\leq nT$ and the minimum integral squared error may be written as

$$\bar{y}_1=(p_1r+q_1x_1)/l_1 \tag{5.5}$$

and

$$f_1(x_1, r)=\alpha_1(r-x_1)^2/a \tag{5.6}$$

where

$$\left.\begin{aligned} p_1 &= 2(aT+\varepsilon^{-aT}-1) \\ q_1 &= -(1-\varepsilon^{-aT})^2 \\ l_1 &= p_1+q_1 \\ \alpha_1 &= aT-p_1^2/2l_1 \end{aligned}\right\} \tag{3.7}$$

Using Eqs. (5.2), (5.3) and (5.6), the terms, $A_2(x_2, r)$, $B_2(x_2, r)$ and C_2 in Eq. (3.10) are determined as follows:

$$\left.\begin{aligned} A_2(x_2, r) &= [aTr^2-2(1-\varepsilon^{-aT})x_2r+(1-\varepsilon^{-aT})x_2^2/2 \\ &\quad +\alpha_1(r^2-2\varepsilon^{-aT}x_2r+\varepsilon^{-aT}x_2^2)]/a \\ B_2(x_2, r) &= [(p_1r+q_1x_2)/2-\alpha_1\{2\varepsilon^{-aT}(1-\varepsilon^{-aT})x_2-2(1-\varepsilon^{-aT})r\}/2]/a \\ C_1 &= [l_1/2+(1-\varepsilon^{-aT})^2\alpha_1]/a \end{aligned}\right\} \tag{5.8}$$

Hence, in the same manner as described in Section 3, we may determine

$\bar{y}_2, \bar{y}_3, \cdots$ and $f_2(x_2, r)$, $f_3(x_3, r), \cdots$, succesively. Therefore, the optimum control signal $\bar{y}_n$ and the minimum value of the criterion function over the total interval can be obtained as

$$\bar{y}_n = B_n(x_n, r)/C_n = (p_n r + q_n x_n)/l_n \tag{5.9}$$

and

$$f_n(x_n, r) = a_n(r - x_n)^2/a \tag{5.10}$$

where

$$\left.\begin{aligned} p_n &= p_1 + 2(1-\varepsilon^{-aT})a_{n-1} \\ q_n &= q_1 - 2\varepsilon^{-aT}(1-\varepsilon^{-aT})a_{n-1} \\ l_n &= p_n + q_n \\ a_n &= aT + a_{n-1} - p_n^2/2l_n \end{aligned}\right\} \tag{5.11}$$

We can successively determine the sequence of optimum control signals, $(\bar{y}_n, \bar{y}_{n-1}, \cdots, \bar{y}_1)$, by the recurrence formulas of Eqs, (5.9), (5.11) and (5.7).

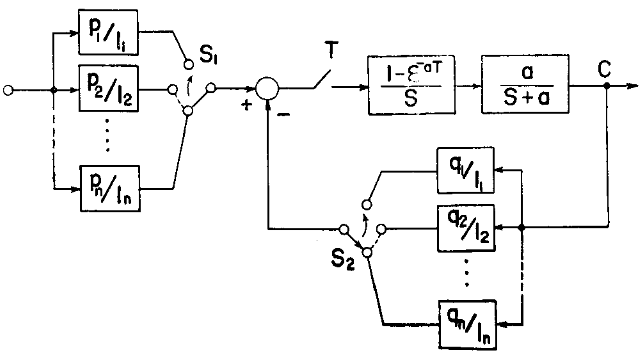

Fig. 5. 2. Realization of optimum control system.

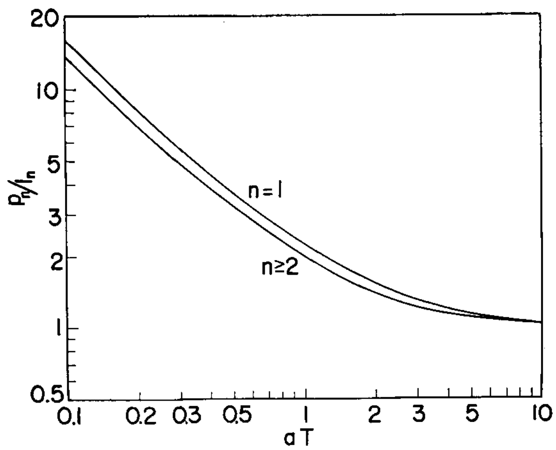

Fig. 5. 3. Condition for an optimum response of the control system shown in Fig. 5. 1.

The block diagram of the optimum control system can be shown in Fig. 5.2, in which switches S_1 and S_2 shift synchronously with the sampler S to the direction of arrows. Figs. 5.3 and 5.4 indicate p_n/l_n and α_n given by Eq. (5.11), respectively. The terms, p_n/l_n and α_n, converge to the costant values, p/l and α, as n increase infinity, then the quasi-optimum controller can be constructed by simple circuits shown in Fig. 5.5 and the quasi-optimum system is time-invariant.

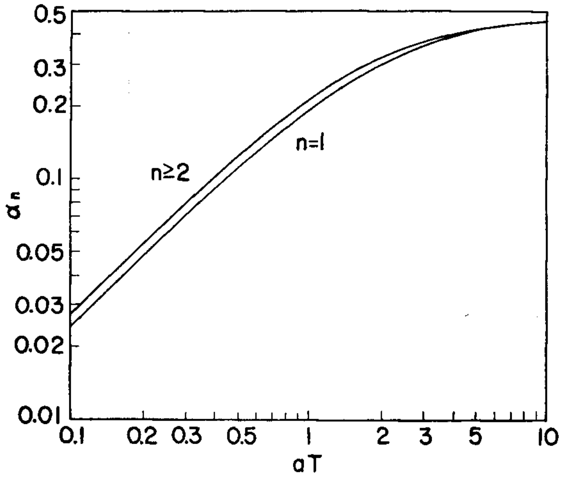

Fig. 5.4. α_k given by Eq. (5.11).

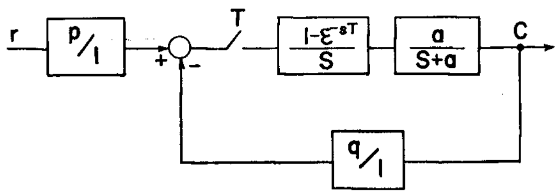

Fig. 5.5. Quasi-optimum control system.

Next, consider obtaining the optimum control law for the stochastic input. We assume that the probability distribution function for the sampled values of the reference input is the normal distribution, $N(\mu, \sigma^2)$. In this example, the terms, X, A_1, B_1, C_1 and h may be written as follows:

$$X = \begin{bmatrix} r \\ x \\ \mu \end{bmatrix}$$

$$A_1 = \frac{1}{a}\begin{bmatrix} aT & -(1-\varepsilon^{-aT}) & 0 \\ -(1-\varepsilon^{-aT}) & (1-\varepsilon^{-2aT})/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} B_1 &= [(aT+\varepsilon^{-aT}-1) \quad -(1-\varepsilon^{-aT})^2/2 \quad 0]/a \\ &\equiv [p_1' \quad q_1' \quad s_1']/a \\ C_n &= \{aT+\varepsilon^{-aT}-1-(1-\varepsilon^{-aT})^2/2\}/a \\ &\equiv l_1'/a = (p_1'+q_1')/a \\ h &= \begin{bmatrix} \varepsilon^{-aT} & 0 & 1-\varepsilon^{-aT} \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}\right\} \tag{5.12}$$

As mentioned in Section 4, the optimum contol signal over the interval $(n-k)T \leq t \leq (n-k+1)T$ may be written as

$$\bar{y}_k = (p_k' r_k + q_k' x_k + s_k' \mu)/l_k' \tag{5.13}$$

By making use of Eq. (4. 17), p_k', q_k', s_k' and l_k' can be determined by the recurrence formulas:

$$\left.\begin{aligned} p_k' &= p_1' \\ q_k' &= q_1'-(1-\varepsilon^{-aT})\varepsilon^{-aT}\gamma_{k-1} \\ s_k' &= s_1'-(1-\varepsilon^{-aT})(\beta_{k-1}+\eta_{k-1}) \\ l_k' &= l_1'+(1-\varepsilon^{-aT})^2\gamma_{k-1} \\ \alpha_k &= aT-p_k'^2/l_k' \\ \beta_k &= -(1-\varepsilon^{-aT})-p_k'q_k'/l_k' \\ \gamma_k &= (1-\varepsilon^{2aT})/2+\varepsilon^{-2aT}\gamma_{k-1}+q_k'^2/l_k' \\ \zeta_k &= -p_k's_k'/l_k' \\ \eta_k &= \varepsilon^{-aT}(\beta_{k-1}+\eta_{k-1})-q_k's_k'/l_k' \\ \xi_k &= \alpha_{k-1}+\xi_{k-1}+2\zeta_{k-1}-s_k'^2/l_k' \end{aligned}\right\} \tag{5.14}$$

where

$$\begin{bmatrix} a_k & b_k \\ b_k' & d_k \end{bmatrix} \equiv \frac{1}{a}\begin{bmatrix} \alpha_k & \beta_k & \zeta_k \\ \beta_k & \gamma_k & \eta_k \\ \zeta_k & \eta_k & \xi_k \end{bmatrix}$$

Fig. 5. 6 shows p_k'/l_k', q_k'/l_k' and s_k'/l_k' as a function of aT. These values converge to the constant values as k approaches infinity, then the quasi-optimum control system can be realized with a simple configuration. In Eq. (5. 13), the coefficient of initial value, q_k'/l_k' is the same as q_k/l_k in Eq. (5. 9) obtained for the deterministic case and $(p_k'+s_k')/l_k'$ is equal to p_k/l_k in Eq. (5. 9). Therefore, if μ is taken equal to r_k in Eq. (5. 13), Eq. (5. 13) coincides with Eq. (5. 9). From Eq. (4. 13), the expected value of the integral squared error, $F_n(x_n)$ is given by

$$\begin{aligned} F_n(x_n) &= \sigma^2\sum_{k=1}^{n} a_k + H_{11}^{(n)}x_n^2+(H_{12}^{(n)}+H_{21}^{(n)})x_n\mu+H_{22}^{(n)}\mu^2 \\ &= \sigma^2\sum_{k=1}^{n} a_k + H_{11}^{(n)}(x_n^2+\mu^2)+2H_{12}^{(n)}x_n\mu \end{aligned} \tag{5.15}$$

where

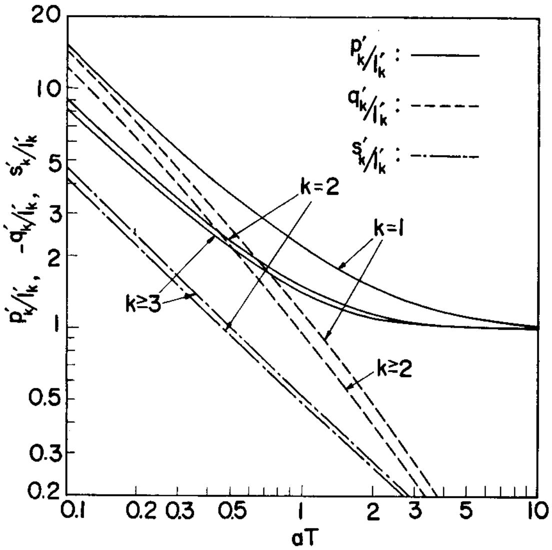

Fig. 5.6. Conditions for an optimum response of the control system shown in Fig. 5.1.

$$H_n = \begin{bmatrix} H_{11}^{(n)} & H_{12}^{(n)} \\ H_{21}^{(n)} & H_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} H_{11}^{(n)} & H_{12}^{(n)} \\ H_{12}^{(n)} & H_{11}^{(n)} \end{bmatrix}$$

and

$$a_k = \alpha_k / a \tag{5.16}$$

when the control system operates for a long time, the expected value of the integral squared error over one sampling period is equal to $a_n\sigma^2$ and it is proportional to a variance of the reference input, σ^2. Fig. 5.7 shows the relation between aT and a_k. For the comparison of performances between the optimum system and the quasi-optimum system, the expected values of the integral squared error of both systems are shown in Fig. 5.8. It is clear that the quasi-optimum system is a good approximation of the optimum system.

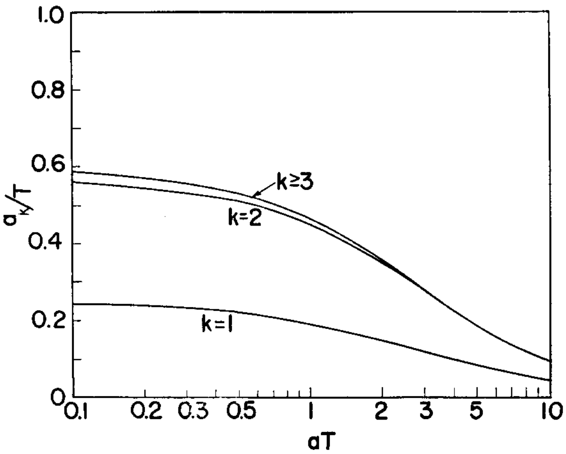

Fig. 5.7. a_k given by Eq. (5.14).

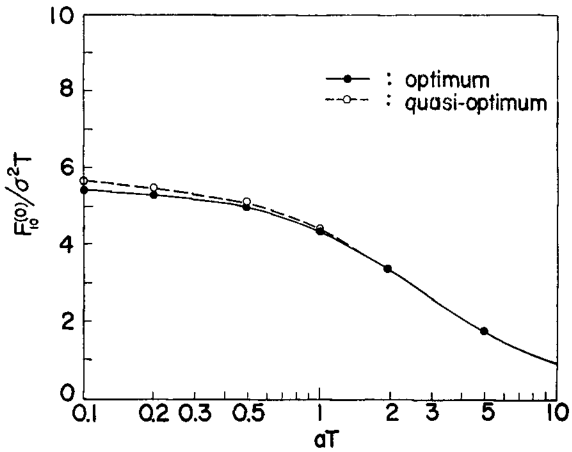

Fig. 5.8. Expected values of integral squared error of optimum and quasi-optimum control systems.

(b) Design for a second order controlled element

Consider the control system with a second order controlled element shown in Fig. 5.9 and assume that the reference input is a step function. The optimum control signal over the interval $(n-k)T \leq t \leq (n-k+1)T$ can be obtained as

$$\bar{y}_k = p_k''(r-x_k)+q_k''\dot{x}_k \quad (5.17)$$

where x_k and $\dot{x}_k$ are the intial values of a system output and its derivative, at $t=(n-k)T$. The terms, p_k'' and q_k'' are also given by the similar recurrence formulas and, as shown in Fig. 5.10, these terms converge to constant valve when k approaches infinity. If we assume that a derivative of the output, $\dot{c}(t)$ is measureable, the optimum and

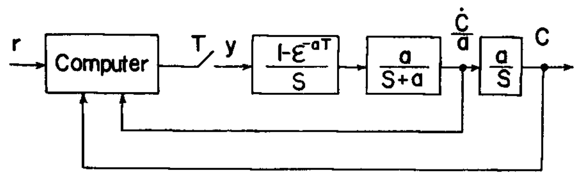

Fig. 5.9. Sampled-data control system with a second order controlled element.

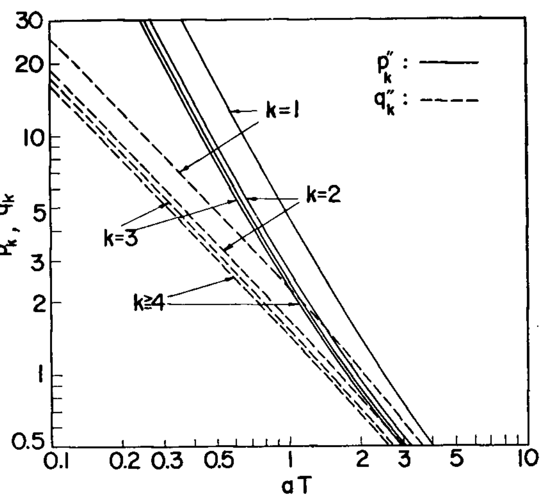

Fig. 5.10 Conditions for an optimum response of the control system shown in Fig. 5.9.

quasi-optimum controls can be achieved by the similar configurations as the previous example.

(c) Design for a ramp input

Consider the optimum design for a ramp input. In this case, the optimum control signals for the system shown in Fig. 5.1 can be obtained as

$$\bar{y}_k = (p_k r_k + q_k x_k + h_k r' T)/l_k \qquad (k = n, n-1, \cdots, 1) \tag{5.18}$$

where r_k and x_k are respectively the reference input and the output at $t=(n-k)T$ and r' is a slop of a reference input. In Eq. (5.18), p_k, q_k and l_k is given by Eq. (5.11) and h_k is obtained as

$$h_k = h_1 - (1-\varepsilon^{-aT})(\zeta_{k-1} - 2a_{k-1}) \tag{5.19}$$

where

$$\zeta_{k-1} = 2(\varepsilon^{-aT} - 1 + aT\varepsilon^{-aT})/aT + (\zeta_{k-2} - 2a_{k-2})\varepsilon^{-aT} + q_{k-1}h_{k-1}/l_{k-1} \tag{5.20}$$

The responses for a ramp input are illustrated in Fig. 5.11, where these responses A and B are corresponding to the optimum systems for a step input and for a ramp input, respectively. In the case of the optimum system for a ramp input, the system has a zero steady state error, but the error is not zero in the optimium system for a step input.

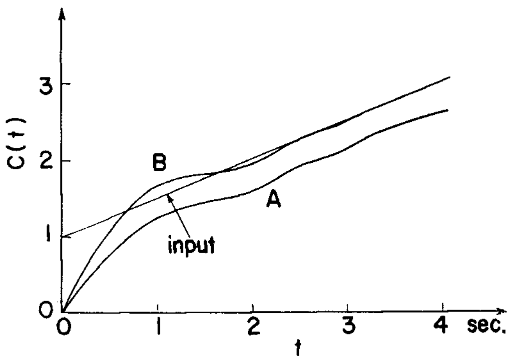

Fig. 5.11. Responses of the control system shown in Fig. 5.1 for a ramp input.

6. Conclusion

A method for designing an optimum sampled-data control system has been introduced. The controller designed in this paper is optimum in the sense of the minimum integral squared error for a deterministic case or the minimum expected value of integral squared error for a stochastic case. By use of dynamic programming technique, it is shown that the optimum control law is a function of the state variables of the system and the over-all optimum system is a time-varying system. However, when the control system operates for a long time, the quasi-optimum control system having a simple configuration can be considered as a good approximation of the optimum system.

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Appendix

From Eq. (3.16), $\bar{y}_n$ and $\bar{y}_{n-1}$ can be written as

$$\bar{y}_n = [d_n^{(0)} d_n^{(1)} \cdots d_n^{(m-1)} d_n] \begin{bmatrix} x_n^{(0)} \\ \vdots \\ x_n^{(m-1)} \\ r \end{bmatrix} \tag{A.1}$$

$$\bar{y}_{n-1} = [d_{n-1}^{(0)} d_{n-1}^{(1)} \cdots d_{n-1}^{(m-1)} d_{n-1}] \begin{bmatrix} x_{n-1}^{(0)} \\ \vdots \\ x_{n-1}^{(m-1)} \\ r \end{bmatrix} \tag{A.2}$$

where $d_n^{(0)}, d_n^{(1)}, \cdots, d_n;\ d_{n-1}^{(0)}, d_{n-1}^{(1)}, \cdots, d_{n-1}$ are constants. By using Eq. (3.3), $x_{n-1}^{(i)}$'s in Eq. (A.2) yields

$$\begin{bmatrix} x_{n-1}^{(0)} \\ x_{n-1}^{(1)} \\ \vdots \\ x_{n-1}^{(m-1)} \end{bmatrix} = \begin{bmatrix} a^{(0)}(T) & b_0^{(0)}(T) & \cdots & b_{m-1}^{(0)}(T) \\ a^{(1)}(T) & b_0^{(1)}(T) & \cdots & b_{m-1}^{(1)}(T) \\ \vdots & \vdots & & \\ a^{(m-1)}(T) & b_0^{(m-1)}(T) & & b_{m-1}^{(m-1)}(T) \end{bmatrix} \begin{bmatrix} \bar{y}_n \\ x_n^{(0)} \\ \vdots \\ x_n^{(m-1)} \end{bmatrix} \tag{A.3}$$

where

$$\left.\begin{aligned} &a^{(0)}(T) = [a(\tau)]_{\tau=T}, \qquad a^{(1)}(T) = \left[\frac{d}{d\tau} a(\tau)\right]_{\tau=T}, \cdots \\ &a^{(m-1)}(T) = \left[\frac{d^{m-1}}{d\tau^{m-1}} a(\tau)\right]_{\tau=T}, \qquad b_0^{(0)}(T) = [b_0(\tau)]_{\tau=T}, \\ &b_0^{(1)}(T) = \left[\frac{d}{d\tau} b_0(\tau)\right]_{\tau=T}, \qquad \cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{A.4}$$

Substituting Eq. (A.3) into Eq. (A.2), $\bar{y}_{n-1}$ becomes

$$\bar{y}_{n-1} = \bar{y}_n \sum_{i=1}^{m-1} d_{n-1}^{(i)} a^{(i)}(T) + \left[\sum_{i=0}^{m-1} d_{n-1}^{(i)} b_0^{(i)}(T) \cdots \sum_{i=0}^{m-1} d_{m-1}^{(i)} b_{m-1}^{(i)}(T)\right] \begin{bmatrix} x_n^{(0)} \\ \vdots \\ x_n^{(m-1)} \end{bmatrix} + d_{n-1} r \tag{A.5}$$

By use of Eq. (A.1), Eq. (A.5) yields

$$\bar{y}_{n-1} = [d_n^{(0)} \sum_{i=0}^{m-1} d_{n-1}^{(i)} a^{(1)}(T) + \sum_{i=0}^{m-1} d_{n-1}^{(i)} b_i^{(i)}(T) \cdots d_n^{(n-1)} \sum_{i=0}^{m-1} d_{n-1}^{(i)} a^{(1)}(T)$$
$$+ \sum_{i=0}^{m-1} d_{n-1}^{(i)} b_{m-1}^{(i)}(T) + d_n \sum_{i=0}^{m-1} d_{n-1}^{(i)} a^{(1)}(T) + d_{n-1}] \begin{bmatrix} x_n^{(0)} \\ \vdots \\ x_n^{(m-1)} \\ r \end{bmatrix} \quad \text{(A. 4)}$$

In the same manner, we can derive $\bar{y}_{n-1}, \bar{y}_{n-2}, \cdots, \bar{y}_1$ expressed by a linear combination of r and $x_n^{(i)}$'s.