Low-frequency Oscillation of a Bounded Plasma in an External Magnetic Field

By

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(Received September 29, 1965)

Low-frequency plasma waves with non-axisymmetric modes are generally analyzed under the existence of an ion beam. A boundary condition, that a cylindrical plasma is coaxially immersed in a cylindrical current sheet with a vacuum clearance, is taken into account. Oscillating electromagnetic field and electron and ion motions associated with the waves are also examined. Natural oscillations with a free boundary and cylindrical plasma waves surrounded by a conducting cylinder are found to exist. Ion cyclotron waves of axisymmetric modes found by Stix can be derived as a special case of these modes.

1. Introduction

Low-frequency magnetohydrodynamic waves, which appear in a fully ionized plasma immersed in a strong magnetic field, are of our interest from the view-point of plasma heating, because it offers an effective method of plasma heating to convert the energy of the waves into thermal plasma energy through damping mechanisms of the waves. Stix[1] and Dawson[2] found that low-frequency transverse waves, the frequencies of which are slightly lower than the ion cyclotron frequency, are very useful for this purpose owing to their effective damping termed "cyclotron damping". Stix and his collaborators experimentally verified the thermalization of ion cyclotron waves by using Model B-65 Stellarator[3] and Model B-66 Stellarator[4]. Wilcox and his coworkers[5] oberved the cyclotron damping of torsional Alfvén waves.

These theoretical and experimental works, however, only concern axisymmetric modes of the waves, except for Bernstein and Trehan's theoretical approach[6]. Bernstein and Trehan found a way to obtain the dispersion relation of non-axisymmetric modes of ion cyclotron waves. However, they treated the case without ion beam currents and did not give the detail physical picture of the waves.

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Here, magnetohydrodynamic transverse waves in a plasma, in which an ion beam is flowing, are generally analyzed. The approach presented here is akin to the Bernstein and Trehan's treatment. The method to introduce the phase relation of electromagnetic field in the waves is also similar to Stix's treatment.

Physical enviroment considered is as follows.

(i) A cylindrical plasma is immersed in a strong magnetic field, the direction of which is parallel to the centre axis of the cylinder.

(ii) Oscillation frequency, ω, is sufficiently low compared with both electron plasma frequency, ω_{pe}, and electron cyclotron frequency, ω_e, .

$$\omega^2 \ll \omega_{pe}^2 + \omega_e^2 \tag{1}$$

(iii) Electron mass, M_e, can be neglected in comparison with ion mass, M_i. That is,

$$\frac{M_e}{M_i} \simeq 0, \tag{2}$$

(iv) The plasma is so rarefied that interparticle collisions are negligible and therefore particles interact with each other through long range electromagnetic interactions.

(v) We neglect the plasma pressure, but an ion beam with velocity, u, parallel to the centre axis is taken into account.

(vi) In equilibrium state, the magnetic field is uniform and static. The first order perturbation from the equilibrium is considered.

In Section **2**, we drive basic equations applicable to our problem. The dispersion relation in the general case is given in Section **3**. The physical picture of the waves, describing ion motion, electron motion and relative phase relation between fields, is also discussed in Section **4**. Finally, oscillations of a bounded plasma are investigated in Section **5**.

2. Basic Equations

In equilibrium state, an externally applied static magnetic field, B_0, is imposed parallel to the z-axis of cylindrical coordinates (r, θ, z). An infinitely long cylindrical uniform plasma is placed coaxially with the z-axis. $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{J}$ and $\boldsymbol{v}$ are the first order perturbed quantities of magnetic field, electric field, current density and mass velocity, respectively, which arise from the oscillation.

The linearized equations of motion are

$$\frac{\partial \boldsymbol{v}_i}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{v}_i = \frac{Z_i e}{M_i}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\times\boldsymbol{B} + \frac{1}{c}\boldsymbol{v}_i\times\boldsymbol{B}_0\right), \tag{3}$$

$$-\frac{\partial \boldsymbol{v}_e}{\partial t} = \frac{e}{M_e}\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}_e\times\boldsymbol{B}_0\right), \tag{4}$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}(\boldsymbol{J}+\boldsymbol{J}_0), \tag{5}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{6}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{7}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sigma. \tag{8}$$

where suffices i and e correspond to ion and electron, and $Z_i e$ and e are the charge on an ion and on an electron. σ is the charge density of the first order. And also

$$\boldsymbol{J} = n_i Z_i e \boldsymbol{v}_i - n_e e \boldsymbol{v}_e \simeq n_e e(\boldsymbol{v}-\boldsymbol{v}_e), \tag{9}$$

$$\boldsymbol{J}_0 \simeq n_e e \boldsymbol{u}, \tag{10}$$

$$\boldsymbol{v} = \frac{n_i M_i \boldsymbol{v}_i + n_e M_e \boldsymbol{v}_e}{n_i M_i + n_e M_e} \simeq \boldsymbol{v}_i,$$

where n_i and n_e are the number densities of ions and electrons. In the equilibrium state, there is the relation

$$n_i Z_i = n_e, \tag{11}$$

therefore $\sigma=0$. We have neglected the displacement current, since we are interested in low-frequency oscillations.

To obtain the dispersion relation, we shall introduce some covenient formulae from above equations. Combination of equations (3) and (4) yields

$$\frac{1}{n_e e^2}\frac{\partial \boldsymbol{J}}{\partial t}+\frac{1}{e}(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}_i = \left(\frac{Z_i}{M_i}+\frac{1}{M_e}\right)\boldsymbol{E}+\frac{1}{c}\left\{\frac{Z_i}{M_i}(\boldsymbol{u}\times\boldsymbol{B}+\boldsymbol{v}_i\times\boldsymbol{B}_0)+\frac{1}{M_e}\boldsymbol{v}_e\times B_0\right\}$$

$$\simeq \frac{1}{M_e}\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}_e\times\boldsymbol{B}_0\right), \tag{12}$$

$$\rho\left\{\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\right\} = \frac{1}{c}(\boldsymbol{J}+\boldsymbol{J}_0)\times\boldsymbol{B}_0, \tag{13}$$

where we have used the approximation (2), and ρ is the mass density of the plasma given by

$$\rho \simeq n_i M_i. \tag{14}$$

Considering the relation

$$\boldsymbol{v}_e = \boldsymbol{v}-\frac{1}{n_e e}\boldsymbol{J}, \tag{15}$$

we get from equations (12) and (15)

$$\frac{4\pi}{\omega_{pe}^2}\left\{\frac{\partial \boldsymbol{J}}{\partial t}+(\boldsymbol{J}_0\cdot\nabla)\boldsymbol{v}\right\} = \boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_0-\frac{1}{cn_e e}\boldsymbol{J}\times\boldsymbol{B}_0, \tag{16}$$

where

$$\omega_{pe}^2 = \frac{4\pi n_e e^2}{M_e}.$$

In our approximation, equation (16) can be rewritten as

$$\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_0-\frac{1}{cn_e e}\boldsymbol{J}\times\boldsymbol{B}_0 = 0. \tag{17}$$

Equation (13), (17) and Maxwell's equations (5) to (8) are the basic equations in our problem.

3. Dispersion Relation of the General Modes

To obtain the dispersion relation, we adopt a normal mode analysis and therefore we assume that the space and time dependence of all oscillating quantities conforms to the form $\exp i(m\theta+kz-\omega t)$.

From equations (7) and (17), we get

$$\frac{\partial^2\boldsymbol{B}}{\partial t^2} = \boldsymbol{\nabla}\times\left(\frac{\partial\boldsymbol{v}}{\partial t}\times\boldsymbol{B}_0-\frac{1}{n_e e}\frac{\partial\boldsymbol{J}}{\partial t}\times\boldsymbol{B}_0\right). \tag{18}$$

Using equations (5) and (13), we readily find that equation (18) becomes

$$-\frac{\omega^2}{A^2}\boldsymbol{B}=\boldsymbol{\nabla}\times\left[\{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\hat{z}\}\times\hat{z}+i\frac{\omega-ku}{\omega_i}\{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\hat{z}\}+\frac{\omega_i u}{A^2}\boldsymbol{B}\right], \tag{19}$$

where we have assumed that the contribution of the electron motion to transverse plasma current is negligible compared with that of the ion motion, as shown afterward. Also, ω_i is the ion cyclotron frequency and A is the phase velocity of Alfvén wave given by

$$A^2 = \frac{B_0}{4\pi\rho}. \tag{20}$$

Each component of equation (19) is readily found to be;

r-component:

$$\left(k^2-\frac{\omega^2}{A^2}\right)B_r = ik\left(k\Omega^*-\frac{\omega_i u}{A^2}\right)B_\theta-i\left\{k\frac{\partial}{\partial r}+\frac{m}{r}\left(k\Omega^*-\frac{\omega_i u}{A^2}\right)\right\}B_z, \tag{21}$$

θ-component:

$$\left(k^2-\frac{\omega^2}{A^2}\right)B_\theta = -ik\left(k\Omega^*-\frac{\omega_i u}{A^2}\right)B_r+\left\{\left(k\Omega^*-\frac{\omega_i u}{A^2}\right)\frac{\partial}{\partial r}+\frac{m}{r}k\right\}B_z, \tag{22}$$

z-component:

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\omega^2}{A^2}-\frac{m^2}{r^2}\right)B_z = \left[k\frac{m}{r}-\left(k\Omega^*-\frac{\omega_i u}{A^2}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\right]B_\theta$$
$$-i\left[-\left(k\Omega^*-\frac{\omega_i u}{A^2}\right)\frac{m}{r}+k\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\right]B_r, \tag{23}$$

where

$$\Omega^* = \frac{\omega - ku}{\omega_i}$$

and the exponent term is supressed. From above three equations, we get finally

$$B_r = \frac{-i}{F^2 - k^2G^2}\left[\frac{m}{r}G(F-k^2)B_z + k(F-G^2)\frac{\partial B_z}{\partial r}\right], \tag{24}$$

$$B_\theta = \frac{1}{F^2 - k^2G^2}\left[\frac{m}{r}k(F-G^2)B_z + G(F-k^2)\frac{\partial B_z}{\partial r}\right], \tag{25}$$

$$B_z = CJ_m(hr)\exp i(m\theta + kz - \omega t), \tag{26}$$

where

$$h^2 = -\frac{F^2 - k^2G^2}{F - G^2}, \tag{27}$$

$$\left.\begin{aligned} G &= k\Omega^* - \frac{\omega_i u}{A^2}, \\ F &= k^2 - \frac{\omega^2}{A^2}, \end{aligned}\right\} \tag{28}$$

and J_m is the Bessel Function of the m-th order. C is an arbitrary multiplicative constant. Equation (27) is the dispersion relation just to be solved and can be also rewritten as

$$h^2 = -\frac{\left(k^2 - \frac{\omega^2}{A^2}\right)^2 - k^2\left(k\Omega^* - \frac{\omega_i u}{A^2}\right)^2}{k^2 - \frac{\omega^2}{A^2} - \left(k\Omega^* - \frac{\omega_i u}{A^2}\right)^2}. \tag{27'}$$

The radial wave number, h, is determined from boundary conditions as discussed in Section 5. The azimuthal wave number, m, does not appear explicitly in equation (27) or (28) but m and h have a relation each other through boundary equations or characteristic equations.

In the case $u=0$, the dispersion formula agrees with that obtained by Stix or Bernstein and Trehan, which is

$$h^2 = -\frac{\left(k^2 - \frac{\omega^2}{A^2}\right)^2 - k^4\Omega^2}{k^2 - \frac{\omega^2}{A^2} - k^2\Omega^2}, \tag{29}$$

where

$$\Omega = \frac{\omega}{\omega_i},$$

Oscillation near Ion Cyclotron Frequency

If we assume $\omega_i \simeq \omega$ and the plasma density is tenuous enough to satisfy the relation

$$\frac{\omega_i^2}{k^2A^2} = \frac{\omega_{pi}^2}{k^2c^2} \ll 1\,, \qquad \left(\omega_{pi}^2 = \frac{4\pi n_i Z_i^2 e^2}{M_i}\right) \tag{30}$$

then equation (27′) may be approximated to

$$\left(\frac{\omega-ku}{\omega_i}\right)^2 \simeq 1-\frac{\omega^2}{A^2k^2}-\frac{\omega^2}{A^2(k^2+h^2)}\,. \tag{31}$$

This is the dispersion relation of the ion cyclotron waves with an ion beam.

In the case $h=0$ and $u=0$, equation (31) becomes

$$\omega-ku = \pm\omega_i\left(1-\frac{\omega_{pi}^2}{c^2k^2}\right). \tag{32}$$

Above equation represents the dispersion relation in one dimensional case. For large k according as

$$c^2k^2 \gg \omega_i^2 \sim \omega_{pi}^2\,,$$

equation (32) may be expressed by another form

$$\omega^2-c^2k^2-\omega_{pi}^2\frac{\omega-ku}{\omega-ku\mp\omega_i} = 0\,. \tag{33}$$

When many ion beams are flowing, equation (33) may be modified to

$$\omega^2-c^2k^2-\sum_s\frac{\omega_{pi\,s}^2(\omega-ku_s)}{\omega-ku_s\pm\omega_i} = 0\,, \tag{34}$$

which is just the dispersion relation obtained by Berger et al[2].

4. Physical Picture of the waves

In this section, the electromagnetic field, the ion currents and the electron currents in the waves are considered so as to get detail information on the physical picture of the waves.

Electric Field

Substituting equations (24), (25) and (26) into equation (7) we get

$$\left.\begin{aligned} E_\theta &= -i\frac{\omega}{kc}\frac{1}{F^2-k^2G^2}\left[\frac{m}{r}G(F-k^2)B_z+k(F-G^2)\frac{\partial B_z}{\partial t}\right], \\ E_r &= -\frac{\omega}{kc}\frac{1}{F^2-k^2G^2}\left[\frac{m}{r}k(F-G^2)B_z+G(F-k^2)\frac{\partial B_z}{\partial t}\right], \\ E_z &\simeq 0\,. \end{aligned}\right\} \tag{35}$$

The sense of the field rotation may be examined by calculating iE_r/E_θ, which may be called a phase relation between the transverse components of the electric field. From equations (35) and (26), we find

$$\frac{iE_r}{E_\theta} = \frac{m(k+G)(F-kG)J_m(hr) - G(F-k^2)hrJ_{m+1}(hr)}{m(k+G)(F-kG)J_m(hr) - k(F-G^2)hrJ_{m+1}(hr)}. \tag{36}$$

The above relation is valid for the general case in our approximation.

For Alfvén wave with $u=0$,

$$\frac{iE_r}{E_\theta} \simeq 1. \tag{37}$$

where we have made use of the dispersion relation of Alfvén wave

$$\omega^2 = k^2A^2. \tag{38}$$

On the other hand, the phase relation for the ion cyclotron wave without the ion beam can be get as

$$\frac{iE_r}{E_\theta} \simeq -\left[1+\frac{h^2}{k^2}\left\{1-\frac{2m}{hr}\frac{J_m(hr)}{J_{m+1}(hr)}\right\}\right], \tag{39}$$

with the aid of the dispersion relation (31). The electric field is generally elliptically polarized. Besides the sense of the field rotation depends on the sign in the brackets of equation (39). For large k, however, the field rotates in the same sense as that of the ion Larmar gyration, since the phase relation of an ion gyration in the magnetic field is

$$\frac{iw_r}{w_\theta} = -1,$$

where w is the individual ion velocity. Equation (39) for the axisymmetric mode of the ion cyclotron waves is reduced to

$$\frac{iE_r}{E_\theta} \simeq -\left(1+\frac{h^2}{k^2}\right), \tag{40}$$

which agrees with Stix's result.

Ion Motion

In our approximation, the ordered motion of an ensemble of ions may be regarded as the mass motion of plasma. Eliminating $\boldsymbol{J}$ from equations (13) and (17), we get

$$\boldsymbol{v}\times\hat{\boldsymbol{z}}+\frac{i}{\omega_i}(\omega-ku)\boldsymbol{v}-\frac{i}{\omega_e}ku\boldsymbol{v}+u\hat{\boldsymbol{z}}\times\frac{\boldsymbol{B}}{B_0}+\frac{c\boldsymbol{E}}{B_0} = 0, \tag{41}$$

where we have made use of equation (10). This equation may be disolved into each component as

$$i\left(\frac{\omega-ku}{\omega_i}-\frac{ku}{\omega_e}\right)v_r+v_\theta-\frac{B_\theta}{B_0}u+\frac{cE_r}{B_0} = 0, \tag{42}$$

$$-v_r+i\left(\frac{\omega-ku}{\omega_i}-\frac{ku}{\omega_e}\right)v_\theta+\frac{B_r}{B_0}u+\frac{cE_\theta}{B_0} = 0, \tag{43}$$

Finally, we have

$$v_r = \frac{1}{1-\Omega^{*2}}\frac{c}{B_0}\left\{(i\Omega^* B_\theta + B_r)\frac{u}{c} + (E_\theta - i\Omega^* E_r)\right\}, \tag{44}$$

$$v_\theta = \frac{-i}{1-\Omega^{*2}}\frac{c}{B_0}\left\{(\Omega^* B_r + iB_\theta)\frac{u}{c} + (\Omega^* E_\theta - iE_r)\right\}, \tag{45}$$

since $\omega_e \gg \omega_i$. Thus, the calculation of the phase relation between v_r and v_θ is straightforward, which is

$$\frac{iv_r}{v_\theta} = -\frac{1-\Omega^*\dfrac{iE_r}{E_\theta}}{\Omega^* - \dfrac{iE_r}{E_\theta}}, \tag{46}$$

with the aid of equation (7). In the case $u=0$, equation (46) agrees with Stix's result, which was derived only for $m=0$ mode.

Next, we shall consider the ion currents in the waves. From equation (13), we have

$$\boldsymbol{J}_\perp = i\Omega^*(n_i Z_i e \boldsymbol{v} \times \hat{z}) + \frac{\boldsymbol{B}_\perp}{B_0} J_0, \tag{47}$$

where

$$\boldsymbol{J}_\perp = J_r \hat{r} + J_\theta \hat{\theta},$$
$$\boldsymbol{B}_\perp = B_r \hat{r} + B_\theta \hat{\theta}.$$

For the ion cyclotron wave in sufficiently rarefied plasma, Ω^* is close to one. Whence $iv_r/v_\theta \simeq -1$ and equation (47) may be simplified to be

$$\boldsymbol{J}_\perp = n_i Z_i e\left(\boldsymbol{v}_i + \frac{\boldsymbol{B}_\perp}{B_0} u\right). \tag{48}$$

Thus we can say that the transverse component of the plasma current is mainly carried by the ions. The second term, $\boldsymbol{B}_\perp u/B_0$, in the parentheses of equation (48) arises from the transverse bend of the ion beam, which is caused by the transverse component of oscillating magnetic field, $\boldsymbol{B}$,.

Electron Motion

The transverse electron motion, $\boldsymbol{v}_{e\perp}$, in the wave is a drift given by

$$\boldsymbol{v}_{e\perp} = c\frac{\boldsymbol{E} \times \hat{z}}{B_0}. \tag{49}$$

Therefore, the resultant electron current is

$$\boldsymbol{J}_{e\perp} = -\frac{n_e e c E_r}{B_0}\left(\frac{E_\theta}{E_r}\hat{r} - \hat{\theta}\right). \tag{50}$$

From equations (42) and (7), we have

$$\frac{\dfrac{E_r}{B_0}}{\dfrac{v_\theta}{c}} = -\frac{\Omega^*\dfrac{iv_r}{v_\theta} + 1}{1 + \dfrac{ku}{\omega}}. \tag{51}$$

The numerator on the right hand side of equation (51) is rewritten by using equations (36) and (46) as

$$\Omega^*\frac{iv_r}{v_\theta}+1 = -\frac{(1-\Omega^{*2})\{-m(k+G)(F-kG)J_m(hr)+G(F-k^2)hrJ_{m+1}(hr)\}}{(1-\Omega^*)\{m(k+G)(F-kG)J_m(hr)\}+\{G(F-k^2)-\Omega^*k(F-G^2)hrJ_{m+1}(hr)\}}\,. \tag{52}$$

The azimuthal component of the electron current is

$$J_{e\theta} = n_e e\frac{\dfrac{E_r}{B_0}}{\dfrac{v_\theta}{c}}v_\theta\,, \tag{53}$$

in which the factor cE_r/B_0v has been already given by equation (52).

For the ion cyclotron wave in the case $u=0$ and $m=0$, equation (52) may be approximated to $-\omega_i^2/A^2k^2$, whence we have

$$J_{e\theta} = -n_e e\frac{\omega_i^2}{A^2k^2}v_{i\theta} = -\frac{\omega_i^2}{A^2k^2}J_{i\theta}\,. \tag{54}$$

Therefore, it becomes evident that, for the ion cyclotron wave, the contribution of the electron motion to the transverse current is very little in comparison with the ion contribution. This is a Stix's conclusion.

5. Bounded Plasma Oscillations

The previous discussion have been made under the assumption that the radial and azimuthal wave numbers are given. However, these wave numbers correlate with each other through boundary equations. In this section, we shall deal with bounded plasma oscillations and derive characteristic equations to determine the correlation between h and m.

Boundary situation considered is schematically shown in Fig. 1. The cylindrical plasma is coaxially immersed in a cylindrical current sheet and has a vacuum clearance between the plasma and the sheet current.

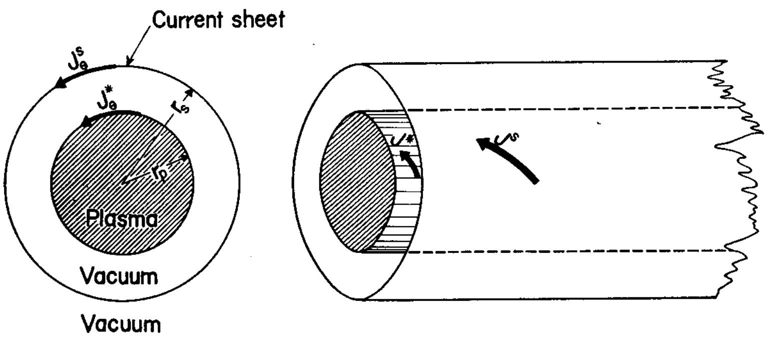

Fig. 1. Boundary situation of the plasma.

Again the normal mode analysis is applied to the following discussion, so that we assume that all oscillating quantities vary as $\exp i(m\theta+kz-\omega t)$. Let the radii of the plasma and the current sheet r_p and r_s, respectively, as shown in Fig. 1. The surface current of the plasma is denoted by J^* and the sheet current by J^s.

The magnetic field in each space region may be expressed as;

for $r_s<r$, (outer vacuum region)

$$\left.\begin{aligned} B_r^w &= kSK_m'(kr), \\ B_\theta^w &= \frac{im}{r}SK_m(kr), \\ B_z^w &= ikSK_m(kr), \end{aligned}\right\} \tag{55}$$

for $r_p<r<r_s$, (vacuum clearance region)

$$\left.\begin{aligned} B_r^v &= k\{LI_m'(kr)+QK_m'(kr)\}, \\ B_\theta^v &= \frac{im}{r}\{LI_m(kr)+QK_m(kr)\}, \\ B_z^v &= ik\{LI_m(kr)+QK_m(kr)\}, \end{aligned}\right\} \tag{56}$$

for $r<r_p$, (plasma region)

$$\left.\begin{aligned} B_r^p &= \frac{-i}{F^2-k^2G^2}\left\{\frac{m}{r}G(F-k^2)B_z^p+k(F-G^2)\frac{\partial B_z^p}{\partial r}\right\}, \\ B_\theta^p &= \frac{1}{F^2-k^2G^2}\left\{\frac{m}{r}k(F-G^2)B_z^p+G(F-k^2)\frac{\partial B_z^p}{\partial r}\right\}, \\ B_z^p &= CJ_m(hr), \end{aligned}\right\} \tag{57}$$

where S, L, Q and C are constants to be determined. We have supressed the exponent term and adopted the superscripts p, v and w to signify the quantities referring to the plasma, the vacuum clearance and the outer vacuum region, respectively.

Next, we shall derive boundary equations. From Maxwell's equations, we have

$$\triangle(\boldsymbol{r}\cdot\boldsymbol{B}) = 0 \tag{58}$$

and

$$-\triangle(\boldsymbol{r}\times\boldsymbol{B}) = \frac{4\pi}{c}\boldsymbol{J}^* \quad \left(\text{or } \frac{4\pi}{c}\boldsymbol{J}^s\right). \tag{59}$$

Also, magnetic pressure balance at the interface gives the relation

$$\triangle\left(\frac{B^2}{8\pi}\right) = 0, \quad \text{at } r=r_p. \tag{60}$$

Here, $\triangle(x)$ denotes the change of some quantity, x, across the surface. Then the linearized forms of these equations are

$$B_r^p = B_r^v \quad \text{at } r=r_p, \tag{61}$$

$$B_z^p = B_z^v \quad \text{at } r=r_p, \tag{62}$$

$$B_r^v = B_r^w \quad \text{at } r=r_s, \tag{63}$$

$$B_z^v - B_z^w = \frac{4\pi}{c} J_\theta^s \quad \text{at} \quad r = r_s. \tag{64}$$

Using above boundary equations (61) to (64), we can determine the constant S, L, Q and C. Finally, we can write dawn the magnetic field in each region as;

for $r_s < r$,

$$\left.\begin{aligned} B_r^w &= iN^s kr_s \{I'_m(kr_s) - YK'_m(kr_s)\} K'_m(kr), \\ B_\theta^w &= -N^s \frac{mr_s}{r} \{I'_m(kr_s) - YK'_m(kr_s)\} K_m(kr), \\ B_z^w &= -N^s kr_s \{I'_m(kr_s) - YK'_m(kr_s)\} K_m(kr), \end{aligned}\right\} \tag{65}$$

for $r_p < r < r_s$,

$$\left.\begin{aligned} B_r^v &= iN^s kr_s K'_m(kr_s) \{I'_m(kr) - YK'_m(kr), \\ B_\theta^v &= -N^s \frac{mr_s}{r} K'_m(kr_s) \{I_m(kr) - YK_m(kr)\}, \\ B_z^v &= -N^s kr_s K'_m(kr_s) \{I_m(kr) - YK_m(kr)\}, \end{aligned}\right\} \tag{66}$$

for $r < r_p$,

$$B_z^p = -N^s \frac{kr_s K'_m(kr_s)}{J_m(hr_p)} \frac{1}{HK_m(kr_p) - kr_p K'_m(kr_p)} J_m(hr), \tag{67}$$

where

$$Y = \frac{HI_m(kr_p) - kr_p I'_m(kr_p)}{HK_m(kr_p) - kr_p K'_m(kr_p)}, \tag{68}$$

$$H = \frac{k}{F^2 - k^2 G^2} \left\{ m(F - k^2)G + k(F - G^2) hr_p \frac{J'_m(hr_p)}{J_m(hr_p)} \right\}, \tag{69}$$

$$N^s = \frac{4\pi J_\theta^s}{c}, \tag{70}$$

and we have again suppressed the exponent term. Equations (65) to (70) give the complete solution for steady state excitation. The electric field in each space region is easily obtainable from equations (65) to (70) and Maxwell's induction equation.

Cylindrical Plasma Wave Guide

Natural wave modes inside a conducting cylinder can be examined by using equations (65) to (67). Outside the completely conducting cylinder, there is no oscillating field. Then we can set $\boldsymbol{B}^w = 0$ for $r > r_s$. Furthermore, for simplification, we shall consider the case $r_s = r_p$ (i.e. a plasma wave guide). In this case

$$[\boldsymbol{B}^w]_{r_p = r_s} = 0. \tag{71}$$

Above relation is equivalent to

$$H = 0$$

or

$$m(F - k^2)G + k(F - G^1) hr_p \frac{J'_m(hr_s)}{J_m(hr_s)} = 0, \tag{72}$$

where we have made use of the fact that $K_m'(x)$ is finite if $x \neq 0$. This equation is the characteristic equation relating h, m and k.

For the ion cyclotron waves, equation (72) is transformed to the simple form

$$\frac{hr_s J_{m+1}(hr_s)}{J_m(hr_s)} = -m\frac{h^2}{k^2}, \tag{73}$$

where we have assumed $u=0$ and $\omega_i^2/A^2k^2 \ll 1$. Especially for the waves of sufficiently short wave length, equation (73) may be expressed approximately by

$$J_{m+1}(hr_s) = 0 \tag{74}$$

Vacuum Boundary

Natural modes of a plasma surrounded by vacuum can be examined by setting $\boldsymbol{J}^s=0$. The condition that $\boldsymbol{B}^p$ does not vanish leads to the relation

$$\frac{k}{F^2-k^2G^2}\left\{m(F-k^2)G+k(F-G^2)hr_p\frac{J_m'(hr_p)}{J_m(hr_p)}\right\} = \frac{kr_pK_m'(kr_p)}{K_m(kr_p)}. \tag{75}$$

When $u=0$, above equation becomes

$$\frac{k^2}{\left(k^2-\frac{\omega^2}{A^2}\right)^2-\Omega^2k^4}\left[-m\Omega\frac{\omega^2}{A^2}+\left(k^2-\frac{\omega^2}{A^2}-k^2\Omega^2\right)\frac{hr_pJ_m'(hr_p)}{J_m(hr_p)}\right] = \frac{kr_pJ_m'(kr_p)}{K_m(kr_p)}, \tag{76}$$

which is in agreement with Bernstein and Trehan's result.

For the axisymmetric ion cyclotron waves, equation (76) can be reduced to

$$\frac{J_0'(hr_p)}{hJ_0(hr_p)} = -\frac{K_0'(kr_p)}{kK_0(kr_p)} \tag{77}$$

or in another form

$$\frac{hr_pJ_1'(hr_p)}{J_1(hr_p)} = \frac{kr_pK_1'(kr_p)}{K_1(kr_p)} \tag{78}$$

Equation (78) agrees with Stix's result.

6. Summary

Low-frequency waves in a plasma, which is immersed in an external magnetic field and cylindrically bounded, were discussed in the case such that an ion beam is flowing and their modes are non-axisymmetric. The dispertion relation applicable to these waves was derived and also the detail structures in the waves, such as ion currents, electron currents, electromagnetic field and phase relations between the components of the field, were formulated.

The application of the non-axisymmetric ion cyclotron waves to plasma heating may be promising because of the fact that the dispersion relation for the non-axisymmetric modes has apparently the same formula as that for the axisymmetric mode though really the radial wave number is correlated with the azimuthal wave number through the boundary equation. If we introduce the case of a multibeam in a plasma and then take the limit of an infinite number of beams, we may examine the cyclotron damping rate for the general mode.

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