Near-Optimal Control of Non-Linear Dynamic Plants with a Quadratic Performance Index

By

Yoshikazu SAWARAGI*, Yoshifumi SUNAHARA* and Koichi INOUE*

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In this paper, a near-optimal control strategy is studied for high-order multi-variable non-linear dynamic plants under a quadratic performance index. The principal line of attack is to introduce the concept of instantaneous linearization and to apply the dynamic optimization technique originated by R. Bellman. The resulting configuration of the near-optimal control system presented here becomes a feedback one containing an on-line digital computer as a main control device. The general procedure proposed here is illustrated by the example of establishing the near-optimal control strategy for a dynamic plant with a non-linear characteristic. The control performance of the system is also discussed by comparing it with that of the system with the precisely optimal control strategy.

List of Principal Symbols

$\boldsymbol{x}(t)$: vector representation of state variables of a plant
$\boldsymbol{m}(t)$: vector representation of control signals
$\boldsymbol{A}(t)$: coefficient matrix of a plant
$\boldsymbol{B}(t)$: driving matrix of a plant
$\boldsymbol{\phi}(t, t_0)$: transition matrix of a plant
$\boldsymbol{G}(k)$: driving matrix of a plant in discrete form
$\boldsymbol{\phi}(k)$: transition matrix of a plant in discrete form
t_0 and t_e : initial and final control instants of time respectively
t_k : k-th sampling instant
T : sampling period
$\boldsymbol{Q}(t)$ and $\boldsymbol{R}(t)$: performance weights
$\boldsymbol{x}(k)$ and $\boldsymbol{m}(k)$: abbreviated symbols for $\boldsymbol{x}(t_k)$ and $\boldsymbol{m}(t_k)$ respectively
$f_{N-k}[\boldsymbol{x}(k)]$: minimum value of the performance index
$\boldsymbol{f}$ and $\boldsymbol{g}$: non-linear functions of state variables
a, b and c : plant parameters
p : adjoint variable
t : time variable

* Department of Applied Mathematics and Physics.

1. Introduction

Up to the present, in spite of the important fact that actual plants, which are to be controlled, are inevitably endowed with non-linear characteristics in their dynamic relation between inputs and outputs, methods of the dynamic optimization have been accepted relying upon mathematically linearized models for actual non-linear plants which are established by focussing our attention to the small domain around the equilibrium point.

As we often observe in such practical cases as industrial plants subjected to disturbances with considerably large magnitude and chemical reaction plants at the starting-up period, since fluctuations of all physical variables of actual plants are not always small, then the use of linearized models mentioned above can not play an important role in the aspect of effective dynamic optimization with high degree of accuracy. This fact reveals that non-linear characteristics of actual plants must be taken into account from the analytical viewpoint in the large.

On the other hand, new approaches to the design problem of control systems have been developed based on such mathematical concepts as R. Bellman's Dynamic Programming[1], L. S. Pontryagin's Maximum Principle[2] and others[3], which are widely called optimization techniques. Using these newly developed optimization techniques, it is, in general, easy to formulate the design problems, provided that the inputs and outputs characteristics of a plant are described by a set of differential equations, and that the performance index is mathematically specified. Although some limited classes of problems have been solved in closed form, it is very difficult to solve analytically these formulated design problems. It is, hence, desirable to develop a synthesis technique for non-linear plants based on an extended concept by which even if not optimal but near-optimal control systems will be easily obtained.

2. Fundamental State Description of High-Order Multi-Variable Linear System

First, we consider a linear dynamical system as a controlled element as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{m}(t), \tag{2.1}$$

where $\boldsymbol{x}(t)$ is a state vector with n components and $\boldsymbol{m}(t)$ is a control signal vector with r components. $\boldsymbol{A}(t)$ is an $n \times n$ matrix referred to as the coefficient matrix of the plant; $\boldsymbol{B}(t)$ is an $n \times r$ matrix called the driving matrix, which may, in general, be time-varying.

The solution of Eq. (2.1) is given by

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \lambda)B(\lambda)m(\lambda)d\lambda, \qquad (2.2)$$

where $\phi(t, t_0)$ is the transition matrix of the linear system described by Eq. (2.1), and this is the matrix solution of the homogeneous equation;

$$\dot{x}(t) = A(t)x(t), \qquad x(t_0) = I, \qquad (2.3)$$

where I is the unit vector.

If the coefficient matrix $A(t)$ is not time-varying, the transition matrix $\phi(t, t_0)$ depends only on the time-difference $t-t_0$ and can easily be described by

$$\phi(t, t_0) = \exp A\cdot(t-t_0). \qquad (2.4)$$

For the covenience of present discussion, we describe the control signal vector as

$$m(t) = m(t_k), \qquad \text{for } t_k \leq t \leq t_{k+1}, \qquad (2.5)$$

where t_k is a sampling instant.

From Eqs. (2.2) and (2.5), the state-transition equation between the state vectors $x(k+1)$ at $t=t_{k+1}$ and $x(k)$ at $t=t_k$ is given by

$$x(k+1) = \phi(k)x(k) + G(k)m(k), \qquad (2.6)$$

where

$$\left.\begin{aligned} \phi(k) &= \phi(t_{k+1}, t_k) \\ G(k) &= \int_{t_k}^{t_{k+1}} \phi(t_{k+1}, \lambda)B(\lambda)d\lambda \end{aligned}\right\}. \qquad (2.7)$$

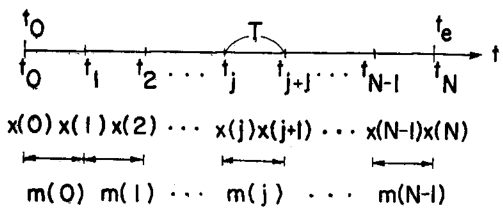

Fig. 1. Illustration of mutual relation between the state vector $x(k)$ and the control vector $m(k)$.

We divide the given control time interval $[t_0, t_e]$ into N equal sub-intervals and denote the sampling period by T as shown in Fig. 1. Then Eq. (2.7) becomes

$$\left.\begin{aligned} \phi(k) &= \phi(\overline{k+1}T, kT) \\ G(k) &= \int_{kT}^{\overline{k+1}T} \phi(\overline{k+1}T, \lambda)B(\lambda)d\lambda \end{aligned}\right\}. \qquad (2.8)$$

It must be noted here that if both the coefficient matrix $A(t)$ and the driving matrix $B(t)$ are not time-varying, then both $\phi(k)$ and $G(k)$ depend only on T, because the transition matrix $\phi(t, t_0)$ is the function of the time-difference $t-t_0$ as described by Eq. (2.4).

3. Determination of Near-Optimal Control Signal Vector

Although a non-linear plant is, in general, described by

$$\dot{x}(t) = F[x(t), m(t), t], \qquad (3.1)$$

we restrict our attention, in this paper, to a particular form

$$\dot{x}(t) = f[x(t), t] + g[x(t), t]m(t) . \tag{3.2}$$

where both f and g are non-linear functions of the state vector $x(t)$.

The performance index chosen to be minimized is of the quadratic form integrated over the given control interval $[t_0, t_e]$;

$$I = \int_{t_0}^{t_e} [x^T(t)Q(t)x(t) + m^T(t)R(t)m(t)]dt . \tag{3.3}$$

where $x^T(t)$ denotes the transpose of $x(t)$. In Eq. (3.3), it is assumed, for simplicity, that the target is always the origin of the state space. $Q(t)$ and $R(t)$ are, in general, time-varying and positive definite symmetric matrices, called the performance weights, of which values must be suitably determined from engineerig viewpoints.

The optimal design problem is, thus, stated as follows; determine the control signal vector $m(t)$, $t_0 \leq t \leq t_e$, which minimizes the performance index specified by Eq. (3.3) subjected to the relation of Eq. (3.2), for any arbitrary initial state $x(t_0)$. Although many optimization techniques are applicable for the optimal design problem stated as above, they must inevitably rely on very difficult methods by reason of the non-linearity of the plant equation (3.2), which are not so available in practice.

Hoping to invoke a digital computer, the performance index (3.3) is approximated as

$$I_N = \sum_{k=1}^{N} [x^T(k)Q(k)x(k) + m^T(k-1)R(k-1)m(k-1)] . \tag{3.4}$$

We consider that we are now at the time $t = t_j$ and that the state vector $x(j)$ is sampled from the actual non-linear plant described by Eq. (3.2). At this situation, we establish a linear model based on this observed state vector $x(j)$ as the following form[4],

$$\dot{x}(t) = A\{x(j), t\}x(t) + B\{x(j), t\}m(t) , \tag{3.5}$$
$$x(t_j) = x(j) , \qquad t_j \leq t \leq t_e ,$$

where from Eq. (3.2)

$$A\{x(j), t\} = \frac{f[x(j), t]}{x(j)}, \; B\{x(j), t\} = g[x(j), t] . \tag{3.6}$$

We also assume that the furture dynamics of the plant is governed by this model of Eq. (3.5) from the present time $t = t_j$ until the final control instant $t = t_e$. The control signal vector $m(j)$, which is to be applied at the present time $t = t_j$, is easily calculated by the use of the discrete Dynamic Programming technique[5] as follows.

First, from Eqs. (2.1), (2.6) and (2.7), the state-transition equation is described by

$$x(k+1) = \phi(k, x(j))x(k)+G(k, x(j))m(k), \qquad [k = j, j+1, \cdots, N-1]. \tag{3.7}$$

It should be noticed here that both ϕ and G are dependent on the state vector $x(j)$ observed at $t=t_j$, because in Eq. (3.5), A and B are functions of $x(j)$.

We consider an arbitrary time $t=t_k$ where $t_j \leq t_k \leq t_e$ and t_e expresses the final instant of time, and denote the minimum value of the performance index (3.4) between the time $t=t_k$ and the final control instant $t=t_e$ as

$$f_{N-k}[x(k)] = \underset{\substack{m(k)\\ \vdots \\ m(N-1)}}{\text{Min}} \left\{ \sum_{i=k+1}^{N} [x^T(i)Q(i)x(i)+m^T(i-1)R(i-1)m(i-1)] \right\}, \tag{3.8}$$

and from Eq. (3.3), apparently

$$f_0[x(N)] = 0. \tag{3.9}$$

Invoking Bellman's Principle of Optimality, Eq. (3.8) becomes

$$f_{N-k}[x(k)] = \underset{m(k)}{\text{Min}} \{x^T(k+1)Q(k+1)x(k+1)+m^T(k)Q(k)m(k) + f_{N-\overline{k+1}}[x(k+1)]\}. \tag{3.10}$$

Since the functional f has the quadratic form with respect to the state vector x, then it can be assumed that

$$\left.\begin{aligned} f_{N-k}[x(k)] &= x^T(k)P(N-k, x(j))x(k) \\ f_{N-\overline{k+1}}[x(k+1)] &= x^T(k+1)P(N-\overline{k+1}, x(j))x(k+1) \end{aligned}\right\}. \tag{3.11}$$

where P's are positive definite symmetric $n\times n$ matrices.

Using Eqs. (3.11), Eq. (3.8) becomes

$$x^T(k)P(N-k, x(j))x(k) = \underset{m(k)}{\text{Min}} [x^T(k+1)S(N-\overline{k+1}, x(j))x(k+1) + m^T(k)R(k)m(k)], \tag{3.12}$$

where

$$S(N-\overline{k+1}, x(j)) = Q(k+1)+P(N-\overline{k+1}, x(j)). \tag{3.13}$$

By substituting Eq. (3.7) into Eq. (3.12), we obtain

$$x^T(k)P(N-k, x(j))x(k) = \underset{m(k)}{\text{Min}} \{[\phi(k, x(j))x(k)+G(k, x(j))m(k)]^T \times S(N-\overline{k+1}, x(j))[\phi(k, x(j))x(k)+G(k, x(j))m(k)]+m^T(k)R(k)m(k)\}. \tag{3.14}$$

Letting

$$\left.\begin{aligned} L_{\phi\phi}(N-\overline{k+1}, x(j)) &= \phi^T(k, x(j))S(N-\overline{k+1}, x(j))\phi(k, x(j)) \\ L_{\phi G}(N-\overline{k+1}, x(j)) &= \phi^T(k, x(j))S(N-\overline{k+1}, x(j))G(k, x(j)) \\ L_{G\phi}(N-\overline{k+1}, x(j)) &= G^T(k, x(j))S(N-\overline{k+1}, x(j))\phi(k, x(j)) \\ L_{GG}(N-\overline{k+1}, x(j)) &= G^T(k, x(j))S(N-\overline{k+1}, x(j))G(k, x(j)) \end{aligned}\right\}, \tag{3.15}$$

Then Eq. (3.12) becomes

$$\begin{aligned} x^T(k)P(N-k, x(j))x(k) &= \underset{m(k)}{\text{Min}} \{x^T(k)L_{\phi\phi}(N-\overline{k+1}, x(j))x(k) \\ &+ x^T(k)L_{\phi G}(N-\overline{k+1}, x(j))m(k) + m^T(k)L_{G\phi}(N-\overline{k+1}, x(j))x(k) \\ &+ m^T(k)[L_{GG}(N-\overline{k+1}, x(j)) + R(k)]m(k)\} . \end{aligned} \tag{3.16}$$

The minimization procedure may easily be carried out by differentiating the right hand side of Eq. (3.16) with respect to $m(k)$ and equating it equal to zero:

$$L_{G\phi}(N-\overline{k+1}, x(j))x(k) + [L_{GG}(N-\overline{k+1}, x(j)) + R(k)]m^*(k) = 0 . \tag{3.17}$$

Eq. (3.17) can be expressed as

$$m^*(k) = D(N-k, x(j)) \cdot x(k) , \tag{3.18}$$

where

$$D(N-k, x(j)) = -[L_{GG}(N-\overline{k+1}, x(j)) + R(k)]^{-1} L_{G\phi}(N-\overline{k+1}, x(j)) . \tag{3.19}$$

Substituting Eq. (3.18) for $m(k)$ in Eq. (3.16), we have

$$\begin{aligned} x^T(k)P(N-k, x(j))x(k) &= x^T(k)L_{\phi\phi}(N-\overline{k+1}, x(j))x(k) \\ &+ x^T(k)L_{\phi G}(N-\overline{k+1}, x(j))D(N-k, x(j))x(k) \\ &+ x^T(k)D^T(N-k, x(j))L_{G\phi}(N-\overline{k+1}, x(j))x(k) \\ &+ x^T(k)D^T(N-k, x(j))[L_{GG}(N-\overline{k+1}, x(j)) + R(k)]D(N-k, x(j))x(k) . \end{aligned} \tag{3.20}$$

Using Eq. (3.19), Eq. (3.20) yields

$$\begin{aligned} &x^T(k)P(N-k, x(j))x(k) \\ &= x^T(k)[L_{\phi\phi}(N-\overline{k+1}, x(j)) + L_{\phi G}(N-\overline{k+1}, x(j))D(N-k, x(j))]x(k) . \end{aligned} \tag{3.21}$$

Comparison of both sides of Eq. (3.21) gives

$$\begin{aligned} &P(N-k, x(j)) \\ &= L_{\phi\phi}(N-\overline{k+1}, x(j)) + L_{\phi G}(N-\overline{k+1}, x(j))D(N-k, x(j)) . \end{aligned} \tag{3.22}$$

Near-optimal control signal vector $m^*(j)$ can, therefore, be calculated from Eq. (3.18) as

$$m^*(j) = D(N-j, x(j)) \cdot x(j) , \tag{3.23}$$

where the feedback matrix $D(N-j, x(j))$ is determined by recurrence relations, Eqs. (3.13), (3.15), (3.19) and (3.22), by starting with the following equation obtained from Eqs. (3.9) and (3.11) as

$$P(0, x(j)) = 0 . \tag{3.24}$$

Detailed procedures are summerized as follows;

(1) Preliminary procedures

i) Compute $\phi(k, x(j))$, $[k=j, j+, \cdots, N-1]$ by solving the differential equation:

$$\dot{\phi}(t, t_j) = A\{x(j), t\}\phi(t, t_j), \quad \phi(t_j, t_j) = I.$$

ii) Compute $G(k, x(j))$, $[k=j, j+1, \cdots, N-1]$ by

$$G(k, x(j)) = \int_{kT}^{\overline{k+1}T} \phi(\overline{k+1}T, \lambda)B\{x(j), \lambda\}d\lambda.$$

(2) Routine procedures

i) Given $P(N-\overline{k+1}, x(j))$, compute $S(N-\overline{k+1}, x(j))$ by

$$S(N-\overline{k+1}, x(j)) = Q(k+1)+P(N-\overline{k+1}, x(j)).$$

ii) Compute $L_{\phi\phi}(N-\overline{k+1}, x(j))$, $L_{\phi G}(N-\overline{k+1}, x(j))$, $L_{G\phi}(N-\overline{k+1}, x(j))$ and $L_{GG}(N-\overline{k+1}, x(j))$ by

$$\left.\begin{aligned} L_{\phi\phi}(N-\overline{k+1}, x(j)) &= \phi^T(k, x(j))S(N-\overline{k+1}, x(j))\phi(k, x(j)) \\ L_{\phi G}(N-\overline{k+1}, x(j)) &= \phi^T(k, x(j))S(N-\overline{k+1}, x(j))G(k, x(j)) \\ L_{G\phi}(N-\overline{k-1}, x(j)) &= G^T(k, x(j))S(N-\overline{k+1}, x(j))\phi(k, x(j)) \\ L_{GG}(N-\overline{k+1}, x(j)) &= G^T(k, x(j))S(N-\overline{k+1}, x(j))G(k, x(j)) \end{aligned}\right\},$$

iii) Compute $D(N-k, x(j))$ by

$$D(N-k, x(j)) = -[L_{GG}(N-\overline{k+1}, x(j))+R(k)]^{-1}L_{G\phi}(N-\overline{k+1}, x(j)).$$

iv) Compute $P(N-k, x(j))$ by

$$P(N-k, x(j)) = L_{\phi\phi}(N-\overline{k+1}, x(j))+L_{\phi\phi}(N-\overline{k+1}, x(j))D(N-k, x(j)).$$

The feedback matrix $D(N-j, x(j))$ in Eq. (3.23) can be evaluated, after $N-j$ times repetitions of the routine procedures i)~iv) listed above, starting with $P(0, x(j))=0$. The near-optimal control signal vector $m^*(j)$ which should be applied to the plant at the present time $t=t_j$ is, thus, calculated on the basis of the state vector $x(j)$ observed at $t=t_j$. The plant is translated from $x(j)$ to $x(j+1)$ by the control signal vector $m^*(j)$ obtained by the above procedures. When $t=t_{j+1}$, the new state vector $x(j+1)$ is obtained by the observation of the plant, and then procedures (1) and (2) are again repeated, that is, the new linearized model is constructed by using the newly observed state vector $x(j+1)$ as

$$\begin{aligned} \dot{x}(t) &= A\{x(j+1), t\}x(t)+B\{x(j+1), t\}m(t), \\ x(t_{j+1}) &= x(j+1), \quad t_{j+1}\leq t\leq t_e. \end{aligned} \tag{3.25}$$

This procedure means up-dating of the linearized model. Naturally, the procedures mentioned here are successively extended to the final control instant of time. In Fig. 2, the flow chart to calculate the near-optimal control signal vector $m^*(j)$ $[j=0, 1, \cdots, N-1]$ is shown.

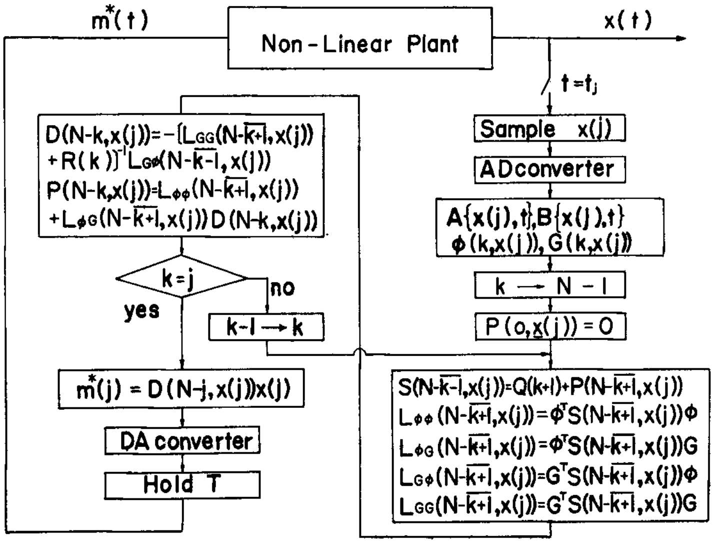

Fig. 2. Flow chart to calculate the near-optimal control vector $m^*(j)$.

It is well known that if the plant is linear, the corresponding optimal control signal vector $m^*(j)$ becomes linear with respect to the state vector $x(j)$. This can be observed from the fact that the feedback matrix D in Eq. (3.23) becomes independent of $x(j)$. However, the near-optimal control signal vector $m^*(j)$ obtained here becomes non-linear, namely, the feedback matrix D becomes dependent on the state vector $x(j)$, reflecting the non-linear characteristics of the plant. This is due to the procedure that the renewed linear model is successively constructed at each sampling instant.

4. Simulation Studies

The principal line of numerical studies in this section is to compare the response of the near-optimal control system with that of the optimal control system.

We consider a first-order non-linear plant as shown in Fig. 3, which is described by

$$\dot{x} = ax + cx^3 + bm\,, \qquad x(t_0) = c_1\,, \tag{4.1}$$

where a, b and c are constants, and c_1 is an arbitrary given initial state.

As the performance index to be minimized, we consider

$$I = \frac{1}{2}\int_{t_0}^{te}(x^2 + m^2)dt\,. \tag{4.2}$$

To obtain the true optimal response, we will resort to the well-known

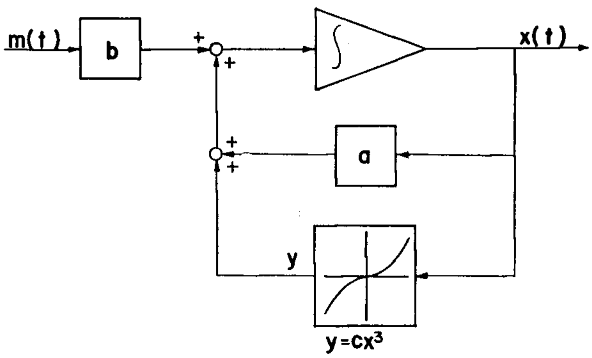

Fig. 3. Block diagram of a non-linear plant.

Hamiltonian treatment. Introducing an adjoint variable p, the Hamiltonian type of function H is defined by

$$H = \frac{1}{2}(x^2+m^2)+p(ax+cx^3+bm). \tag{4.3}$$

The true optimal control signal m^0 is, therefore, obtained by $\underset{m}{\mathrm{Min}}\, H$ as

$$m^0 = -bp. \tag{4.4}$$

Defining the minimum value of H by H^0, we have

$$H^0 = \frac{1}{2}(x^2+b^2p^2)+(ax+cx^3-b^2p). \tag{4.5}$$

The true optimal response is, therefore, determined by

$$\dot{x} = \frac{\partial H^0}{\partial p}, \qquad \dot{p} = -\frac{\partial H^0}{\partial x}. \tag{4.6}$$

In this case, from Eq. (4.6), we have

$$\left.\begin{aligned} \dot{x} &= ax+cx^3-b^2p \\ \dot{p} &= -x-(a+3cx^2)p \end{aligned}\right\} \tag{4.7}$$

Apparently, the boundary conditions for x and p are

$$x(t_0) = c_1, \qquad p(t_e) = 0. \tag{4.8}$$

This is the well-known two-point boundary-value problem of non-linear differential equations, which can only be solved by iterative numerical methods, and which also require lengthy and difficult computations. Therefore, in spite of giving initial condition for $x(t_0)=c_1$, assuming a suitable terminal condition for $x(t_e)$, and solving the non-linear differential equation (4.7) backwards by the Runge-Kutta's method, then we have $x(t_0)=c_1'$, which is regarded as the initially given boundary condition for $x(t_0)$. We call the response obtained by the above-mentioned method as "optimal response."

On the other hand, if a designer decides that the region of the fluctuation of x is only limited in the neighbourhood of $x=0$, he may neglect the second term cx^3 of the right hand side of Eq. (4.1). If it is also assumed that the control time interval $[t_0, t_e]$ is sufficiently large, the design problem turns out to be a very easy way. It is interesting to consider how the control system which is designed on the basis of such two assumptions behaves, when the considerably large initial value $x(t_0)=c_1$ is pre-assigned to the plant. When the controller is designed based on the two assumptions mentioned above we have

$$m^{+} = -\frac{a+\sqrt{a^2+b^2}}{b}x \tag{4.9}$$

as the optimal control signal. This means a time-invariant linear feedback control. The response for this control signal is given by

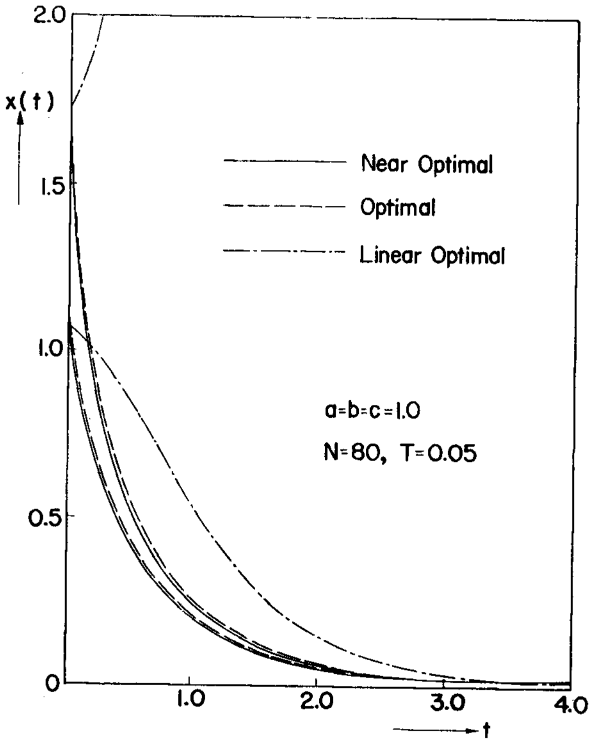

Fig. 4. Comparison of optimal, near-optimal and linear optimal responses.

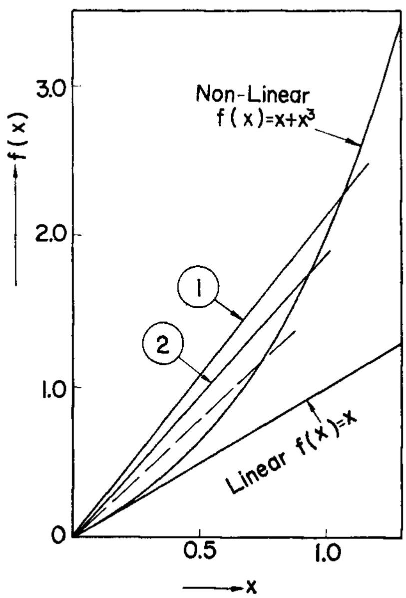

Fig. 5. Successive construction of linear models.

$$\begin{aligned}\dot{x} &= ax+cx^3-(a+\sqrt{a^2+b^2})x \\ &= cx^3-\sqrt{a^2+b^2}x\,,\end{aligned} \tag{4.10}$$

which is named "linear optimal response." In this control system, it should be noticed that if the initial value $x(t_0)=c_1$ satisfies the inequality

$$c_1(cc_1^3-\sqrt{a^2+b^2}c_1)\geq 0 \qquad \text{i.e.,}\ cc_1^2-\sqrt{a^2+b^2}\geq 0\,, \tag{4.11}$$

the system becomes unstable.

Simulations are performed with numerical data as $[t_0, t_e]=[0, 4.0]$, $T=0.05$, $N=80$ and $a=b=c=1.0$.

In Fig. 4, optimal, linear optimal and near-optimal response curves are shown for two initial values. It is shown in Fig. 5 how the non-linear plant is linearized at successive sampling instants in the case where $c_1=1.068$. In Fig. 5, initial linearization is shown by the straight line ① combining the initial state (1.068, 2.287) with the origin. Then the near-optimal control signal is calculated as $m^*(t)=-4.540$, $0\leq t\leq 0.05$. When the plant is driven by this control signal, the state variable x becomes $x(t)=0.9426$ at $t=0.05$. A new linearization shown by the straight line ② is performed at $t=0.05$, which gives $m^*(t)$, $0.05\leq t\leq 0.1$.

5. Conclusions

A method of obtaining a near-optimal control strategy for an integrated quadratic performance index is developed based on the instantaneous linearization method, when the plant is described by non-linear differential equations. Since present results have been obtained in the discrete form, then the design technique proposed here may play an important role in realizing a computer control system, in particular, a system with non-linear high-order characteristics.

The near-optimal control signal obtained is, of course, non-linear in state variables, reflecting the non-linearity of the plant.

It has also been checked that the near-optimal controller is effective, as the simulation studies revealed previously.

An application is also possible in such a case as industrial plants whose non-linear functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in Eq. (3.2) are given by numerical data.

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